Calculus I

Lecture 7

Basic Theorems in Differential Calculus

Lecturer: Kahenya, N.P

Introduction to lecture 7

Lecture 7 will introduce the basic theorems that are important to understanding the concept of differential and by extension integration. So far, we have looked at the intermediate value theorem that we noted it can be used in determining the zeros in a given polynomial function. Other basic theorems for differential calculus are the Fermat's, Rolle's, and Mean Value theorems.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain the basic theorems in differential calculus.

(ii) Apply the basic theorems in problem solving.

References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Stewart, 2012; Sullivan & Miranda, 2019).

Extreme Value theorem

The extreme value theorem states that if a function f is defined over a closed interval $[\mathrm{a},\mathrm{b}]$, then it attains an absolute maximum value $\mathrm{f}(\alpha)$ and an absolute minimum value $\mathrm{f}(\beta)$ at some points α and β in the closed interval $[\mathrm{a},\mathrm{b}]$.

Fermat's Theorem

States that if a function $\mathrm{f}(\mathrm{x})$ have a relative extrema (i.e. it reaches its minimal or maximal value) at point $\mathrm{x}=\mathrm{c}$ and it is differentiable at this point $\mathrm{x}=\mathrm{c}$ i.e. $\mathrm{f}'(\mathrm{c})$ exists, then this point is a critical point of the function $\mathrm{f}(\mathrm{x})$ such that;

$$\mathrm{f}'(\mathrm{c})=0$$

Proof (for relative maximum)

Suppose we have a relative maximum at point $x = c$ then $f(c) \geq f(x)$ for all x that are sufficiently close to this point.

Then for all δx that are sufficiently close to zero then we have

$$f(c) \geq f(c + \delta x) \cdots (*)$$

Let assume $\delta x > 0$ and rewrite (i) and divide it with this δx to get;

$$\frac{f(c + \delta x) - f(c)}{\delta x} \leq 0$$

Next we determine the limits of the right-hand side to get;

$$\lim_{\delta x \to 0^+} \frac{f(c + \delta x) - f(c)}{\delta x} \leq \lim_{\delta x \to 0^+} 0 = 0$$

We can now assume that $f'(c) = \lim_{\delta x \to 0^+} \frac{f(c+\delta x) - f(c)}{\delta x} \leq 0 \Rightarrow f'(c) \leq 0 \cdots$ (i)

Now let assume $\delta x < 0$ and we rewrite (i) and divide it with $\delta x < 0$ to get;

$$\frac{f(c + \delta x) - f(x)}{\delta x} \geq 0$$

(Recall that when we divide by a negative value the inequality sign changes).

Assuming again the $f'(c)$ exists, we have;

$$f'(c) = \lim_{\delta x \to 0^-} \frac{f(c + \delta x) - f(c)}{\delta x} \geq \lim_{\delta x \to 0^-} 0 = 0$$

$$\Rightarrow f'(c) \geq 0 \cdots \text{(ii)}$$

From (i) and (ii) we can conclude $f'(c) = 0$ which implies that $x = c$ and it must be a critical point.

Similarly, it can be shown for relative minimum.

Geometrical interpretation for Fermat's theorem

The tangent line to the graph of the function $f(x)$ at the critical point $x = c$ is parallel to the horizontal axis (x-axis) and the slope of this tangent line is zero i.e. $f'(c) = 0$ (see figure 1 below). The equation of this tangent line is $y = f(c)$. The equation of the normal to the tangent line is given by $x = c$.

Function $f(x)$ with extreme values at the points in the domain of $f(x)$ for which $f'(x) = 0$. These points are called stationary points.

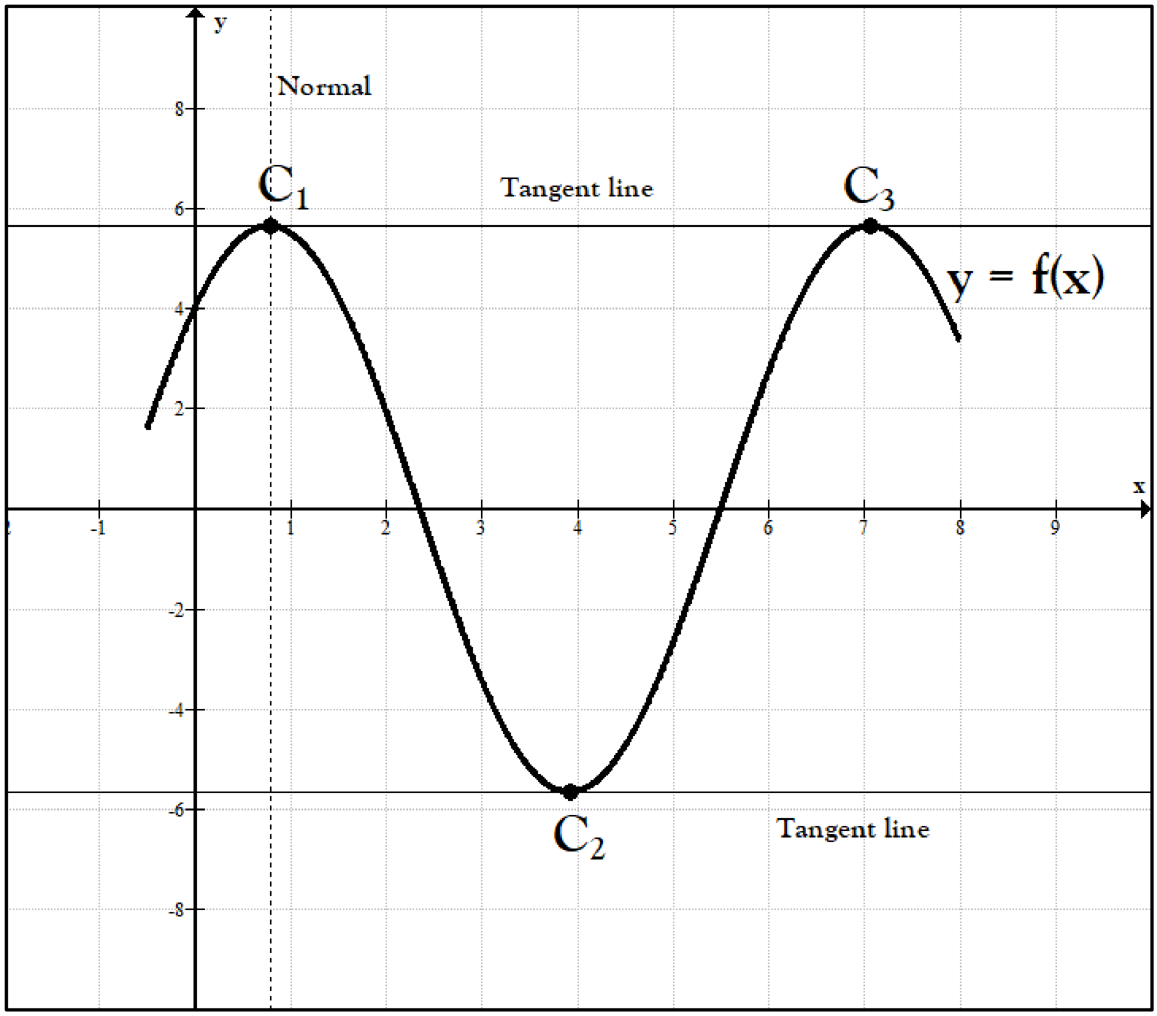

Figure 1

Remark 1: Even if $f'(c) = 0$ it is not a must that there is a minimum and maximum at point c.

Remark 2: A critical number $c \in \mathbb{D}(f)$ is such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 1: Given a function $\mathrm{f(x) = x^3}$ then $\mathrm{f(c) = c^3 \Rightarrow f'(c) = 3c^2}$. Hence to satisfy the Fermat's theorem we have;

$$\mathrm{f'(c) = 0 \Rightarrow 3c^2 = 0 \therefore c = 0}$$

The critical point is at $x = 0$ however the function has no maximum or minimum at this point (see figure 2 below).

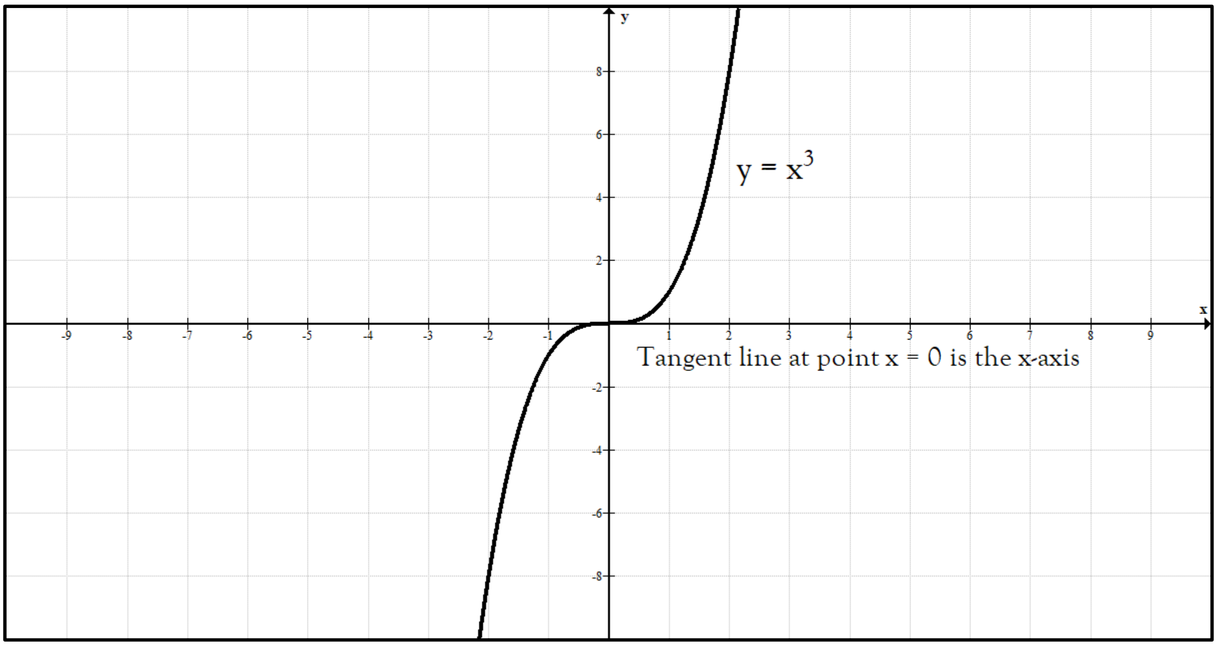

Figure 2

Example 2: Consider the function $f(x) = |x|$ i.e. the absolute value function. This function has its minimum at point $x = 0$ (See figure 3 below) However the limit of this function does not exist at point $x = 0$, and hence the derivative of f at $x = 0$ doesn't exists. Therefore, we cannot use the Fermat's theorem that show that $f'(c) = 0$ at the critical point.

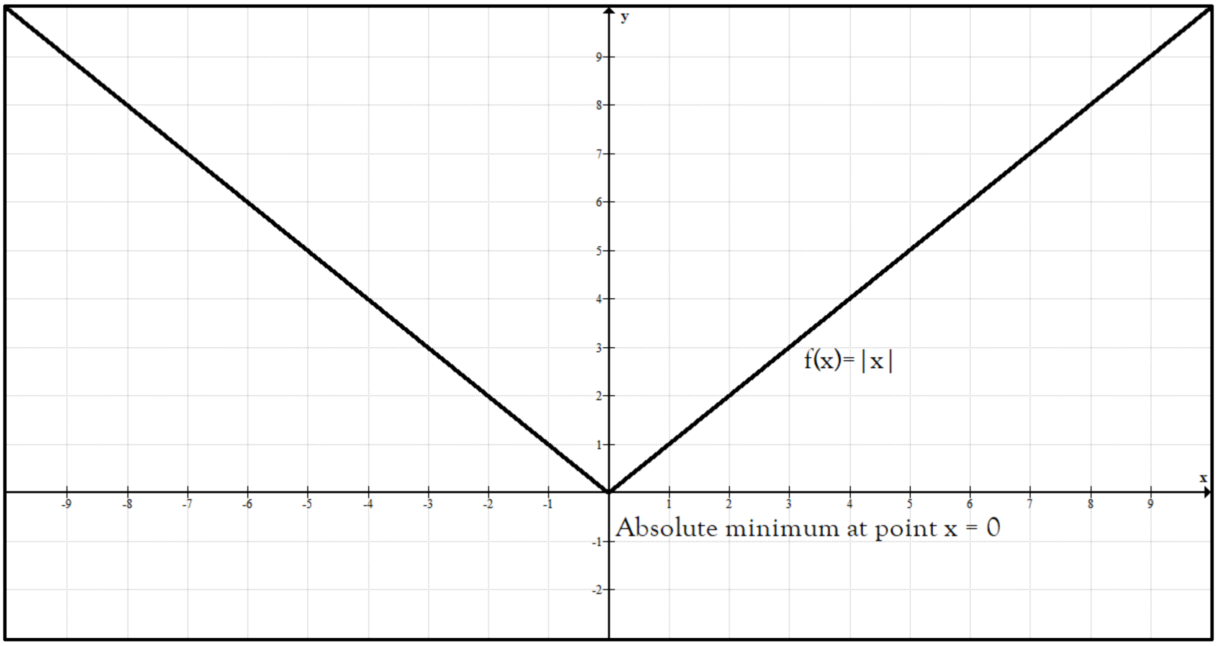

Figure 3

Rolle's Theorem

States that if a function $f(x)$ satisfy the following hypotheses;

(i) It is continuous over the closed interval $[a, b]$.

(ii) It is differentiable over the open interval (a, b).

(iii) $f(a) = f(b)$

Then there exists a point $c \in (a, b)$such that $f'(c) = 0$.

Proof (for case $\mathrm{f(x) < f(a)}$ for some $\mathrm{x \in (a, b)}$ as in figure 4 below).

Applying the extreme value theorem f has a minimum value in the closed interval $[a, b]$ since $\mathrm{f(a) = f(b)}$. The function attains its minimum value at $\mathrm{c \in (a, b)}$. Using Fermat's theorem $f'(c) = 0$.

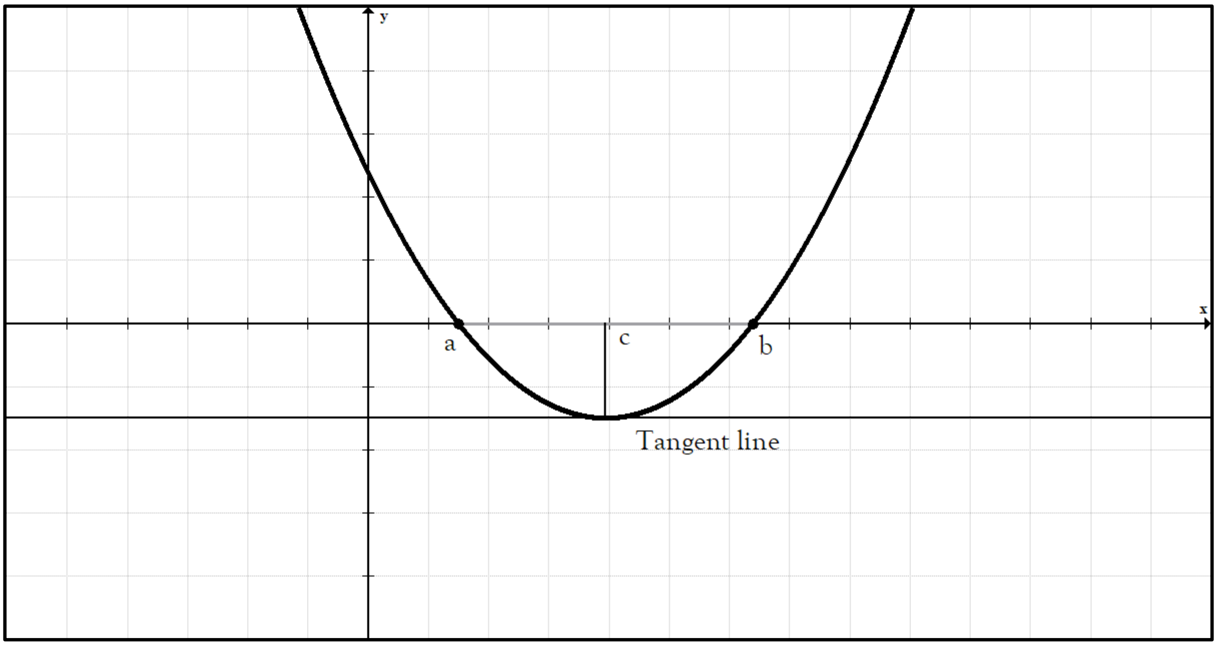

Figure 4

Example 1: Verify the hypotheses of Rolle's theorem for the function below over the interval $\mathrm{[a, b] = [-2, 3]}$ and hence determine a point in the interval that satisfy the conclusion of the theorem.

$$\mathrm{f(x) = x^3 - 2x^2 - 5x + 6}$$

Solution: The first and the second hypotheses are satisfied since the function is a polynomial function. Polynomial functions are continuous and differentiable everywhere.

For the 3rd hypothesis i.e. is $\mathrm{f(a) = f(b)}$? we have;

$$\mathrm{f(a) = f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0}$$

$$\mathrm{f(b) = f(3) = 3^3 - 2(3)^2 - 5(3) + 6 = 0}$$

$$\therefore \mathrm{f(a) = f(b) = 0}$$

Hence, we need to find a $\mathrm{c} \in (\mathrm{a}, \mathrm{b})$ such that $\mathrm{f}'(\mathrm{c}) = 0$.

$$\mathrm{f}(\mathrm{c}) = \mathrm{c}^3 - 2\mathrm{c}^2 - 5\mathrm{c} + 6 \Rightarrow \mathrm{f}'(\mathrm{c}) = 3\mathrm{c}^2 - 4\mathrm{c} - 5$$

Next, we have; $3\mathrm{c}^2 - 4\mathrm{c} - 5 = 0$

$$\Rightarrow \mathrm{c} = \frac{4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{4 \pm \sqrt{76}}{6} \approx -0.7863; 2.1196$$

Therefore, the function satisfies the Rolle's theorem over the interval $[-2,3]$.

Example 2: Verify the hypotheses of Rolle's theorem for the function below over the given interval; $\mathrm{f}(\mathrm{x}) = \sqrt{1 - \cos x}$, $[\mathrm{a}, \mathrm{b}] = [0, 2\pi]$ and hence determine a point in the interval that satisfy the conclusion of the theorem.

Proof: The first and the second hypotheses are satisfied since the function is a sine function. Such functions are continuous and differentiable over the defined interval.

For the 3rd hypothesis i.e. is $\mathrm{f}(\mathrm{a}) = \mathrm{f}(\mathrm{b})$? we have;

$$\mathrm{f}(\mathrm{a}) = \mathrm{f}(0) = \sqrt{1 - \cos 0} = 0$$

$$\mathrm{f}(\mathrm{b}) = \mathrm{f}(2\pi) = \sqrt{1 - \cos 2\pi} = 0$$

Therefore $\mathrm{f}(\mathrm{a}) = \mathrm{f}(\mathrm{b})$

Hence we need to find a $\mathrm{c} \in (\mathrm{a}, \mathrm{b})$ such that $\mathrm{f}'(\mathrm{c}) = 0$.

$$\mathrm{f}(\mathrm{c}) = \sqrt{1 - \cos c}$$

Let $\mathrm{u} = 1 - \cos \mathrm{c} \therefore \frac{\mathrm{du}}{\mathrm{dc}} = \sin \mathrm{c}$. Again, $\mathrm{f}(\mathrm{c}) = \mathrm{u}^{\frac{1}{2}} \Rightarrow \frac{\mathrm{df}}{\mathrm{du}} = \frac{1}{2}\mathrm{u}^{-\frac{1}{2}}$

Therefore;

$$\mathrm{f}'(\mathrm{c}) = \frac{\mathrm{df}}{\mathrm{du}} \times \frac{\mathrm{du}}{\mathrm{dc}} = \frac{1}{2}\mathrm{u}^{-\frac{1}{2}} \times (\sin\ \mathrm{c}) = \frac{\sin c}{2\sqrt{1 - \cos c}}$$

Next we have;

$$\mathrm{f}'(\mathrm{c}) = 0 = \frac{\sin c}{2\sqrt{1 - \cos c}}$$

$\Rightarrow \sin c = 0 \therefore c = 0, 2\pi$

The function satisfies Rolle's theorem at the given interval.

Mean Value Theorem/Lagrange Theorem

States that if a function $f(x)$ satisfies the following hypotheses;

(i) It is continuous over the closed interval $[a, b]$

(ii) It is differentiable over the open interval (a, b)

Then there exists at least a point $c \in (a, b)$ such that;

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation of Mean value theorem

There exists a point $c \in (a, b)$ where the slope of the tangent , $f'(c)$, to the graph of $f(x)$ is the same as the gradient of the secant line AB, $\frac{f(b)-f(a)}{b-a}$ (see figure below). That is;

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

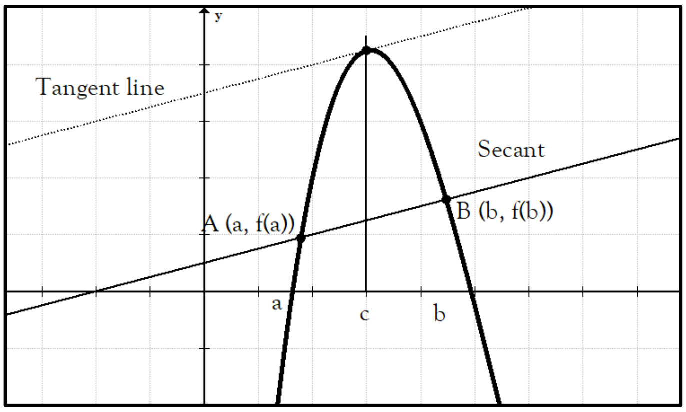

Figure 5

Proof

The proof of mean value theorem uses the Rolle's theorem. Consider the figure below.

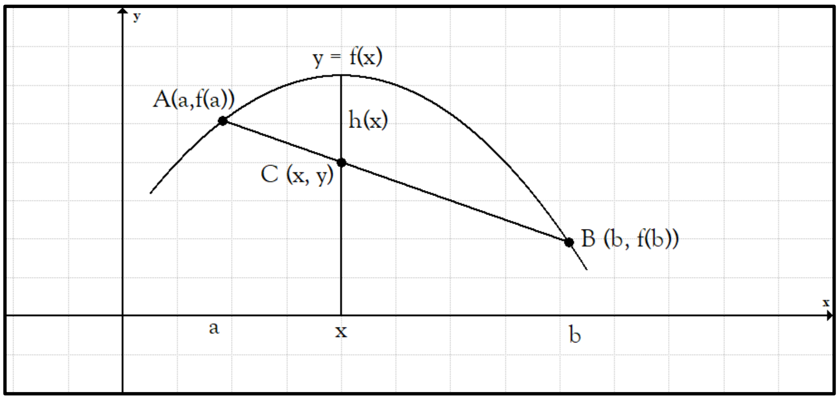

Figure 6

Let the function $h(x)$ be difference between $f(x)$ and the graph of the secant line AB…(i)

To get the equation of the secant line AB consider point $C(x, y)$.

$$\text{Gradient of AC} = \frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\therefore\ y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \cdots \text{(ii)} - \text{Equation of the graph of secant line AB}$$

From the statement (i) $h(x) = f(x) - \text{(ii)}$ i.e.,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \cdots \text{(iii)}$$

Next we need to show that the function $h(x)$ satisfies the Rolle's theorem:

Hypothesis 1: Is $h(x)$ continuous on $[a, b]$? $h(x)$ is sum of two continuous functions that are polynomials, hence it is continuous.

Hypothesis 2: Is $h(x)$ differentiable on (a, b)? $h(x)$ is sum of two differentiable functions that are polynomials, hence it is differentiable. From equation (iii) we can see that;

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Hypothesis 3: Is $h(a) = h(b)$?

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a)$$

$$= f(b) - f(a) - f(b) + f(a) = 0$$

Hence $h(a) = h(b)$

Since we have seen that the three hypotheses are satisfied it implies that there exists a $c \in (a, b)$ such that $h'(c) = 0$. Thus

$$h'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Therefore;

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 1: If $f'(c) = 0$ for all $x \in (a, b)$, then the function $f(x)$ is a constant function on this interval i.e. $f(x) = c, c \in \mathbb{R}, \forall x \in (a, b)$.

Corollary: If $f'(x) = g'(x)$ for all $x \in (a, b)$ the function $(f - g)(x)$ is constant in the interval (a, b) i.e. $f(x) = g(x) + c, c \in \mathbb{R}, \forall x \in (a, b)$

Example 1: Suppose the function $\mathrm{s} = \mathrm{f}(\mathrm{t})$ represents trajectory of a point in motion. The derivative of the function will represent the velocity of the point at time t. The average velocity of the point at time interval $[\mathrm{t}_0, \mathrm{t}_1]$ is;

$$\frac{\mathrm{f}(\mathrm{t}_1) - \mathrm{f}(\mathrm{t}_0)}{\mathrm{t}_1 - \mathrm{t}_0}$$

Then at some moment or time $t_c \in (t_0, t_1)$ the instantaneous velocity will be the same as the average velocity i.e.

$$\mathrm{f}'(\mathrm{t}_c) = \frac{\mathrm{f}(\mathrm{t}_1) - \mathrm{f}(\mathrm{t}_0)}{\mathrm{t}_1 - \mathrm{t}_0}$$

Example 2: Verify the Mean value theorem for the function $\mathrm{f}(\mathrm{x}) = (\mathrm{x} - 2)(\mathrm{x} + 3)(\mathrm{x} - 1)$ over the interval $[-3,3]$ and hence determine a point in the interval that satisfy the conclusion of the theorem.

Solution: The function is a polynomial function and therefore is continuous and differentiable over the given interval. Therefore, we need to determine a point $\mathrm{c} \in (-3,3)$ such that

$$\mathrm{f}'(\mathrm{c}) = \frac{\mathrm{f}(3) - \mathrm{f}(-3)}{3 - (-3)}$$

$$\mathrm{f}(3) = (3 - 2)(3 + 3)(3 - 1) = 12$$

$$\mathrm{f}(-3) = (-3 - 2)(-3 + 3)(-3 - 1) = 0$$

Next we have $\mathrm{f}(\mathrm{c}) = (\mathrm{c} - 2)(\mathrm{c} + 3)(\mathrm{c} - 1) = \mathrm{c}^3 - 7\mathrm{c} + 6 \Rightarrow \mathrm{f}'(c) = 3\mathrm{c}^2 - 7$

Therefore

$$3c^2 - 7 = \mathrm{f}'(\mathrm{c}) = \frac{\mathrm{f}(3) - \mathrm{f}(-3)}{3 - (-3)} = \frac{12}{6} = 2$$

$$3\mathrm{c}^2 - 7 = 2$$

$$3\mathrm{c}^2 = 9 \therefore \mathrm{c} = \pm\sqrt{3} \in (-3,3)$$

The function satisfies the MVT at the interval $[-3,3]$.

Example 3: Verify that the function below satisfies the hypotheses of the Mean Value Theorem on the interval $[1,5]$ and hence determine a point in the interval that satisfy the conclusion of the theorem.

$$\mathrm{f}(\mathrm{x}) = \frac{2}{\mathrm{x}}$$

Solution: The function $f(x) = \frac{2}{x}$ is a rational function and continuous over the interval $[1,5]$ and hence it is differentiable over the interval $(1,5)$.

$$f(1) = 2; f(5) = \frac{2}{5} = 0.4$$

$$f(c) = \frac{2}{c} \therefore f'(c) = -\frac{2}{c^2}$$

Hence we get;

$$-\frac{2}{c^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0.4 - 2}{5 - 1} = -0.4$$

$$\Rightarrow \frac{2}{c^2} = 0.4$$

$$c^2 = 5 \therefore c = \sqrt{5}$$

We ignore $-\sqrt{5} \notin [1,5]$

The function satisfies the MVT at the given interval.

Example 4: Consider the function $f(x) = x^{\frac{2}{3}}$ over the interval $[-1,1]$. Verify if the function satisfies the hypotheses of the MVT over the interval.

Solution: The function is continuous over the interval $[-1,1]$.

However, the function is not differentiable at point $x = 0$.

The function fails to satisfy the MVT, however it does not contradict the theorem.

Exercise

1) Prove theorem 1.
2) Prove the corollary to theorem1.
3) Verify the hypotheses of the Rolle's theorem for each function over the indicated interval and hence determine a point in the interval that satisfies the conclusion of the theorem.
   a) $f(x) = x^2 - 2x, [0,2]$
   b) $f(x) = x^2 + x - 2, [-2,1]$
   c) $f(x) = x^3 - 3x^2 - x + 3, [-1,3]$
   d) $f(x) = |x|, [-5,5]$
   e) $f(x) = \cos 2x\,, [0,2\pi]$
4) Verify the hypotheses of the Mean Value Theorem for each function over the indicated interval and hence determine a point in the interval that satisfies the conclusion of the theorem.
   a) $f(x) = \sqrt[3]{x}\,;\ [-2,2$
   b) $f(x) = \frac{1}{x}; [-1,1]$
   c) $f(x) = \frac{3}{x}; [3,9]$
   d) $f(x) = \frac{x-1}{x+1}; [0,3]$
   e) $f(x) = x + 3\sin^2 2x\,; [-\frac{7}{11}\pi, \frac{7}{11}\pi]$

References

Cowen, R. ., Were, J. ., & Vaz, P. . (1990). *An Introduction to Calculus*. Nairobi University Press.

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Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.