Calculus I

Lecture 1

Introduction to functions and limits

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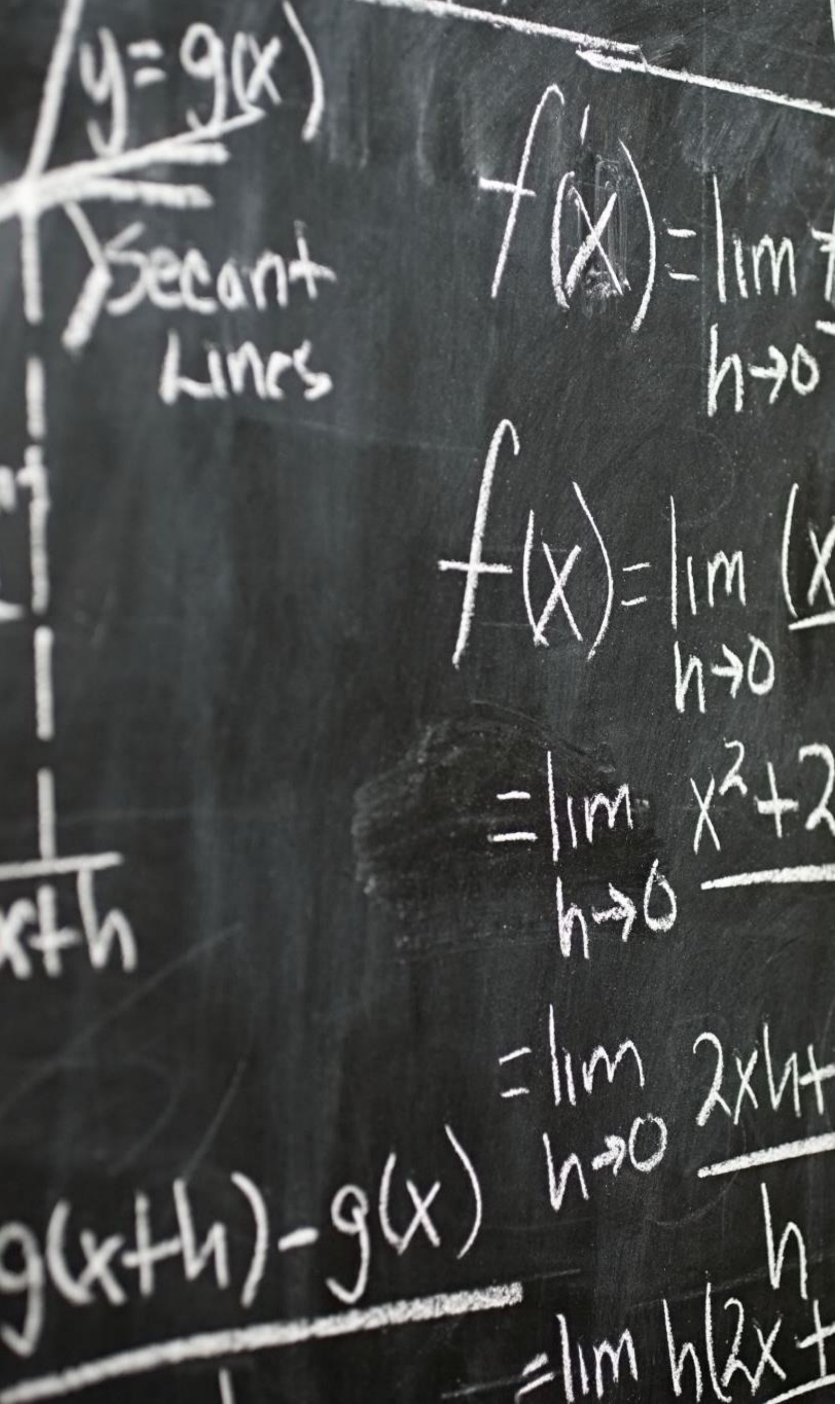


Course description

Calculus I deals with differentiation of single real valued functions.

It focuses on the limits of functions and continuity of functions.

Calculus I introduces the basic techniques of differentiation and key theorems in differential calculus.

The course also introduce the learner to the application of differentiation in rates of changes, related changes, kinematics, and optimization.

Key competencies

To enhance learners' competencies in;

Calculus knowledge and skills;

Critical thinking and analytical skills;

Problem solving;

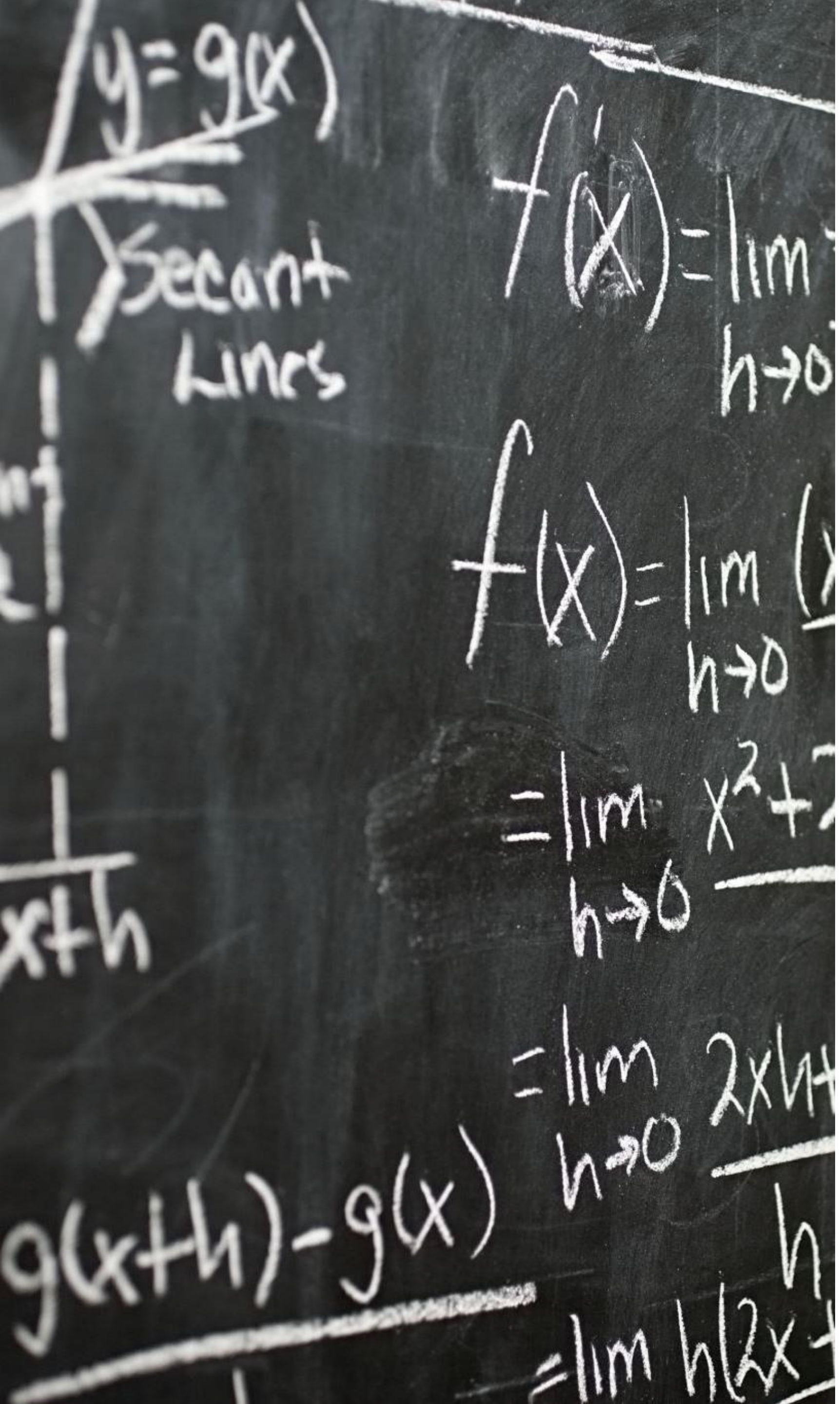
Self-management/efficacy;
Teamwork; and

digital literacy.

Course goals

(i) To equip the learners with basic knowledge of differential calculus in particular, definition of terms, the basic rules of differentiation and their application.

(ii) To enhance key competencies necessary for future learning in calculus.

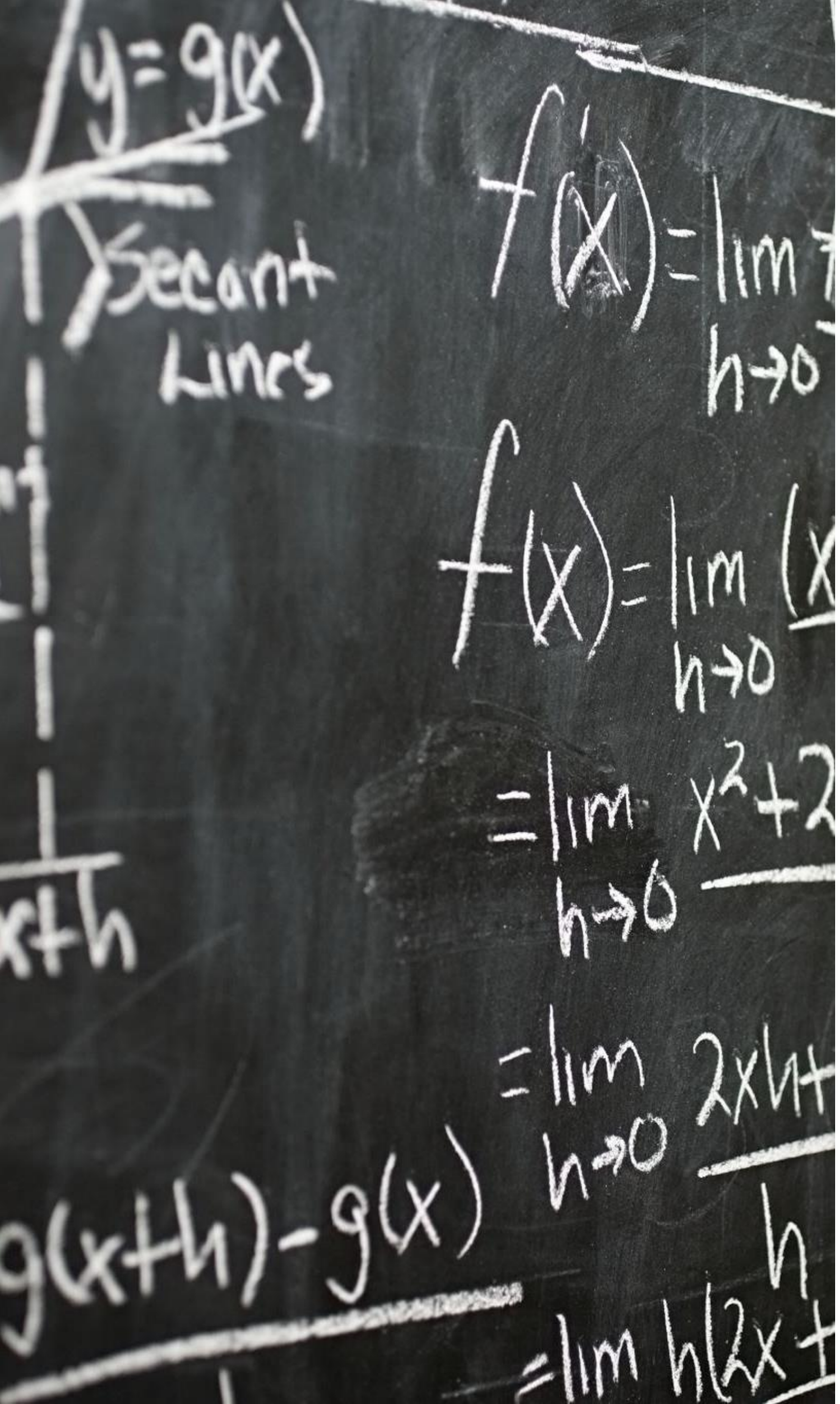


Intended learning outcomes

(i) Explain the concepts of limits and continuity of functions.

(ii) Explain and apply fundamental theorems in differential calculus.

(iii) Apply the rules of differentiation.



Introduction to lecture 1

This lecture will define key concepts and definition of terms that are commonly used calculus.

This lecture will focus on these definitions that will form a foundation in comprehending other concepts and operations in other lectures in differential and integral calculus.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain key terms relating to functions and limits of functions.

(ii) Determine limits of given functions.

References for further reading

The lecture notes have been adopted from relevant topics from (Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019)

Definitions

Definition 1: A relation is a rule that relates a value in the domain with a value in the range.

Functions are situation where one quantity depends on another.

For example, the area A of a circle depends on the radius r i.e.$A = \pi r^2$.

Definition 2: A function f is a rule that assigns to each element x in the Domain $\mathbb{X}$ exactly one element called $f(x)$ in a set $\mathbb{Y}$, called the Range.

An arbitrary number in the domain of a function is called an independent variable, while a number in the range of the function is the dependent variable.

Example 1

In the function $A = \pi r^2$, A depends on r.

Thus, A is the dependent variable and r is the independent variable and therefore we can say that A is a function of r; written $A(r)$.

Definitions

Definition 1: Domain is the set of all the values which are input in function.

For example, in the function $y = f(x)$, the domain is the set of all the x coordinates i.e., the independent variables.

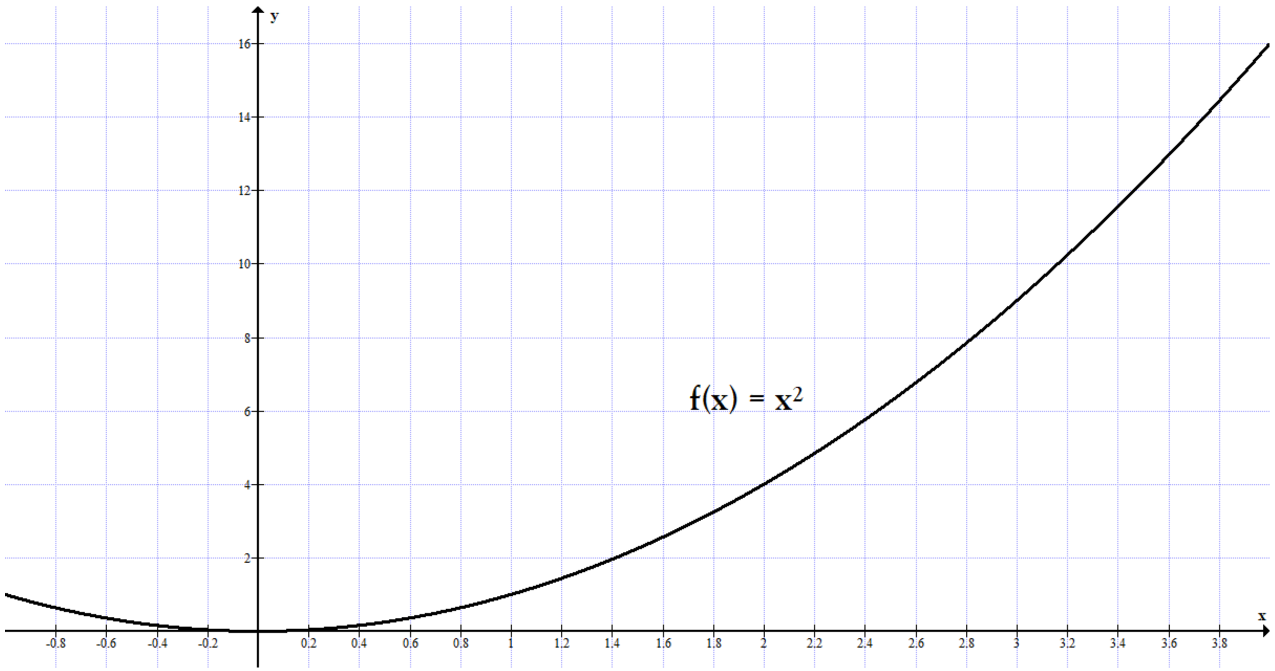
Definition 2: Range is the set of all the values which are output of a function when inputs from the domain are used.

The range constitutes the dependent variables.

For example, in the function $y = f(x)$, the range is the set of all the y coordinates.

Example

Consider the graph of the function y = f(x) = x^2 from x = 0 to x = 4. Note that the domain of the function is $0 \leq x \leq 4$ and the range is $0 \leq f(x) \leq 16$



Definition (a polynomial)

A polynomial function is a function of the form;

$$\mathrm{f(x)} = \mathrm{a_0} + \mathrm{a_1 x} + \mathrm{a_2 x^2} + \mathrm{a_3 x^3} + \cdots \mathrm{a_{n-1} x^{n-1}} + \mathrm{a_n x^n}$$

where $n > 0$ i.e., n is a positive integer, and the coefficients $\mathrm{a_0}, \mathrm{a_1}, \mathrm{a_2}, \ldots \mathrm{a_n}$ are constants real numbers.

If $a_n \neq 0$, then the polynomial ais of degree n e.g., $f(x) = 3x^4 + x^2 + 3x - 1$ is of degree 4 with coefficients $a_4 = 3, a_3 = 0, a_2 = 1, a_1 = 3, a_0 = -1$.

A polynomial function of the form $\mathrm{f(x)} = \mathrm{b}$ is called a constant function e.g., $\mathrm{f(x)} = 3$.

A polynomial of the form $f(x) = a_1 x + a_0$ is called a linear function.

Other functions that these course will delve in include algebraic functions, rational functions, discontinuous functions, trigonometric functions, logarithmic functions, among others.

Definition (Intervals)

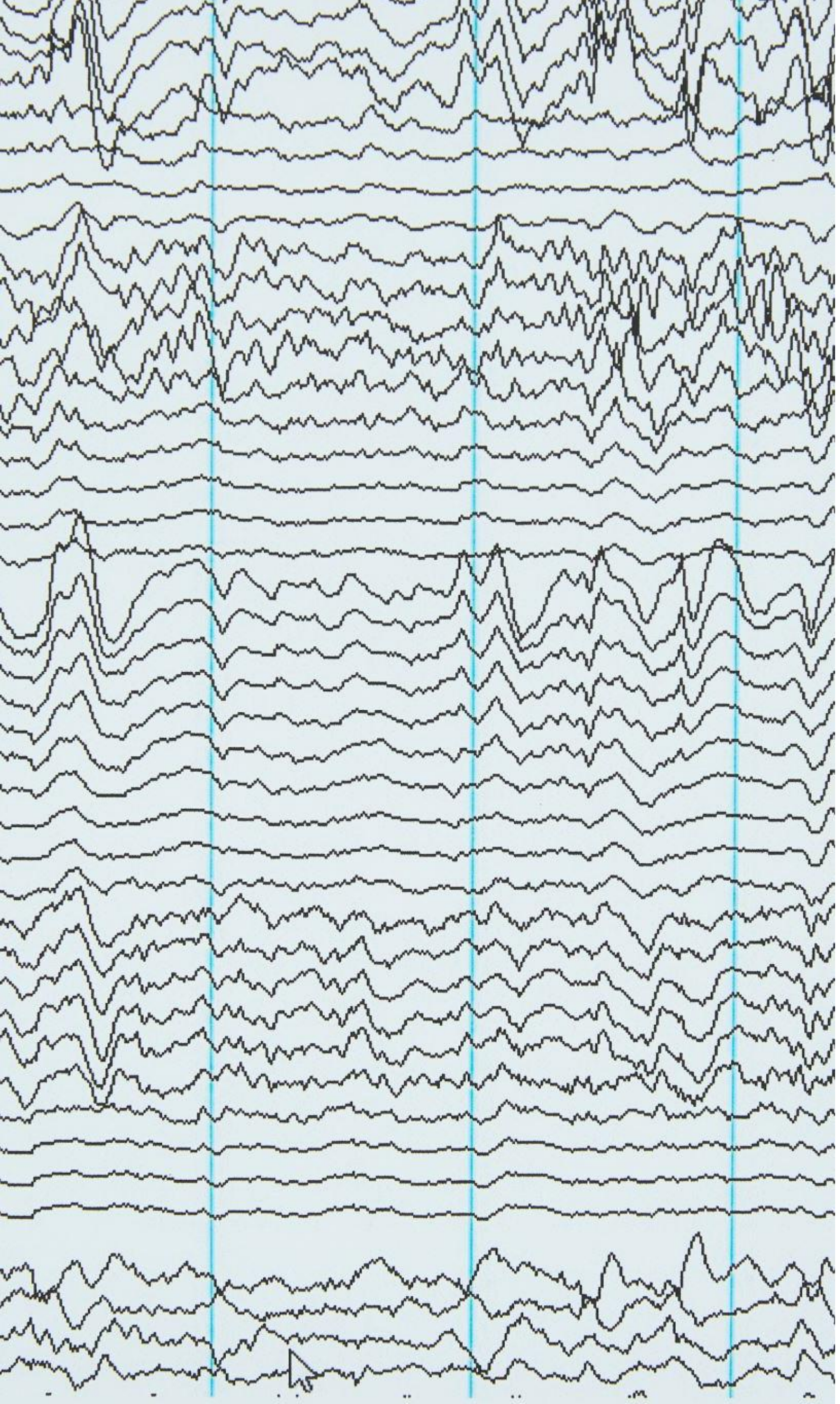
There are different intervals where a function can be defined.

a) **Bounded intervals**

Open intervals: Let a and b be fixed real numbers then the open interval (a, b) is the set of all real numbers x such that $a < x < b$. Real numbers a and b are not part of the interval.

Closed intervals: Let a and b are fixed real numbers then the closed interval $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$. Real numbers a and b are part of the interval.

Half-open intervals: These are intervals that are open on one side and closed on the other e.g. $[a, b)$ or $(a, b]$.



Definition..contd...(Intervals)

Unbounded intervals. Example of unbounded intervals are;

(a, ∞) i.e., $x > a$

$(-\infty, a)$ i.e., $x < a$

$[a, \infty)$ i.e., $x \geq a$

$(-\infty, a]$ i.e., $x \leq a$

Limits and their properties

Limit of a function concept is a fundamental concept in calculus.

Consider a spherical balloon. Suppose one has to pump air into the balloon such that it expands uniformly.

If its radius is r , then its volume is $v = \frac{4}{3}\pi r^3$.

Now, if we let the radius to approach or tends to 12 cm i.e., $x \to 12$, then its volume v approaches 2304π i.e., $v \to 2304\pi$.

We can denote this as;

$$\lim_{r \to 12}\left(\frac{4}{3}\pi r^3\right) = 2304\pi$$

Limits and their properties...contd...

The idea of a function approaching a limit can be defined as the function f(x) approaches the limit l near a point a if one can make f(x) as close as to l as one would like, by having x sufficiently close to, but not equal to a.

This can be denoted as;

$$\lim_{x \to a} f(x) = l$$

Example

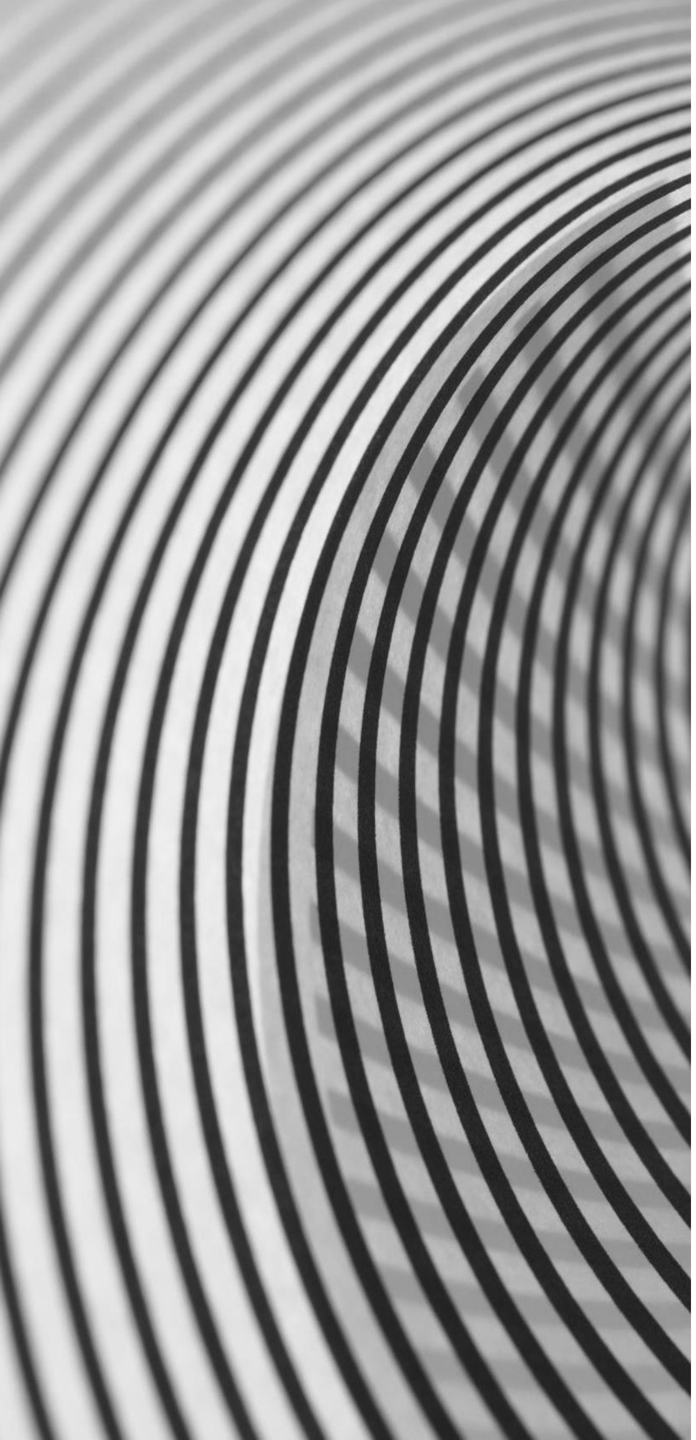
Consider the function $f(x) = x^2 + 1$ for the values of x near 2 but not equal to 2. The table below shows the values of $f(x)$ for the values of x about 2. We investigate the values of $f(x)$ from either sides of 2. That is, when x is approaching 2 from the left and from the right.

From the LHS

x	1	1.5	1.8	1.9	1.999	1.999999
$f(x)$	2	3.25	4.24	4.61	4.996001	4.999996000001

From the RHS

x	3	2.5	2.1	2.01	2.000001	2.0000001
$f(x)$	10	7.25	5.41	5.0401	5.0000040 00001	5.0000040000001



Example…contd…

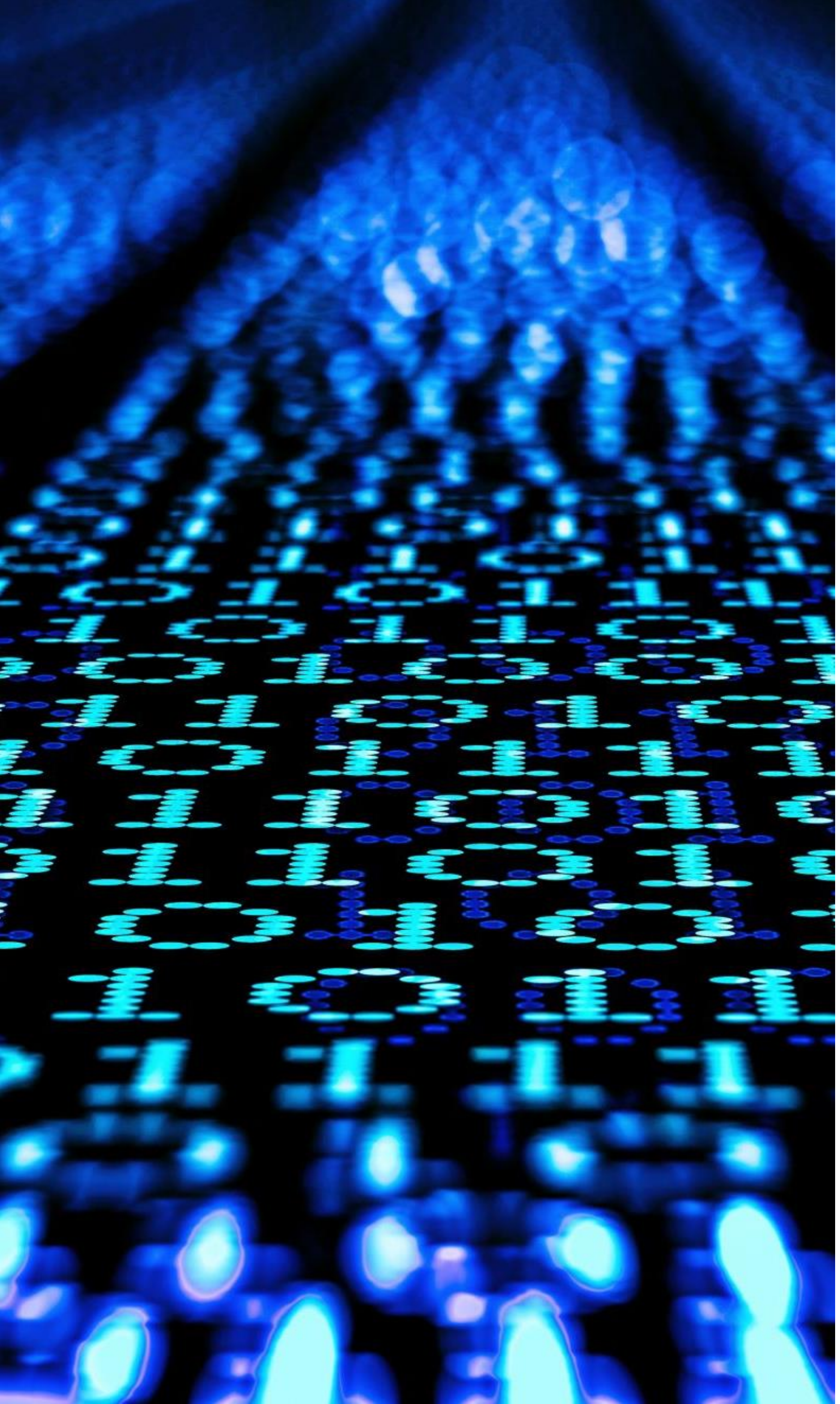
The value of $f(x)$ approaches 5 as x gets closer to 2 from either sides.

One can make the value of $f(x)$ as close as 5 by taking x sufficiently close to 2.

That is, the limit of the function $f(x) = x^2 + 1$ as x approaches 2 is 5 denoted;

$$\lim_{x \to 2}(x^2 + 1) = 5$$

Remark 1: Note that for the limit of $f(x)$ to exist, it is NOT a MUST that $f(x)$ MUST be defined at point a.



Definition

The limit of $f(x)$ is l if the function $f(x)$ approaches to l as x tends to a from either side i.e.,

From the left; $\lim_{x \to a^-} f(x) = l$

From the right; $\lim_{x \to a^+} f(x) = l$

$$\Rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = l$$

Remark 2: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then the limit of the function does not exist.

Theorem

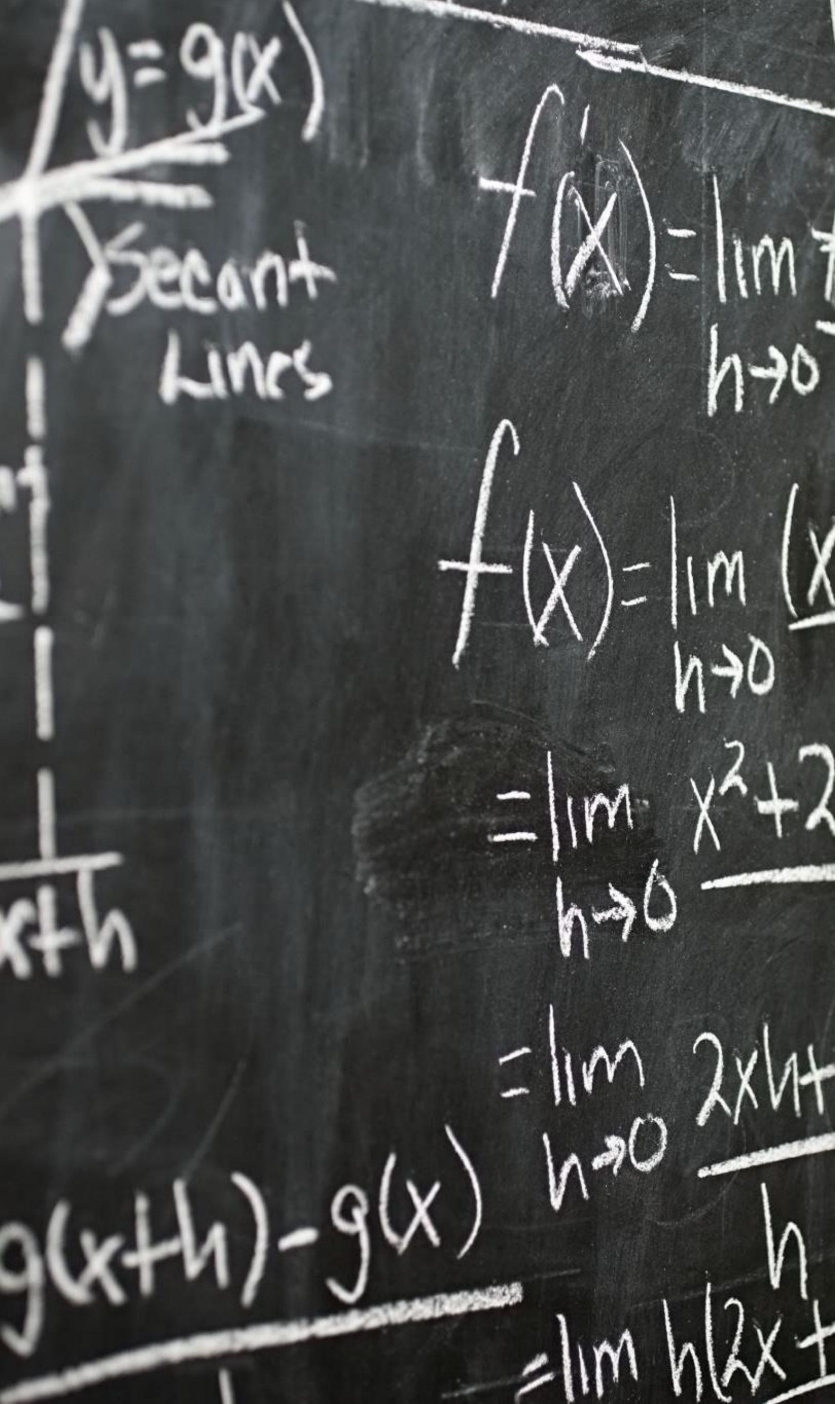
The theorem states that, if $\lim_{x \to a} f(x) = m$ and $\lim_{x \to a} g(x) = n$ then

a) $\lim_{x \to a} (f \pm g)(x) = m \pm n$

b) $\lim_{x \to a} (f \cdot g)(x) = m \cdot n$

c) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{m}{n}$

d) $\lim_{x \to a} [\alpha f(x)] = \alpha \lim_{x \to a} f(x) = \alpha m$, where α is a constant



Methods of finding limit of functions

Limits by direct substitution

Example 1: Evaluate; $\lim_{x \to 3}(3x + 5) = 3(3) + 5 = 14$

Example 2:

Evaluate; $\lim_{X \to -2}(x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2$

Factorization

Direct substitution may be faulty in certain circumstances. Hence one need to first simplify the function or factorize it before substituting it accordingly.

Example: Evaluate; $\lim_{x\to 3}\left(\frac{5x^2-12x-9}{x-3}\right)$

Solution: Suppose we use direct substitution, we get;

$$\lim_{x\to 3}\left(\frac{5x^2-12x-9}{x-3}\right)$$

$$=\frac{45-36-9}{3-3}=\frac{0}{0}-\text{indeterminate form}$$

Factorization …contd…

The indeterminate form doesn't talk much about the limit of the functions. In such a situation one has to options; (i) factorize the numerator or (ii) use the L'Hopital's rule.

Factorizing the numerator to get;

$5x^2 - 12x - 9 = (x-3)(5x+3)$ and hence we have;

$$\lim_{x \to 3}\left(\frac{5x^2 - 12x - 9}{x-3}\right)$$

$$= \lim_{x \to 3}\left(\frac{(x-3)(5x+3)}{x-3}\right) = \lim_{x \to 3}(5x+3) = 18$$

L'Hopital's rule

This rule is used where after direct substitution or otherwise, one gets the indeterminate forms;

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Then the limit exists or not, and it's called an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Definition (L'Hopital's rule)

Let $f(x)$ and $g(x)$ be differentiable functions and $g'(x) \neq 0$ on the open interval I that contains a point x. Suppose;

$$\lim_{x\to a} f(x) = \pm\infty\,;\ \lim_{x\to a} f(x) = 0;\ \lim_{x\to a} f(x) = 0;\ \lim_{x\to} g(x) = 0$$

Then;

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ if the limit on the RHS exists}$$

Remarks

1) Limits not of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ may require another rule rather than the L'Hopital's rule.

2) Indeterminate forms of the type; $0 \cdot \infty; 0 \cdot -\infty;\ 0^0, \infty^0, \text{and } 1^\infty$ can be reduced to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

3) For type $0^0, \infty^0, \text{and } 1^\infty$ apply logs of the expressions involved.

Example

Evaluate; $\lim_{x\to 0^+} x^a \ln x = 0 \cdot -\infty \ (a > 0) = \lim_{x\to 0^+} \frac{\ln x}{x^{-a}} = \frac{-\infty}{\infty}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\frac{1}{x}}{-ax^{-a-1}} = \lim_{x\to 0^+} \frac{x^a}{-a} = 0$$

Example

Evaluate; $\lim_{x \to 0^+} x^x$

Solution: by direct substitution we get; $\lim_{x \to 0^+} x^x = 0^0$ – indeterminate form

Hence, we let $y = x^x \Rightarrow \ln y = x \ln x$. Next, we find the limits of both sides to get;

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0} x \ln x = 0$$

$$\therefore \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} y = e^0 = 1$$

Example

Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Solution: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = 1^{\infty}$. Let $y=\left(1+\frac{2}{x}\right)^x \Rightarrow \ln y = x\ln\left(1+\frac{2}{x}\right)$

Thus; $\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \infty\cdot 0 \Rightarrow = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$

Applying L'Hopital's rule to get;

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{-\frac{2}{x(x+2)}}{-\frac{1}{x^2}} = 2 \therefore \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$$

Rationalizing the function

Example 1: Evaluate the following; $\lim_{x\to 4}(4 - \sqrt{x})$

Solution: $\lim_{x\to 4}(2 - \sqrt{x}) = 0$

Example 2: Evaluate the following; $\lim_{x\to\infty}\left(\frac{\sqrt{x}-5}{x-3}\right)$

Solution: $\lim_{x\to\infty}\left(\frac{\sqrt{x}-5}{x-25}\right) = \lim_{x\to\infty}\frac{\sqrt{x}-5}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim_{x\to\infty}\frac{1}{\sqrt{x}+5} = 0$

Example 3

Evaluate the following; $\lim_{x \to 0} \left(\frac{\sqrt{x+9}-3}{x} \right)$

Solution: Direct substitute we get $\frac{0}{0}$ - indeterminate.

$$\lim_{x \to 0} \left(\frac{\sqrt{x+9}-3}{x} \right) = \lim_{x \to 0} \left(\frac{\sqrt{x+9}-3}{x} \times \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} \right)$$

$$= \lim_{x \to 0} \frac{x+9-9}{x\left(\sqrt{x+9}+3\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(\sqrt{x+9}+3\right)} = \frac{1}{6}$$

Limits at infinity

Suppose the function $f(x)$ gets arbitrarily close to l as x becomes sufficiently large, then we say that $f(x)$ has a limit at infinity denoted; $\lim_{x \to \infty} f(x) = l$

Similarly; Suppose the function $f(x)$ gets arbitrarily close to l for $x < 0$ as $|x|$ becomes sufficiently large, then we say that $f(x)$ has a limit at negative infinity denoted; $\lim_{x \to -\infty} f(x) = l$.

Note that if the function gets arbitrarily close to las $x \to \pm\infty$, the graph of $f(x)$ approaches the line $f(x) = l$ which is a horizontal asymptote of $f(x)$.

Infinite limits at infinity

A function $f(x)$ is said to have a positive (negative) infinite limit at infinity, as $f(x)$ (or $f(x) < 0$ and $|f(x)|$) becomes arbitrarily large for x sufficiently large denoted; $\lim_{x \to \pm\infty} f(x) = \pm\infty$

Example 1: Evaluate the following: $\lim_{x \to \infty} \left(\frac{2x^2+7}{3x^3-8}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^3 to get;

$$\lim_{x \to \infty} \left(\frac{2x^2 + 7}{3x^3 - 8}\right) = \lim_{x \to \infty} \left(\frac{\frac{2x^2}{x^3} + \frac{7}{x^3}}{\frac{3x^3}{x^3} - \frac{8}{x^3}}\right) = \lim_{x \to \infty} \left(\frac{\frac{2}{x} + \frac{7}{x^3}}{3 - \frac{8}{x^3}}\right) = \frac{0 + 0}{3 - 0} = 0$$

Example

Evaluate the following: $\lim_{x\to\infty}\left(\frac{2x^4+11}{3x^3+x+1}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^3 to get;

$$\lim_{x\to\infty}\left(\frac{2x^4+11}{3x^3+x+1}\right) = \lim_{x\to\infty}\left(\frac{\frac{2x^4}{x^3}+\frac{11}{x^3}}{\frac{3x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{2x+\frac{11}{x^3}}{3+\frac{1}{x^2}+\frac{1}{x^3}}\right) = \infty$$

Example

Evaluate the following: $\lim_{x\to\infty}\left(\frac{x^4-1}{3x^4+x^2+1}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^4 to get;

$$\lim_{x\to\infty}\left(\frac{\frac{x^4}{x^4}-\frac{1}{x^4}}{\frac{3x^4}{x^4}+\frac{x^2}{x^4}+\frac{1}{x^4}}\right)=\lim_{x\to\infty}\left(\frac{1-\frac{1}{x^4}}{3+\frac{1}{x^2}+\frac{1}{x^4}}\right)=\frac{1}{3}$$

References

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Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

End of lecture 1

Thank you