Calculus I

Lecture 12

Differentiation and Kinematics

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Introduction to lecture 12

Lecture 12 will introduce yet another important application of differentiation, that is, kinematics. Kinematics deals with the motion of objects without reference to the forces that cause this motion. It is a continuation of the lecture on rate of change and optimization. Most of the applications in kinematics are cases of determining the critical values. It will also build on past definitions and theorems especially Fermat's theorems, first principle of differentiation, and limits among others.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Show the relationship between different quantities using derivatives.

(ii) Solve kinematics problems involving differentiation.

References for further reading

The lecture notes have been adopted from relevant topics from (Briggs et al., 2015; Rogawski et al., 2019; Stewart, 2012; Sullivan & Miranda, 2019).

Definition 1: Displacement (s) vs Time (t)

(i) Velocity v of an object is change in displacement s meters, with respect to time t seconds i.e. $\mathrm{v} = \frac{\mathrm{ds}}{\mathrm{dt}}\ \mathrm{ms}^{-1}$

(ii) If velocity is zero then the object is at rest i.e., $\mathrm{v} = 0 \Rightarrow \frac{\mathrm{ds}}{\mathrm{dt}} = 0$

(iii) If the value of velocity v is negative then the object is moving in the opposite direction to that in which displacement s is measured i.e. $\frac{\mathrm{ds}}{\mathrm{dt}} < 0$

Note that speed is the absolute value of velocity i.e. $|\mathrm{v}(\mathrm{t})|$.

Definition 2: Velocity ($\mathbf{v}$) vs time ($\mathbf{t}$)

i) Acceleration a refers to the change in velocity v with respect to time t i.e.,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\ ms^{-2}$$

ii) If the acceleration of a particle is zero i.e. $a = \frac{d^2s}{dt^2} = 0$ it implies that the velocity of the particle is constant.

iii) If the acceleration of the particle is a positive value i.e. $a = \frac{d^2s}{dt^2} > 0$ then the particle is accelerating, and if the acceleration is a negative value the particle is slowing down or retarding i.e. $a = \frac{d^2s}{dt^2} < 0$.

Definition 3: Motion in a straight line

If a particle is constrained to move on a straight line we can choose the origin to taken at the point where the particle is situated at $t = 0$, we can get expressions for its position s and velocity v with respect to time t and its acceleration a. That is,

(i) $v = u + at$

(ii) $s = ut + \frac{1}{2}at^2$

(iii) $v^2 = u^2 + 2as$ where u is velocity when $t = 0$

Example 1:

A particle is thrown upwards, and its height after t seconds is s meters, where $s = 12t - 5t^2$.

Find the time when the greatest height is reached, the time when it returns to the original level. Its velocity after 2 seconds and the acceleration when t = 1.2 s.

Solution:

i) The greatest height reached

If $s = 12t - 5t^2$ then velocity $v = \frac{ds}{dt} = 12 - 10t$. Note that for the greatest height (A case of maximization) the velocity is zero. That is

$$\frac{ds}{dt} = 0 = 12 - 10t \therefore t = 1.2 \text{ sec}$$

Therefore the particle attains the greatest height at 1.2 sec. This greatest height is;

$$s = 12(1.2) - 5(1.2)^2 = 7.2 \text{ m}$$

ii) When it returns to the original level, $s = 0 \Rightarrow 12t - 5t^2 = 0$

$$t(12 - 5t) = 0 \therefore t = 0 \text{ or } t = 2.4 \text{ sec}$$

Therefore it returns to the original position after 2.4 sec.

iii) when $t = 2s$; $v = 12 - 20 = -8\ m/s$ i.e. the particle is moving in opposite direction.

iv) We have seen that $v = \frac{ds}{dt} = 12 - 10t \therefore a = \frac{d^2s}{dt^2} = -10\ ms^{-2}$ i.e. the particle is decelerating.

Example 2: A missile is shot vertically upwards. It travels a distance m given by $s = 1050t - 15t^2$ in time t seconds. Find the;

(i) velocity with which the missile was shot,

(ii) time taken when it came to rest, and

(iii) distance covered.

Solution:

(i) By definition velocity $v = \frac{ds}{dt} = 1050 - 30t$. Initial time $t = 0 \therefore v = 1050\ m/s$.

(ii) The velocity when the missile comes to rest is $\frac{ds}{dt} = 0 = 1050 - 30t$

Therefore the time it took to come to rest is $t = 35 sec$

(iii) Hence distance $s = 1050(35) - 15(35)^2 = 18375\ m$

Definition 3: Gravity influence on motion

The following equations relate height $s(t)$ and velocity $v(t)$ with time of objects projected in the air above the earth's surface. The equations are attributed to Galileo's work.

(i) $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$. But since $s(0) = 0$ (i.e. when $t = 0$) then $s_0 = 0$ and therefore we have $s(t) = v_0 t - \frac{1}{2}gt^2$

(ii) $v(t) = \frac{ds}{dt} = v_0 - gt$ where constants s_0 and v_0 are initial values, $g \approx 9.8\ ms^{-2}$ is acceleration due to gravity.

Remarks:

a) We are assuming air resistance.

b) From equations (ii) we have $gt = v_0 - v(t) \therefore t = \frac{v_0 - v(t)}{g}$. Therefore equation (i) can be written as;

$$s(t) = v_0 \frac{v_0 - v(t)}{g} - \frac{1}{2} g \left(\frac{v_0 - v(t)}{g}\right)^2 = \frac{2v_0^2 - 2v_0 v(t) - v_0^2 - 2v_0 v(t) + v^2(t)}{2g} = \frac{v_0^2 + v^2(t)}{2g}$$

$$\Rightarrow 2gs(t) = v_0^2 + v^2(t)$$

$$\Rightarrow v^2(t) = v_0^2 - 2gs(t)$$

$$|v(t)| = \sqrt{v_0^2 - 2gs(t)}$$

c) When the particle attains maximum velocity $v = 0 \Rightarrow 0 = v_o^2 - 2gs(t)$

$$\therefore 2gs(t) = v_0^2$$

$$\Rightarrow s(t) = \frac{v_0^2}{2g} - \text{maximum height}$$

Example 1: A metal ball is launched upward from ground level with an initial velocity of $67\ \text{m/s}$. Determine;

(i) Velocity of the ball when time $t = 1$ and $t = 10$.

(ii) Time the ball attains the maximum height.

Solution: Note that the initial distance $S_0 = 0$ and the initial velocity $v_0 = 67$ with $g = 9.8\text{ms}^{-2}$.

(i) We apply Galileo's equations:

$$s(t) = 67t - 4.9t^2; v(t) = 67 - 9.8t$$

When $t = 1$; $v(1) = 67 - 9.8(1) = 57.2\ \text{m/s}$

When $t = 10$, $v(10) = 67 - 9.8(10) = -31\ \text{m/s}$ – the ball is coming back to the ground.

(ii) The greatest height is attained when velocity is zero i.e. $\frac{ds}{dt} = 0 = 67 - 9.8t$

$$\Rightarrow 67 = 9.8t \therefore t \approx 6.84\ \text{sec}$$

The greatest height attained at this time is;

$$s\left(\frac{67}{9.8}\right) = 67\left(\frac{67}{9.8}\right) - 4.9\left(\frac{67}{9.8}\right)^2 \approx 229.03\ \text{m}$$

Alternatively;

$$s\left(\frac{67}{9.8}\right) = \frac{67^2}{2 \times 9.8} \approx 229.03$$

Example 2: A projectile is launched at an angle θ to the horizontal with an initial velocity v_0. The position of the projectile at time t seconds is given by;

$$p(x) = \begin{cases} x = v_0 t \cos\theta + gt \\ y = v_0 t \sin\theta - 2gt^2 \end{cases} \quad t \geq 0$$

where $g = 9.8\ \text{ms}^{-2}, 0 < \theta < \frac{\pi}{2}$.

(i) Express the slope of the tangent line to the motion of the projectile as a function of t..

(ii) Determine the time t when the projectile attains its maximum height.

Solution: The slope of the tangent is by definition $p'(x) = \frac{dy}{dx}$

$$\Rightarrow x'(t) = v_0 \cos\theta + g$$

$$\Rightarrow y'(t) = v_0 \sin\theta - 4gt$$

Therefore;

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{v_0 \sin\theta - 4gt}{v_0 \cos\theta + g}$$

The projectile will attain its maximum height when $\frac{dy}{dx} = 0$. Hence

$$\frac{v_0 \sin\theta - 4gt}{v_0 \cos\theta + g} = 0 \Rightarrow v_0 \sin\theta - 4gt = 0 \therefore t = \frac{v_0 \sin\theta}{4g}\ \text{sec}$$

The maximum height attained is;

$$s(t) = \frac{v_0^2}{2g} = \frac{v_0^2}{19.6}\ \text{m}$$

Definition 4: There are instances where you need to consider the motion of a particle in a plane, say the xy-plane, along a path or trajectory or orbit. The position of the particle at time t seconds is a position vector $\boldsymbol{r}$.

The position vector r is a twice-differentiable vector-valued function. It can be written as;

$$\mathbf{r} = F(t) - \text{Equation of motion}$$

The velocity of the particle at any instant is the vector

$$\mathbf{v} = \frac{dr}{dt} = v\mathbf{T}$$

Where v is the instantaneous speed of the particle and $\mathbf{T}$ is the unit tangent vector to the path of the particle.

Example 1: A particle moves along the curve of $y^2 = 3x$ from the vertex at a speed of 7 units per second. Determine the velocity vector **v** as the particle moves through point $(3,3)$.

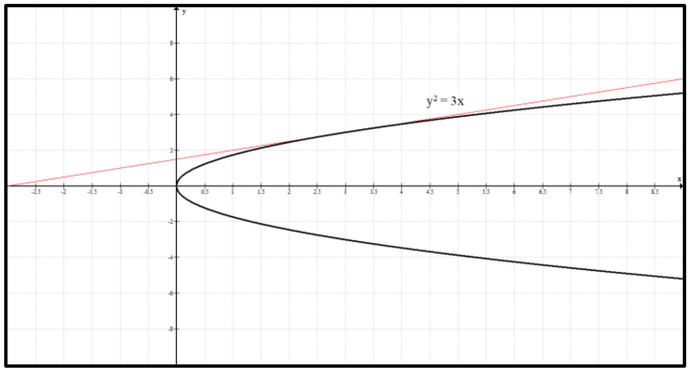

Figure 1

Solution: At point $R(x, y)$, the position vector $r = \overline{OR} = x\mathbf{i} + y\mathbf{j}$

$$\Rightarrow \mathbf{r} = \frac{y^2}{3}\mathbf{i} + y\mathbf{j}$$

$$\frac{d\mathbf{r}}{dy} = \frac{2}{3}y\mathbf{i} + \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dy}\right| = \sqrt{\frac{4}{9}y^2 + 1}$$

$$\boldsymbol{T} = \frac{\frac{d\mathbf{r}}{dy}}{\left|\frac{d\mathbf{r}}{dy}\right|} = \frac{\frac{2}{3}y\mathbf{i} + \mathbf{j}}{\sqrt{\frac{4}{9}y^2 + 1}} - \text{unit tangent vector}$$

At point (3,3) then vector **T** becomes;

$$\mathbf{T} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

Therefore, the velocity of the particle at this point is;

$$\mathbf{v} = v\mathbf{T} = 7 \cdot \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

Remarks:

i) This example is an introductory to vectors and differentiation.

ii) It has assumed some basic knowledge of vector projection.

iii) The problem did not calculate velocity as a function of time. To do this one need to use Newton's second law of motion $f = ma$ where f is force, m is mass, and a is acceleration. There is also an assumption that the mass of the particle is constant.

Exercise

1) An arrow thrown vertically upward in the air from ground level returns back at the point of launch 10 seconds. Find the initial velocity and the greatest height attained by the arrow.

2) A helicopter drops a bag containing tree seedlings (with no parachute attached) from a height of 200 meters. Find the velocity of the bag the moment it hit the ground.

3) A particle moving along a line covers a distance s m in time t seconds from a fixed-point O on the line, where $s = t^3 - 4t^2 - 6t + 5$. Find the time when the acceleration of the particle is $4\ \mathrm{ms}^{-2}$ and the instant velocity at this time.

4) A particle moves on a line following the law $s = t^3 - 3t^2 + 3t - 2$ where s is the distance from a fixed-point O in meters on the line and t is the time in seconds. Find the average acceleration of the particle during the fourth second, where and when the velocity will be in 12 m/s and the distance of the particle from the fixed-point O, when the acceleration vanishes.

5) A projectile is fired into the air from the ground with an initial velocity $\mathrm{v}_0 = 75\ \mathrm{m/s}$. Determine the maximum height it will attain.

6) A particle moves along the upper curve of the function $\frac{y^2}{9} - \frac{x^2}{16} = 1$ from left to right with a constant speed of 7 units per second. Determine the velocity vector at point$(0, 3)$.

7) A ball is thrown vertically upwards from the ground with an initial velocity of 25 m/s. the equation of its motion is $\mathrm{s} = -18\mathrm{t} - 25\mathrm{t}^2$ where t is the time in seconds and s is the height , in meter, of the ball from the starting point at t seconds.

   (i) Find the instantaneous velocity of the ball at the end of 1 second

   (ii) The time taken for the ball to strike the ground and the greatest height attained

   (iii) The velocity with which it strikes the ground and whether the ball will be rising or falling when t = 1 and 4 second.

References

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