Calculus I

Lecture 6

Techniques of Differentiation II

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Introduction to lecture 6

This lecture 6 builds on the previous lecture 5. It will introduce the other techniques of differentiation namely quotient and chain rules, their relationship with the other rules with an emphasis on differentiation of rational and trigonometric functions.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain the basic rules of differentiation; the quotient and chain rule.

(ii) Apply the techniques of differentiation namely the quotient and the chain rules.

References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Stewart, 2012; Sullivan & Miranda, 2019).

Definition1: Quotient Rule

Suppose the functions $f(x)$ and $g(x)$ are differentiable functions of x then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Alternatively, if $f = \frac{u}{v}$ where u and v are functions of x then;

$$f' = \frac{vu' - uv'}{v^2}$$

Proof: Suppose $H(x) = \frac{f(x)}{g(x)}$ then

$$H'(x) = \lim_{h \to 0}\left(\frac{H(x+h) - H(x)}{h}\right)$$

$$\Rightarrow = \lim_{h \to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

Next, we subtract and add the term $f(x)g(x)$ to the numerator to get;

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) \ - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{\frac{g(x)\left(f(x+h) - f(x)\right)}{h} - \frac{f(x)\left(g(x+h) - g(x)\right)}{h}}{g(x+h)g(x)}$$

$$= \frac{\lim_{h \to 0} \frac{g(x)\left(f(x+h) - f(x)\right)}{h} - \lim_{h \to 0} \frac{f(x)\left(g(x+h) - g(x)\right)}{h}}{\lim_{h \to 0} g(x+h)g(x)}$$

$$= \frac{\lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{\left(f(x+h) - f(x)\right)}{h} - \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{\left(g(x+h) - g(x)\right)}{h}}{\lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} g(x)}$$

$$\therefore \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 1: Find the derivative of the function $f(x) = \frac{x^2-5x+1}{2x^3+4}$

Solution: Suppose $u = x^2 - 5x + 1$ then $u' = 2x - 5$ and let $v = 2x^3 + 4 \Rightarrow v' = 6x^2$

Hence;

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{(2x^3 + 4)(2x - 5) - 6x^2(x^2 - 5x + 1)}{(2x^3 + 4)^2}$$

$$f'(x) = \frac{20x^3 - 2x^4 - 6x^2 + 8x - 20}{(2x^3 + 4)^2}$$

Example 2: Find the equation of the tangent to the graph of

$$f(x) = \frac{3\sqrt{x} + x}{2 + x^2} \text{ at } x = 1$$

Solution: Let $u = 3\sqrt{x} + x \Rightarrow u' = 3 \cdot \frac{1}{2}x^{-\frac{1}{2}} + 1 = \frac{3}{2\sqrt{x}} + 1$ and let $v = 2 + x^2 \Rightarrow v' = 2x$

Hence; $f'(x) = \frac{(2+x^2)\left(\frac{3}{2\sqrt{x}}+1\right)-2x\,(3\sqrt{x}+x)}{(2+x^2)^2} = \frac{-x^2-6x\sqrt{x}+2+\frac{1.5x^2+3}{\sqrt{x}}}{(2+x^2)^2}$

At $x = 1$

$$f'(x) = \frac{-1 - 6 + 2 + 1.5 + 3}{9} = -\frac{\frac{1}{2}}{9} = -\frac{1}{18}$$

We use the formula $y - f(x_0) = f'(x_0)(x - x_0)$

Our $f(x_0) = f(1) = \frac{3+1}{2+1} = \frac{4}{3}$

Hence we get; $y - \frac{4}{3} = -\frac{1}{18}(x - 1)$

$$18y - 24 = -x + 1$$

$$x + 18y - 25 = 0$$

Example 3: Show that $\frac{d}{dx}(\tan x) = \sec^2 x$

Solution: We apply the quotient rule since $\tan x = \frac{\sin x}{\cos x}$. We let $u = \sin x$ and $v = \cos x$

$$\Rightarrow u' = \cos x; v' = -\sin x$$

Hence $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\,(\sin x)' - \sin x\,(\cos x)'}{(\cos x)^2}$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Definition 2: Chain rule

Suppose the function $g(x)$ is differentiable at x and the function $h(x)$ is differentiable at $g(x)$, then the composite function $f(x) = h(x) \circ g(x)$ defined by $f(x) = h\big(g(x)\big)$ is differentiable at x and $f'(x)$ is given by the product

$$f'(x) = h'\big(g(x)\big) \cdot g'(x)$$

Alternatively; if $f = h(u)$ where $u = g(x)$ then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Remarks:

(i) The chain rule can be used to find the derivatives of powers of functions.

(ii) The chain rule is used to differentiate composite functions.

(iii) Sometimes you must combine the chain rule with other rules.

Example 1: Find the derivative of the function $f(x) = \ln(x + \cos x)$ with respect to x.

Solution: Let $u = x + \cos x \Rightarrow \frac{du}{dx} = 1 - \sin x$

Also; $f(x) = \ln u \Rightarrow \frac{df}{du} = \frac{1}{u}$ and therefore;

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (1 - \sin x) = \frac{1 - \sin x}{x + \cos x}$$

Example 2: Find the derivative of the function $f(x) = e^{1+x^3}$ with respect to x.

Solution: Let $u = 1 + x^3 \Rightarrow \frac{du}{dx} = 3x^2$

Also; $f(x) = e^u \Rightarrow \frac{df}{du} = e^u$ and therefore;

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^u \cdot 3x^2$$
$$= 3x^2 e^{1+x^3}$$

Example 3: Find the derivative of $y = \tan x^7$

Solution: We let $u = x^7 \Rightarrow \frac{du}{dx} = 7x^6$.

Again $y = \tan u \Rightarrow \frac{dy}{du} = \sec^2 u$.

Therefore;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot 7x^6 = 7x^6 \sec^2 x^7$$

Example 4: Let $y = e^{\cos x^2}$ hence find $\frac{dy}{dx}$.

Solution: Let $u = \cos v$, where $v = x^2$

Our function can be written as $y = e^u \therefore \frac{dy}{du} = e^u$.

$$\frac{du}{dv} = -\sin v; \frac{dv}{dx} = 2x$$

Therefore;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = -e^u \cdot \sin v \cdot 2x = -2xe^{\cos x^2} \sin x^2$$

Example 5: Find the derivative of $y = e^{2x^3-1}$

Solution: Let $u = 2x^3 - 1 \Rightarrow \frac{du}{dx} = 6x^2$

Our function can be written as $y = e^u \therefore \frac{dy}{du} = e^u$

Therefore;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 6x^2 = 6x^2 e^{2x^3-1}$$

Definition 3: If $n \in \mathbb{R}$ and the function $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example 1: Find the derivative of $y = \sin^7 x$

Solution: Let $u = \sin x \Rightarrow y = u^7$ and therefore $\frac{du}{dx} = \cos x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^7 x) = nu^{n-1}\frac{du}{dx} = 7\sin^6 x \cos x.$

Example 2: Find the derivative of $y = \frac{1}{\sqrt[5]{x^3+2}}$

Solution: Let $u = x^3 + 2 \Rightarrow \frac{du}{dx} = 3x^2$. Again $y = u^{-\frac{1}{5}}$

Therefore $\frac{dy}{dx} = nu^{n-1}\frac{du}{dx} = -\frac{1}{5}(x^3+2)^{-\frac{6}{5}} \cdot 3x^2$

$$= -\frac{3}{5}x^2(x^3+2)^{-\frac{6}{5}}$$

Example 3: Find the derivative of $y = (3x^2 + x)^6(3x^4 - 2x^3 + 7x)^3$ with respect to x.

Solution: This problem will require the application of product and chain rules.

Let $u = 3x^2 + x \therefore \frac{du}{dx} = 6x + 1$

Let $v = 3x^4 - 2x^3 + 7x \therefore \frac{dv}{dx} = 12x^3 - 6x^2 + 7$

Now we have $y = u^6v^3$

Let $g = u^6 \therefore \frac{dg}{du} = 6u^5$;

let $h = v^3 \therefore \frac{dh}{dv} = 3v^2$

We can have $y = gh$ where g and h are functions of u and v respectively.

Note that;

$$\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = 6u^5(6x+1)$$
$$= 6(3x^2+x)^5(6x+1)$$

Again;

$$\frac{dh}{dx} = \frac{dh}{dv} \cdot \frac{dv}{dx} = 3(3x^4 - 2x^3 + 7x)^2(12x^3 - 6x^2 + 7)$$

Now since $y = gh$ then

$\frac{dy}{dx} = gh' + hg'$ (With respect to x).

$$= 3(3x^2+x)^6(3x^4 - 2x^3 + 7x)^2(12x^3 - 6x^2 + 7)$$
$$+6(3x^4 - 2x^3 + 7x)^3(3x^2+x)^5(6x+1)$$

Example 4: Find the equation of the tangent to the graph of function below at $x = 1$

$$f(x) = \frac{\sqrt{x}}{1 + x^3}$$

Solution: Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

Let $v = 1 + x^3 \Rightarrow \frac{dv}{dx} = 3x^2$

$$\therefore \frac{df}{dx} = \frac{\frac{1}{2}(1 + x^3)x^{-\frac{1}{2}} - 3x^2\sqrt{x}}{(1 + x^3)^2}$$

At $x = 1$; $\frac{df}{dx} = \frac{\frac{1}{2}(1+1)-3}{2^2} = -\frac{2}{4} = -\frac{1}{2}$

The equation of the tangent is;

$$-\frac{1}{2} = \frac{y - \frac{1}{2}}{x - 1} \Rightarrow y - \frac{1}{2} = -\frac{1}{2}(x - 1) \therefore 2y + x - 2 = 0$$

Derivatives of basic elementary functions

There exist a calculus table for basic elementary functions that need not be solved always. You need to be aware of such functions or be able to use the table for derivative of basic functions. For example, if x is the independent variable then;

(i) $\frac{d}{dx}(\sin x) = \cos x$

(ii) $\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\frac{d}{dx}(\cot x) = \text{cosec}^2 x$

(v) $\frac{d}{dx}(a^x) = a^x \ln a$

(vi) $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

(vii) $\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$

Exercise

1) Prove the chain rule.

2) Show that $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

3) Show that $\frac{d}{dx}(a^x) = a^x \ln a$

4) Find the derivative of following functions with respect to x.

   a) $y = \ln(\ln(\ln x))$

   b) $f(x) = x^x$, with $x > 0$

   c) $f(x) = \frac{x^4 + 3\ln x}{e^{2x}}$

   d) $g(x) = \frac{\cos x}{e^x - 2x}$

   e) $h(x) = \frac{\sin x}{x}$

5) Find the derivative of the following functions with respect to x.

   a) $f(x) = \cos(x^2 \sin x)$

   b) $f(x) = \frac{1 + 3\cos 4x}{1 - \cos 4x}$

   c) $f(x) = \sqrt{\csc^3 x}$

   d) $f(x) = [x^3 + (1 - 5x)^3]^5$

   e) $f(x) = \sqrt{\frac{x^3 + 2}{x^5 + 3}}$

6) Determine the equation of the tangent line to the graph of the below functions at the given points.

   a) $f(x) = \frac{1}{x}$; $x = 1$

   b) $f(x) = \frac{2x+1}{x}$ at $x = 2$

   c) $f(x) = \frac{1}{2x^2 + 3x + 1}$ at $x = -1.5$

References

Cowen, R., Were, J., & Vaz, P. (1990). *An Introduction to Calculus*. Nairobi University Press.

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Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.