Calculus I

Lecture 10

Rates of Change and Related Rates

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Introduction to lecture 10

Lecture 10 introduces some key application of differentiation i.e. rate of changes. It is an application of what we have so far discussed especially the definition of derivative. The lecture will also discuss rate of changes of related variables referred to as related rates problems.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Apply differentiation to rate of changes.

(ii) Solve problems involving rate of changes.

References for further reading

The lecture notes have been adopted from relevant topics from (Briggs et al., 2015; Rogawski et al., 2019; Stewart, 2012; Sullivan & Miranda, 2019).

Definition 1: Rate of change

Suppose $y = f(x)$, and that x changes from x_0 to x_1 then the change in x or the increment in x is given by;

$$\Delta x = x_1 - x_0$$

Since y is a function of x then the corresponding change in y will be;

$$\Delta y = f(x_1) - f(x_0)$$

The average rate of change of y with respect to x is the difference quotient;

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This can be interpreted as the gradient of the secant (see figure below).

Now if let x_1 approach x_0 then Δx will approach 0. The limit of the average rate of change is called the instantaneous rate of change of y with respect to x. This can be interpreted as the gradient of the tangent line to the curve at point x_0 (see figure below).

Hence, the derivative $f'(x_0)$ is the instantenous rate of change of y with respect to x at point x_0 i.e.

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

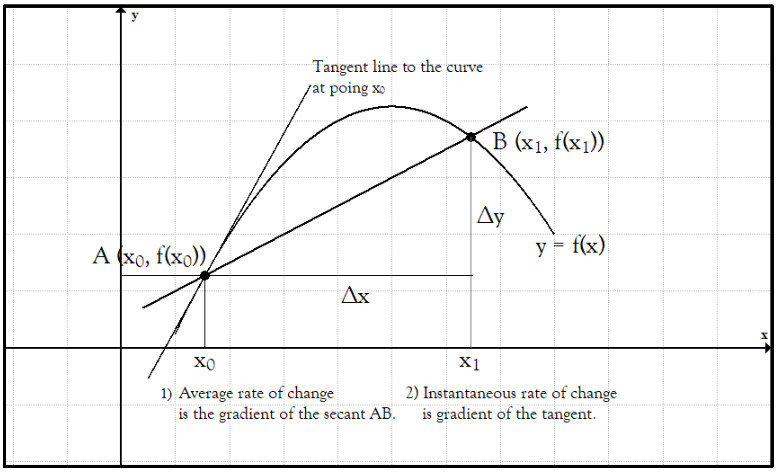

Figure 1

Application examples (Rate of change)

The concept of rate of change is applicable to diverse fields such as economics and business models, social science, population growth models of say viruses, sound, engineering, change in weather patterns or elements among many other fields.

Example 1: The population of a virus is modeled by the function $p(t) = 200\left(\frac{t^2+1}{t^2+7}\right)$ where $t \geq 0$ is time in hours.

i) Determine the instantaneous growth rate of the virus population for $t \geq 0$

ii) Determine the steady-state population.

Solution:

i) $p'(t) = \frac{d}{dt}\left(200\left(\frac{t^2+1}{t^2+7}\right)\right) = \frac{2400t}{(t^2+7)^2}$

ii) The virus population will approach a fixed value or limit over a long period of time. This is what we call the steady-state population. Hence, we need to determine the limit as t approaches ∞.

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\left(200\left(\frac{t^2+1}{t^2+7}\right)\right) = 200\cdot\lim_{t\to\infty}\left(\frac{1+\frac{1}{t^2}}{1+\frac{7}{t^2}}\right) = 200$$

As the population approaches steady state the growth rate approaches zero.

Example 2: The radius of an elastic spherical container is increasing at the rate of $2\ \text{cm/s}$. Determine the rate of change of its volume and surface area when the radius r is 7 cm.

Solution: Let the volume of the sphere at time t seconds be given by;

$$v = \frac{4}{3}\pi r^3$$

It is given that the rate of change in radius is;

$$\frac{dr}{dt} = 2\ \text{cm/s}$$

Our volume is $v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2$

But $\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt} = 4\pi r^2\cdot 2 = 8\pi r^2$

$$\therefore \left.\frac{dv}{dt}\right|_{r=7} = 8\pi(7)^2 = 392\pi\ \text{cm}^3 s^{-1}$$

This means that the volume is increasing at the rate of $392\pi\ \text{cm}^3 s^{-1}$ when the radius of the spherical container is 7 cm. Next the surface area of a sphere is given by $A = 4\pi r^2$, then

$$\frac{dA}{dr} = 8\pi r$$

$$\therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 8\pi r\cdot 2 = 16\pi r$$

At $r = 7$ we have;

$$\left.\frac{dA}{dt}\right|_{r=7} = 16\pi(7) = 112\pi\ \text{cm}^2\text{s}^{-1}$$

This means that the surface area is increasing at rate of $112\pi\ cm^2 s^{-1}$ when the radius of the container is 7 cm.

Example 3: The volume v of a right circular cylinder is given by $v = f(r,h) = \pi r^2 h$ where $r = 7\ cm$ and $h = 12\ cm$ are the radius of the base and the height of the cylinder respectively. Determine;

(i) The instantaneous rate of change of the volume v with respect to height h if the radius remains constant and height varies.

(ii) The rate of change at the instant when $h = 12\ cm$ (radius remains constant).

(iii) The rate of change of the volume with respect to the radius at the instant when $r = 7$ cm

Solution: This is an application of partial derivatives.

(i) $\frac{\partial v}{\partial h} = \pi r^2$

(ii) $\Rightarrow \left.\frac{\partial v}{\partial h}\right|_{r=7} = 49\pi\ cm^3/cm$

(iii) $\frac{\partial v}{\partial r} = 2\pi rh \Rightarrow \left.\frac{\partial v}{\partial r}\right|_{r=7,h=12} = 2\pi(7)(12) = 168\pi\ cm^3/cm$

Example 4: The temperature (in Celcius) of a surface $z = T(x, y)$ at any point (x, y) is given by;

$$T(x, y) = 16(x^2 + y^2 + x)^2$$

i) Find the rate of change in temperature with respect to x at point $(1,3)$.

ii) Find the rate of change in temperature with respect to y a point $(1,3)$ (the equation resembles a water tumbler).

Solution: (i) Let $u = x^2 + y^2 + x \Rightarrow u_x = \frac{\partial u}{\partial x} = 2x + 1$. Also $T = 16u^2 \Rightarrow T_u = \frac{\partial T}{\partial u} = 32u$

$$T_x = \frac{\partial T}{\partial x} = T_u \cdot u_x = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial x} = 32u(2x + 1) = 32(2x + 1)(x^2 + y^2 + x)$$

$$\therefore \left.\frac{\partial T}{\partial x}\right|_{x=1,y=3} = 32(3)(1 + 9 + 1) = 1056^0C$$

The temperature is increasing at the rate of $1056^0 C$ per unit of distance.

Next , Let $u = x^2 + y^2 + x \Rightarrow u_y = \frac{\partial u}{\partial y} = 2y$. Also $T = 16u^2 \Rightarrow T_u = \frac{\partial T}{\partial u} = 32u$

$$T_y = \frac{\partial T}{\partial y} = T_u \cdot u_y = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial y} = 32u(2y) = 64y(x^2 + y^2 + x)$$

$$\therefore \left.\frac{\partial T}{\partial x}\right|_{x=1, y=3} = 192(1 + 9 + 1) = 2112^0 C$$

The temperature is increasing at the rate of $2112^0 C$ per unit of distance.

Definition 2: Approximation

From a previous lecture, we noted that;

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

where Δy and Δx are small quantities in y and x. respectively. Then we have the approximation;

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

This is used to determine the small increments δy in y as x increase by a small amount δx.

Note that the approximation below becomes more accurate as δx becomes smaller:

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

Example 1: Suppose $y = \sqrt{x}$, find the approximate increase in y if x is increased from 16.0 to 16.001.

Solution: The increase in x; $\delta x = 0.001$ when $x = 16.0$. It is given that;

$$y = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

by definition, for small values δx;

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \Rightarrow \delta y \approx \frac{dy}{dx} \cdot \delta x = \frac{1}{2\sqrt{x}} \cdot \delta x = \frac{1}{2\sqrt{16}} \times 0.001 = \frac{1}{8} \times 0.001 = 0.000125$$

$$\delta y \approx 0.000125$$

The approximate increase in y is 0.000125

Example 2: Find the approximate change in y, where $y = \frac{3}{5}x^5 + 2x$ when x decrease from 7.0 to 6.999.

Solution: The increase in x, $\delta x = 6.999 - 7.0 = -0.001$ when $x = 7.0$

We have $y = \frac{3}{5}x^5 + 2x \Rightarrow \frac{dy}{dx} = 3x^4 + 2$

For small values of δx, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

$$\Rightarrow \delta y \approx \frac{dy}{dx} \cdot \delta x = (3x^4 + 2)\delta x$$

At $\delta x = -0.001; x = 7.0$ we have;

$\delta y \approx -0.001(3(7)^4 + 2) = -7.205 \ \therefore \delta y \approx -7.205$

That is y decreases approximately by 7.205

Example 3: Two roads intersect at right-angles. A police patrol car is 0.6 km from the intersection and is heading towards it at a speed of 100 kph. A suspect car is on the other road and 1.2 km away from the intersection. The suspect car is moving away from the intersection at 84 kph. Determine the instantaneous rate of change in distance between the police patrol car and the suspect car.

Solution: Consider the sketch diagram below of the relative positions of the two cars.

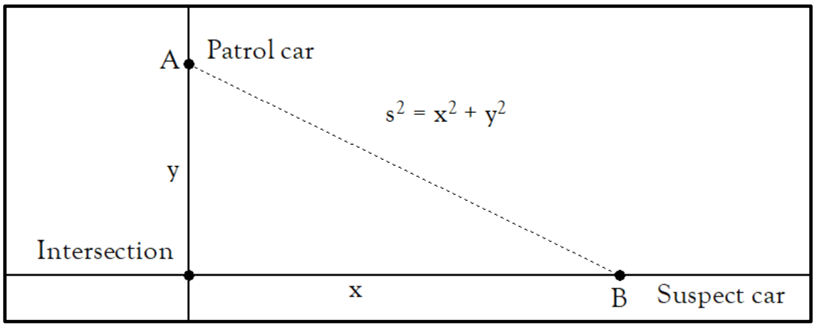

Figure 2

If y is the distance of the patrol car from the intersection while x is the distance of the suspect car from the intersection then the shortest distance between the two cars at time t is

$$s = \sqrt{x^2 + y^2}$$

The rate of change of distance with respect to time is speed. Hence we have;

$$\frac{ds}{dt}=\frac{d}{dt}\left(\sqrt{x^2+y^2}\right)$$

Let $u=x^2+y^2 \Rightarrow \frac{du}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$

But $s=u^{\frac{1}{2}} \Rightarrow \frac{ds}{du}=\frac{1}{2\sqrt{u}}$

Therefore

$$\frac{ds}{dt}=\frac{ds}{du}\cdot\frac{du}{dt}=\frac{1}{2\sqrt{u}}\cdot\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)=\frac{1}{2s}\cdot 2\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)=\frac{1}{s}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$

The police patrol car is moving towards the intersection at 100 kph he we have;

$$\frac{dy}{dt}=-100 \text{ (since it is moving towards the negatives of y axis).}$$

The suspect car is moving away from the origin (intersection) at 84 kph, hence

$$\frac{dx}{dt}=84$$

At the instant when the patrol car is 0.6 km from the intersection and the suspect car is 1.2 km from the same intersection, then the distance between them is;

$$s=\sqrt{0.6^2+1.2^2}=\sqrt{1.8}\approx 1.34 \text{ km}$$

Therefore;

$$\frac{ds}{dt}=\frac{1}{s}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)=\frac{1}{1.34}((1.2)(84)+(0.6)(-100))\approx 30.45 \text{ kph}$$

Example 4: An observer is tracking a rocket launched vertically into the air from a launching pad 9 km away. At the moment when the angle of elevation is $\theta=\frac{\pi}{4}$, the angle is changing at a rate of 0.75 radians per minute. Determine the velocity of the rocket at that moment.

Solution: Let y be the height of the rocket at time t minutes.

Our interest is to determine the velocity $v=\frac{dy}{dt}$ when $\theta=\frac{\pi}{4}\cdots$ (i)

It is given that $\frac{d\theta}{dt}=0.75 \text{ rad/min}\cdots$ (ii)

We need to establish a relation between equations (i) and (ii) above i.e.

$$\tan\theta = \frac{y}{9} \cdots \text{(iii)}$$

Equation (iii) establish a relation between the vertical height of the rocket, from the launchpad, and the angle of elevation.

We next differentiate equation (iii) with respect to time t to get;

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{9}\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 9\sec^2\theta \frac{d\theta}{dt} = \frac{9}{\cos^2\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{9}{\cos^2\frac{\pi}{4}} \cdot 0.75 = 13.5 \text{ km/min} \text{ or } 810\text{kph}$$

Exercise

1) The population growth rate of some species is given by $p(t) = \frac{500t}{t+3}$ for $t \geq 0$ hours. Determine the instantaneous growth rate of the population for $t \geq 0$; the instantaneous growth rate at $t = 7$; and the steady-state population.
2) Water is poured into an inverted circular cone-shaped tank of base radius 8 m and height 13 m at the rate of 2 liters per second. Find the;
   a) Rate of increase of the height of the water level.
   b) Rate of increase of the surface of the water when the water level is 7 m high.
3) Given that $y = \sqrt[3]{x}$ find an approximation for $\sqrt[3]{27.01}$
4) A plane flies at a height of 10 km in the direction of a ground observer at a speed of 540 kph. Find the rate of change of the angle of elevation of the plane from the observer at the instant when the angle is 45^0.
5) If $y = 2\sqrt{x}$ find the approximate increase in y if x is increased from 25 to 25.002
6) Given the area of triangle ABC as $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$. Determine the;
   (i) rate of change of A with respect to a when b and c are held constant.
   (ii) Rate of change of A with respect to b when a and c are held constant.
   (iii) Rate of change with respect to c when a and b are held constant.
   (iv) Evaluate (i) , (ii), and (iii) above when $a = 3, b = 4, c = 5$

7) Psychologist L Thurstone proposed the function for the time T it takes one to memorize a list of n words as $T = f(n)$ where $T = cn\sqrt{n-b}$ where C and b are constants depending on the person and task.

a) Determine the rate of change of time T with respect to the number n of words to be memorized.

b) Suppose that for a certain person and certain task, $c = 3, b = 3$ Find $\frac{dT}{dn}$ when memorizing 20 words, 50 words.

c) Interpret the above results

References

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