Calculus I

Lecture 11

Optimization

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Introduction to lecture 11

Lecture 11 will introduce another important application of differentiation, that is, optimization. It will build on past definitions and theorems especially Fermat's theorems, first principle of differentiation, and limits among others.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain the concept of optimization.

(ii) Solve optimization problems.

References for further reading

The lecture notes have been adopted from relevant topics from (Briggs et al., 2015; Rogawski et al., 2019; Stewart, 2012; Sullivan & Miranda, 2019).

Definition 1: Optimization

Optimization is the process of determining the best solution to a problem by either maximizing or minimizing a certain quantity. One need to model optimizing problems using functions.

Example 1:

- A farmer will aim at minimizing cost of production and maximizing profit, is a case of optimization.
- What is the shape of a paint container that minimizes the cost of its production?

Definition 2: Let $x_0 \in \mathbb{D}$, domain of a function $f(x)$. Then $f(x_0)$ is;

(i) Absolute maximum/global maximum value of $f(x)$ if $f(x_0) \geq f(x)$ for all x in the domain.

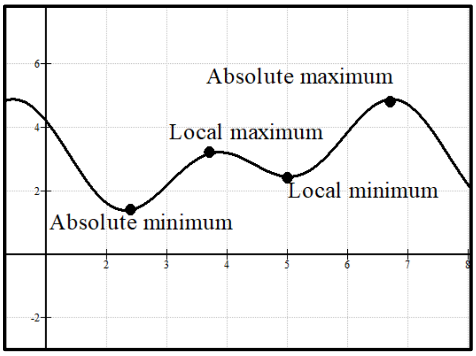
(ii) Absolute minimum/global minimum value of $f(x)$ if $f(x_0) \leq f(x)$ for all x in the domain.

(iii) The maximum and minimum values of $f(x)$ are also referred to as extreme values of $f(x)$.

Definition 3: The value $f(x_0)$ is a;

(i) Local maximum value of $f(x)$ if $f(x_0) \geq f(x)$ where x is near point x_0.

(ii) Local minimum value of $f(x)$ if $f(x_0) \leq f(x)$ where x is near point x_0.



Theorem 1: Fermat's Theorem

If $f(x)$ has a local maximum or minimum at point x_0, and if $f'(x_0)$ exists then $f'(x_0) = 0$.

Definition 1: A critical value of a function $f(x)$ is a value x_0 in the domain of $f(x)$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Definition 2: If a function $f(x)$ has a local maximum or minimum at point x_0, then x_0 is a critical number of $f(x)$.

Definition 3: The critical points are also referred to as stationary or turning points. However not all turning points have $f'(x) = 0$. The turning points maybe point of local maximum, local minimum, or inflexion/inflection.

Definition 4: Closed Interval Method

Suppose $f(x)$ is defined on the closed interval [a, b] then to obtain the absolute maximum and minimum values find;

(a) The values of $f(x)$ at the critical numbers of $f(x)$ in the open interval (a, b).

(b) The values of $f(x)$ at the endpoint of the interval.

(c) The largest/smallest of the values in (a) and (b) is the absolute maximum/minimum values.

Example 1: Consider the function $f(x) = x^{\frac{2}{3}}$. Find its critical number(s).

Solution: $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

At point $x = 0$ the $f'(x)$ does not exists. Hence $x = 0$ is a critical value.

Example 2: Determine the absolute maximum and minimum values of the function,

$f(x) = \frac{1}{2}x^3 + x^2 + 1$ over the interval $-3 \le x \le 2$

Solution: $f'(x) = \frac{3}{2}x^2 + 2x \Rightarrow$ the critical values occur when $f'(x) = \frac{3}{2}x^2 + 2x = 0$

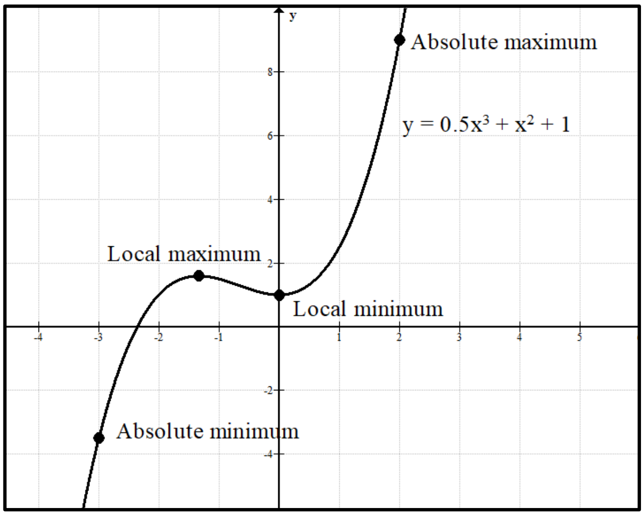
$$\Rightarrow x\left(\frac{3}{2}x + 2\right) = 0 \therefore x = 0 \text{ or } x = -\frac{4}{3}$$

Thus; $f(0) = 1; f\left(-\frac{4}{3}\right) = \frac{43}{27}$. The values of the function at the endpoints is;

$f(-3) = -3.5; f(2) = 9$. Considering these four points i.e.

$f(-3) = -3.5; f(0) = 1; f\left(-\frac{4}{3}\right) = \frac{43}{27}$; and $f(2) = 9$

It is clear that;



$f(2) = 9$ – absolute maximum

$f(-3) = -3.5$ – absolute mimimum

$f(0) = 1$ – local minimum

$f\left(-\frac{4}{3}\right) = \frac{43}{27}$ – local maximum

(See graph on the left)

Figure 1

Example 2: Investigate the turning points for the function $f(x) = x^3 - 7x + 6$.

Solution: By definition, at the turning points $f'(x) = 3x^2 - 7 = 0$

Solving the equation, we get; $x = \pm\sqrt{\frac{7}{3}}$ It is at these two points where the graph has the turning points. Next, we determine the nature of the turning points.

Method 1: We investigate the signs of the gradients just before and after these points.

x	-2	$-\sqrt{\frac{7}{3}}$	-1	1	$\sqrt{\frac{7}{3}}$	2
$\frac{dy}{dx}$	5	0	-4	-4	0	5
Sign	+ve	0	−ve	−ve	0	+ve
Sketch of tangent						

From the table at point $x = -\sqrt{\frac{7}{3}}$ we have a local maximum, and at point $x = \sqrt{\frac{7}{3}}$ we have a local minimum.

Method 2: Second derivatives

a) If at point x_0 of the curve of $y = f(x)$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ (i.e. negative), the point is a point of local maximum.

b) If at point x_0 of the curve of $y = f(x)$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ (i.e. positive), the point is a point of local minimum.

c) If $\frac{d^2y}{dx^2} = 0$ then no definite conclusion can be made.

In our case, $f''(x) = 6x$ and at point $x = \sqrt{\frac{7}{3}}$

$\frac{d^2y}{dx^2} = 6\sqrt{\frac{7}{3}} > 0$ hence a point of local minimum

Again, at point $x = -\sqrt{\frac{7}{3}}$; $\frac{d^2y}{dx^2} = -6\sqrt{\frac{7}{3}} < 0$ hence a point of local maximum.

Example 3: An artisan intends to construct an open box from a rectangular sheet of metal of sides 28 meters by 20 meters. He has four equal square portions removed at the corners and the sides are then turned up to form an open box. Determine its maximum volume.

Solution: Let the square portions be of side x units as seen in the figure below.

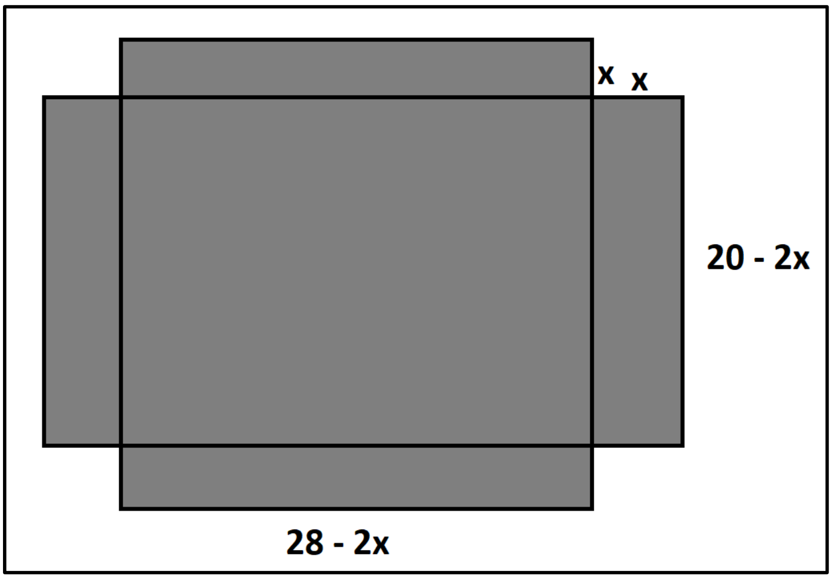


Figure 2

The volume of the metal box $v = x(20 - 2x)(28 - 2x) = 4x^3 - 96x^2 + 560x$

For maximum volume;

$$\frac{dV}{dx} = 12x^2 - 192x + 560 = 0$$

$$12x^2 - 192x + 560 = 0 \Rightarrow x = \frac{192 \pm \sqrt{192^2 - 4(12)(560)}}{24} \approx 3.84, 12.16$$

We can use second derivatives to determine;

$$\frac{d^2y}{dx^2} = 24x - 192\Big|_{x=3.84} = -99.84 < 0 - \text{local maximum}$$

$$\frac{d^2y}{dx^2} = 24x - 192\Big|_{x=12.16} = 99.84 > 0 - \text{local minimum}$$

Hence for maximum volume $x = 3.84$ metres.

Example 4: The sum of two numbers is 49. Find two numbers with this sum such that their product is a maximum.

Solution: Let the numbers be x and y then $x + y = 49 \Rightarrow y = 49 - x \cdots$ (i)

Let the product of the two numbers be $f = xy \Rightarrow f = x(49 - x) = 49x - x^2$

$$\therefore \frac{df}{dx} = 49 - 2x$$

For maximum value; $\frac{df}{dx} = 49 - 2x = 0$

$$\Rightarrow 49 = 2x \therefore x = 24\frac{1}{2}$$

Example 5: A manufacturer needs to produce a cylindrical can with a capacity of $500\ cm^3$. The cost for production of top and bottom parts costs 0.03 KES per unit square cm, while the curved part will cost 0.02 KES per square cm. Find the dimensions that will minimize the cost of producing the container.

Solution: The total surface area A is given by $A = 2\pi r^2 + 2\pi rh$.

Hence the cost C of producing the can in KES, is given by;

$$C = 2\pi r^2 + 2\pi rh = 0.06\pi r^2 + 0.04\pi rh \cdots \text{(i)}$$

Again $v = \pi r^2 h = 500 \Rightarrow h = \frac{500}{\pi r^2} \cdots$ (ii)

Equation (i) can also be written as;

$$C = 0.06\pi r^2 + 0.04\pi r\left(\frac{500}{\pi r^2}\right) = 0.06\pi r^2 + \frac{20}{r}$$

This is the function that needs to be minimize. Note that the manufacturer must produce a can and hence $r > 0$.

Thus;

$$\frac{dC}{dr} = 0.12\pi r - \frac{20}{r^2} = \frac{0.12\pi r^3 - 20}{r^2} = 0 \Rightarrow r \approx 5.5 \text{ cm}$$

Our $h = \frac{500}{\pi r^2} = \frac{500}{\pi 5.5^2} \approx 5.26$ cm

The minimum cost of the container is;

$$C = 0.06\pi(5.5)^2 + 0.04\pi(5.5)(5.26) \approx 535 \text{ KES}$$

Example 6: A pirates' boat is 70 nautical miles due east of a coast patrol boat. The pirates' boat is heading south at 20 knots while the patrol boat is heading eastward at 15 knots. The patrol boat intends to fire a warning shot when the distance between it and the pirates' boat is at its minimum. Find when this occurs, assume the two boats maintain their respective courses.

Solution: We can draw a sketch diagram to understand the relative positions of the boats

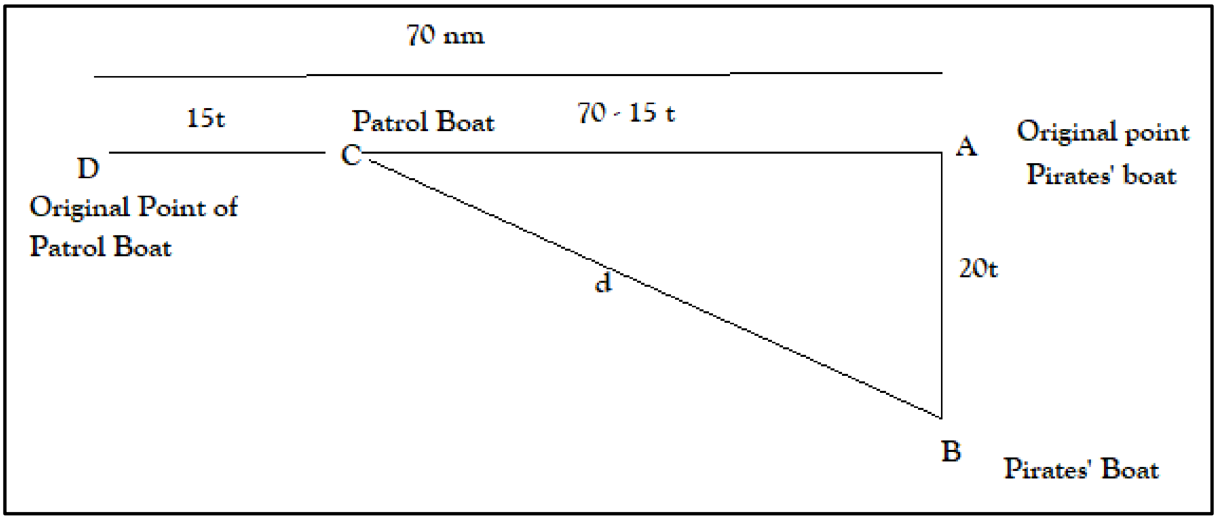


Figure 3

Let the patrol boat fire the warning shot t hours later.

After t hours the patrol boat will have traveled $15t$ nautical miles while the pirates' boat will have traveled $20t$ nautical miles.

Using the Pythagoras theorem we find the distance d between them after t hours to be;

$$y^2 = (20t)^2 + (70 - 15t)^2$$

$$\Rightarrow y = \sqrt{400t^2 + 4900 - 2100t + 225t^2} = \sqrt{625t^2 - 2100t + 4900}$$

The shortest distance possible is when this function is minimum.

Note that considering the quantity or expression $P = 625t^2 - 2100t + 4900$ is sufficient to get the minimum value of y.

Hence we have;

$$\frac{dP}{dt} = 1250t - 2100 = 0 \therefore t = \frac{2100}{1250} = 1.68$$

Note that the absolute minimum distance between the two boats will be when $t = 1.68$ hours. Hence

$$y = \sqrt{625(1.68)^2 - 2100(1.68) + 4900} = \sqrt{3136} \approx 56 \text{ nm}$$

Exercise

1) Identify different scenarios where optimization is applicable.

2) A rectangular piece of land is to be fenced by 1200 meters of fencing. One side of the land is a straight riverbank that requires no fencing. Determine the dimensions of the land to be fenced to give maximum area.

3) Otieno lives in an island in L. Victoria that is 8 km away from a straight beach. His friend Koech lives 3 km up from the beach. Otieno uses a boat to reach the beach. He can row it at 4 kph and walk at a speed of 5 kph on the beach. Determine the minimum time Otieno can take to reach Koech's home from the island.

4) The sum of the two number is 101. Determine the numbers if their product is to be a maximum.

5) A closed cylindrical tank is to have an internal capacity of 1000 liters. If the internal radius is r show that the internal area A of the tank is

$$A = \frac{2\pi r^3 + 2000}{r}$$

Hence find the value of r that will give a minimum area.

6) Find the critical values for the following functions;

a) $f(x) = x^3 - 3x + 2$

b) $f(x) = x^2 + 2x - 1$

c) $f(x) = \sqrt{2 - x^2}$

d) $f(x) = x^{\frac{1}{3}} - x^3$

e) $f(\theta) = 2\sin\theta + \cos^2\theta$

f) $f(\theta) = 3\theta - \tan\theta$

References

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