Calculus I

Lecture 5

Techniques of Differentiation I

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Introduction to lecture 5

This lecture 5 builds on the previous lecture 4. It will introduce rules of differentiation and the techniques of differentiation namely power and product rules.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain the basic rules of differentiation for algebraic functions.

(ii) Apply the techniques of differentiation namely the power and the product rules.

References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Stewart, 2012; Sullivan & Miranda, 2019).

Rules of differentiation for algebraic functions

Rule I – Constant rule

The derivative of a constant function is the zero function i.e., given $f(x) = k$ where k is any constant, then $\frac{d(k)}{dx} = 0$. For example, suppose $y = f(x) = 7$, then $\frac{dy}{dx} = 0$

Theorem (constant rule)

If k is a constant and $f(x)$ is the constant function defined by $f(x) = k$, then f is differentiable at every point x and $f'(x)$ is the function defined by $f'(x) = 0$.

Proof: Suppose $y = f(x) = k$ then taking a small change in y we get that

$$y + \Delta y = k \Rightarrow \Delta y = k - y$$

But $y = k \Rightarrow \Delta y = k - k = 0$

Hence;

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x}$$

Taking the limits of both sides as $\Delta x \to 0$ we get;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = 0 \therefore \frac{d}{dx}(k) = 0$$

Rule II – Identity rule

The derivative of the identity function is the constant function 1 i.e.

Given $f(x) = x$, then $\frac{d(x)}{dx} = 1$

Rule III - Homogeneous rule

Let $y = f(x) = k \cdot u$ where k is a known constant and u is a differentiable function of x, then $\frac{dy}{dx} = k \cdot \frac{du}{dx}$.

Example 1: Let $y = f(x) = 7x^3$

$$=> \frac{dy}{dx} = 7\frac{dy}{dx}(x^3) = 7 \cdot 3x^2 = 21x^2$$

Rule IV – Sum and difference rule

Given $f(x) = u + v$ where u and v are differentiable functions of x, then;

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v).$$

Theorem 1: Sum rule

Let $g(x)$ and $h(x)$ be functions differentiable at a point x. Let $f(x) = g(x) + h(x)$, then $f(x)$ is also differentiable at x and $f'(x) = g'(x) + h'(x)$.

Proof: By definition $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{[g(x + \Delta x) + h(x + \Delta x)] - [g(x) + h(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[g(x + \Delta x) - g(x) + h(x + \Delta x) - h(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x + \Delta x) - g(x)}{\Delta x} + \frac{h(x + \Delta x) - h(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

$$= g'(x) + h'(x)$$

Example 1: Work out;

$\frac{d}{dx}(2x^4 + 3x^2) = \frac{d}{dx}(2x^4) + \frac{d}{dx}(3x^2)$

$$= 2\frac{d}{dx}(x^4) + 3\frac{d}{dx}(x^2)$$

$$= 2 \cdot 4x^3 + 3 \cdot 2x$$

$$= 8x^3 + 6x$$

Rule V – Power rule

If $y = x^n, n \in \mathbb{R}$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

where $n \neq 0$ when $x = 0$.

Proof: Let $y = x^n$

A small increment in both y and x yields;

$$(y + \Delta y) = (x + \Delta x)^n$$

Expanding the RHS using the binomial theorem we get;

$$y + \Delta y = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots$$

Hence, we have

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots$$

Divide both sides by Δx to get;

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots$$

Next, we find the limits of both sides as $\Delta x \to 0$ i.e.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots \right)$$

To get;

$$\frac{dy}{dx} = nx^{n-1}$$

Example 1: Given the function $y = x^5$ then $\frac{dy}{dx} = \frac{d(x^5)}{dx} = 5x^4$

Example 2: Given the function $y = 3x^{\frac{1}{3}}$ then

$$\frac{dy}{dx} = 3 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}}$$

Example 3: Given $f(x) = 5x^{-3}$ then

$$f'(x) = 5 \cdot (-3)x^{-3-1} = -15x^{-4} = -\frac{15}{x^4}$$

Example 4: Determine the equation of the tangent and normal to the graph of the function $f(x) = 2x^3 + x^2 + 1$ at $x = 1$.

Solution: The gradient function is $f'(x) = 6x^2 + 2x$. Hence the gradient of the graph at;

$x = 1$ is $f'(1) = 6 + 2 = 8$

In our previous lecture 4 we noted that the equation of a tangent to the graph of $f(x)$ at a point x_0 is given as

$$y - f(x_0) = f'(x_0)(x - x_0)$$

our $x_0 = 1$; $f'(x_0) = 8$; $f(x_0) = f(1) = 4$ and therefore we have;

$$y - 4 = 8(x - 1)$$

$$y = 8x - 4 \ - \text{equation of the tangent to the graph at } x = 1$$

The normal is perpendicular to the tangent line and hence its gradient is $-\frac{1}{8}$

The normal line is given by;

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$$

$$y - 4 = -\frac{1}{8}(x - 1)$$

$$y = -\frac{1}{8}x + \frac{1}{8} + 4 = \frac{33}{8} - \frac{1}{8}x$$

$$\therefore 8y + x = 33 - \text{Equation of the Normal}$$

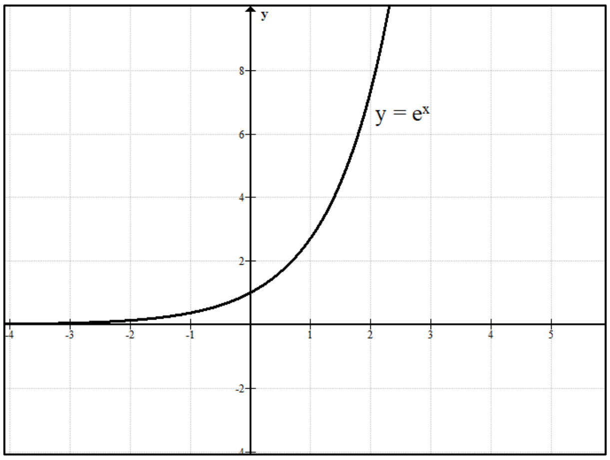
Remark

Rules I – IV are basically application of power rule.

Rule VI – Derivative of the exponential function $f(x) = e^x$

The derivative of the exponential function $f(x) = e^x$ is given by;

$$f'(x) = e^x$$



Example 1: Find the derivative of the function $f(x) = 5e^x - x^4$

Solution: $\frac{d}{dx}(5e^x - x^4) = \frac{d}{dx}(5e^x) - \frac{d}{dx}(x^4) = 5e^x - 4x^3$

Rule VII - Product Rule

Suppose $y = f(x) = uv$ where u and v are differentiable functions of x, then;

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Theorem 1: Let $g(x)$ and $h(x)$ be differentiable functions and if $f(x) = g(x) \cdot h(x)$ then $f(x)$ then $f(x)$ is differentiable, and the derivative of the product $f(x)$ is;

$$f'(x) = [g(x) \cdot h(x)]' = g(x)h'(x) + g'(x)h(x)$$

Proof: by definition;

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{g(x + \delta x)h(x + \delta x) - g(x)h(x)}{\delta x}$$

We add and subtract $g(x + \delta x)h(x)$ to the numerator and then apply the sum rule to simplify i.e.

$$f'(x) = \lim_{\delta x \to 0} \frac{g(x + \delta x)h(x + \delta x) - g(x + \delta x)h(x) + g(x + \delta x)h(x) - g(x)h(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[g(x + \delta x)\frac{h(x + \delta x) - h(x)}{\delta x} + h(x)\frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} g(x + \delta x) \cdot \lim_{\delta x \to 0} \frac{h(x + \delta x) - h(x)}{\delta x} + \lim_{\delta x \to 0} h(x) \cdot \lim_{\delta x \to 0} \frac{g(x + \delta x) - g(x)}{\delta x}$$

$$\therefore f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

Theorem 2:

If $y = uv$ then $y' = uv' + vu'$

Proof

Let $y = uv$ then a small change in y, u , v is denoted by $\Delta y,\ \Delta u,\ \Delta v$ respectively and hence we have;

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

Subtract y from both sides (note that $y = uv$)

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv$$

$$\Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v - uv$$

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$$

$$\frac{\Delta y}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{\Delta u\Delta v}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \lim_{\Delta x \to 0} \Delta v$$

$$= uv' + vu' + u'0 = uv' + vu'$$

Example 1:

The rule for the product of two functions can be extended to the product of any finite number of functions.

Let $y = uvw$ then we have

$$(uvw)' = uvw' + uwv' + wvu'$$

Example 2:

Given $y = (x^2 + 3)(2x - 1)(x^3 + 7x)$ find $\frac{dy}{dx}$

Solution:

Let $u = x^2 + 3 \Rightarrow u' = 2x; v = 2x - 1 \Rightarrow v' = 2;$ and $w = x^3 + 7x \Rightarrow w' = 3x^2 + 7$

Then

$$y' = (uvw)' = wvu' + uwv' + uvw'$$

$$= 2x(2x - 1)(x^3 + 7x) += 2(x^2 + 3)(x^3 + 7x) + (x^2 + 3)(2x - 1)(3x^2 + 7)$$

Example 3: Suppose $f(x) = (x^3 + 2x - 7)(4x^5 - 3x^2 + x)$ find $f'(x)$ at $x = 0$.

Solution: We let $u = x^3 + 2x - 7x^0 => \frac{du}{dx} = 3x^2 + 2$

Also, we let $v = 4x^5 - 3x^2 + x => \frac{dv}{dx} = 20x^4 - 6x + 1$

Thus;

$\frac{d}{dx}(x^3 + 2x - 7)(4x^5 - 3x^2 + x)$

$= (x^3 + 2x - 7)(20x^4 - 6x + 1) + (3x^2 + 2)(4x^5 - 3x^2 + x)$

$$= 32x^7 + 48x^5 - 155x^4 + 4x^3 - 18x^2 + 46x - 7$$

Hence at $x = 0$ then $f'(x) = -7$

Example 4: Determine the first and the second derivatives of $y = t^3x^2 \cos x$ with respect to x.

Solution: The function y is a product of two functions.

Suppose $y = uv$ then $u = t^3x^2$ and $v = \cos x$. Then $u'(x) = 2xt^3; v'(x) = -\sin x$

$$\Rightarrow \frac{dy}{dx} = u(x)v'(x) + u'(x)v(x)$$

$$= -t^3x^2 \sin x + 2xt^3 \cos x$$

$$= 2xt^3 \cos x - t^3x^2 \sin x)$$

$$= xt^3(2 \cos x - x \sin x)$$

Next, we determine the second derivative i.e.

$$\frac{d^2y}{dx^2} = 2xt^3(\cos x)' + 2t^3x' \cos x - [t^3x^2(\sin x)' + t^3(x^2)' \sin x]$$

$$= -2xt^3 \sin x + 2t^3 \cos x - t^3x^2 \cos x - 2t^3x \sin x$$

$$= t^3 \cos x(2 - x^2) - 4xt^3 \sin x$$

Exercise

1) Find the first and second derivatives of the following functions with respect to x

   a) $f(x) = 3x^3 + 7x^2e^{2x}$

   b) $f(x) = x^3 \sin x$

   c) $f(x) = 7 - 2x + 4x^{-3}$

   d) $f(x) = t^3 \cos x + 3x\tan(xt)$

   e) $f(x) = \tan x + \sec^2 x$

2) Find the equation of the tangent and normal lines to the graph of the following functions at the given point.

   a) $f(x) = 2x + 1; x = 3$

   b) $f(x) = 3x^2 + 2x + 1; x = 1$

   c) $f(x) = \sin x; x = \frac{\pi}{2}$

   d) $f(x) = \frac{2}{x}; x = 4$

   e) $f(x) = 3 - x - 5x^2; x = 2$

3) Find the points on the graph of $y = x^4 - 4x^2 + 2$ where the tangent line is parallel to the line $y = 0$.

4) Evaluate $\frac{d}{dx}(x^7 + 3x^5 - 3x^2 - 9x + 2)$. State the rules that you have applied in evaluating this sum.

References

Cowen, R., Were, J., & Vaz, P. (1990). *An Introduction to Calculus*. Nairobi University Press.

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.