Calculus I

Lecture 4

First Principle of Differentiation

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Introduction to lecture 4

This lecture is a continuous of lectures 1,2, and 3.

It will show the relationship between limits and differentiation.

Intended learning outcomes

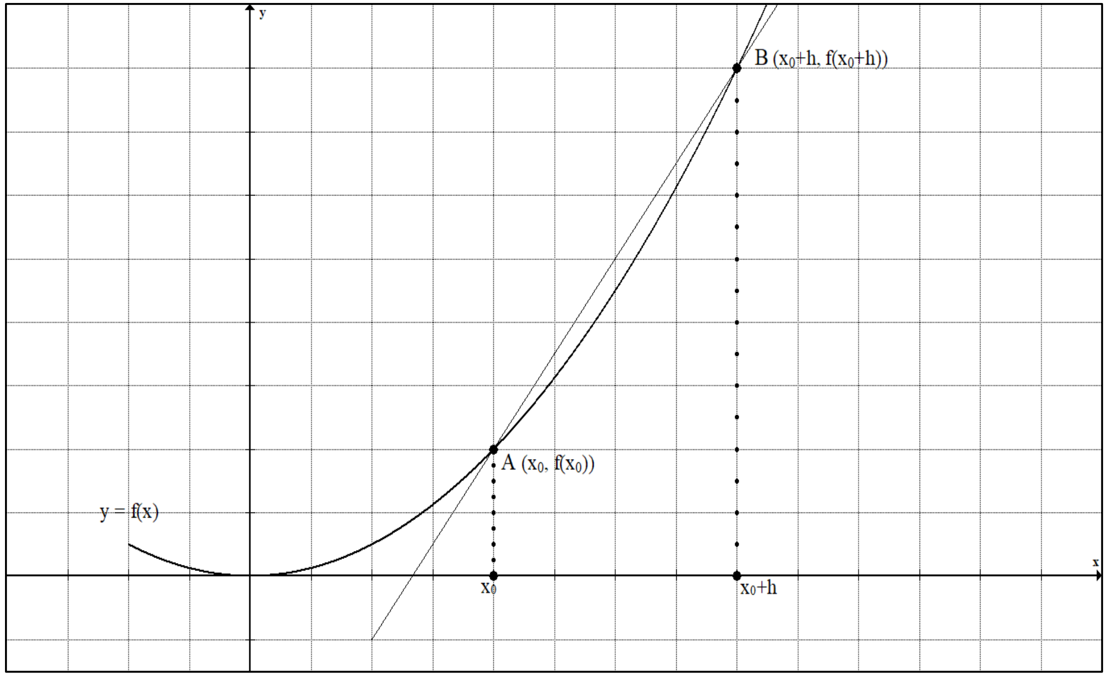
Be	**At the end of this lecture, you will be able to;**
Explain	Explain the first principle of differentiation.
Apply	Apply the first principle of differentiation to determine the derivatives of given functions.

References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019).

Introduction

Consider the curve of the graph $y = f(x)$. The secant line from the point $(x_0, f(x_0))$ to the point $(x_0 + h, f(x_0 + h))$ is an approximation of the curve at the point $(x_0, f(x_0))$ as h tends to zero.



Introduction...contd...

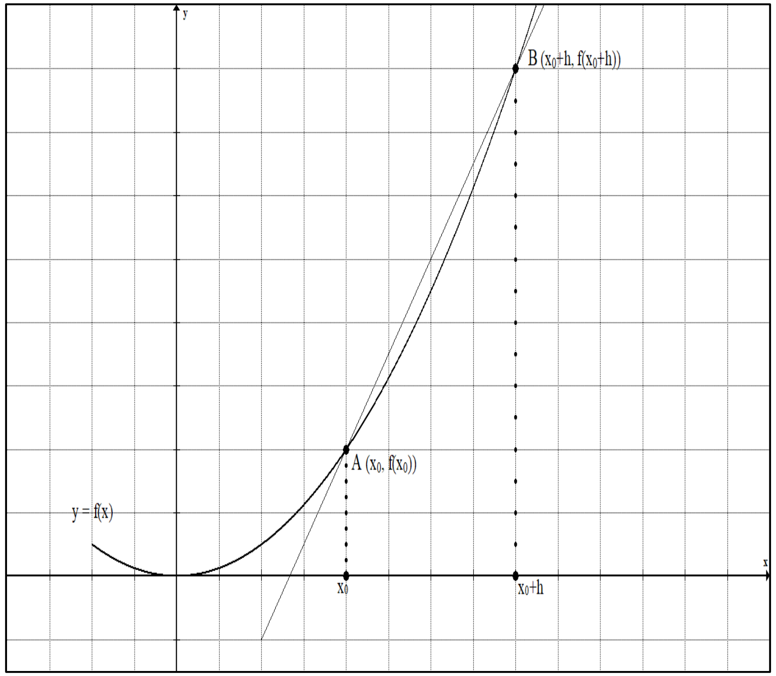
The slope or gradient of the secant AB is given by;

$$\frac{f(x_0+h)-f(x_0)}{h} \quad \dots\dots (i)$$

As h approaches zero i.e., $h \to 0$, we get a better approximate of the curve.

Note that h is a small change along the x axis, and we denote it as Δx.

Therefore, by definition; slope $= \frac{\text{change in y−coordinates}}{\text{change in x coordinates}}$



Introduction...contd...

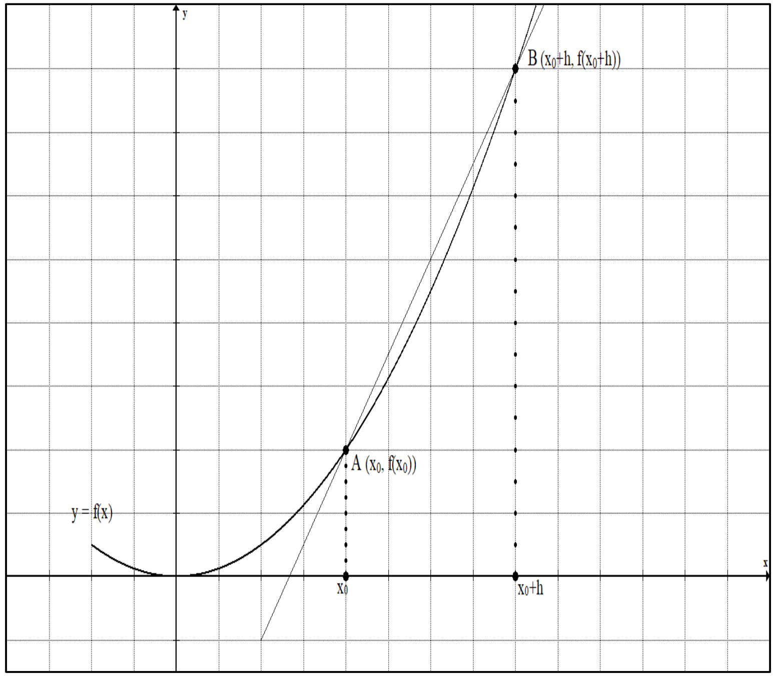
This implies $f(x_0 + \Delta x) - f(x_0)$ is the change in y coordinates and $(x_0 + \Delta x) - x_0$ is the change in x coordinates.

Therefore;

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0}$$

As $\Delta x \to 0$, point B will move along the curve and approach point A,

such that the secant line AB approach the tangent line at point A.



Introduction…contd…

The slope or gradient of the tangent line is the same as the gradient of the curve at that point A.

That is;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{(x_0+\Delta x)-x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Definition 1: Derivative of a function by first principle

Given a function $f(x)$ then the function $f'(x)$ or $\frac{dy}{dx}$ is the derivative of $f(x)$ at point x_0 defined by;

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Definition

Alternatively; (difference quotient)

$$\frac{dy}{dx} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right)$$

When the limit exists, we say that $f(x)$ is differentiable at point x_0.

Definition 2: Derivative Notation

Differentiation is the operation of finding the derivative function $f'(x)$ or $\frac{dy}{dx}$.

$\frac{dy}{dx}$ is called the Leibniz notation.

Example 1

Find the gradient of the tangent line to the curve $f(x) = x^2$ at point $x = 3$.

By definition

$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, Our $x_0 = 3$

Thus;

$$\lim_{\Delta x \to 0} \frac{f(3+\Delta x)-f(3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(3+\Delta x)^2-9}{\Delta x} = \lim_{\Delta x \to 0} \frac{9+6\Delta x+(\Delta x)^2-9}{\Delta x}$$

Example 1...contd...

$$= \lim_{\Delta x \to 0} \frac{9+6\Delta x+(\Delta x)^2-9}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6\Delta x+(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(6+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0}(6 + \Delta x) = 6$$

That is, the gradient of the tangent to the curve $f(x) = x^2$ at $x = 3$ is 6. (Alternatively, the gradient of the curve at point $x = 3$ is 6).

Example 2

Use the definition $f'(x) = \lim_{\delta x \to 0} \left(\frac{f(x_0 + \delta x) - f(x_0)}{\delta x} \right)$ to find the derivative of;

$$f(x) = 2x^2 - 7$$

Solution:

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x_o + \delta x) - f(x_o)}{\delta x} = \lim_{\delta x \to o} \frac{(2(x_0 + \delta x)^2 - 7) - (2x_0^2 - 7)}{\delta x}$$

Example 2…contd…

$$= \lim_{\delta x \to 0} \left(\frac{2x_0^2 + 4x_0 \delta x + 2(\delta x)^2 - 7 - 2x_0^2 + 7}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{4x_0 \delta x + 2(\delta x)^2}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\delta x(4x_0 + 2\delta x)}{\delta x} \right) = \lim_{\delta x \to 0} (4x_0 + 2\delta x) = 4x_0$$

Clearly;

$$\frac{dy}{dx} = 4x$$

Example 3

Use the definition $f'(x) = \lim_{\delta x \to 0} \left(\frac{f(x_0+\delta x)-f(x_0)}{\delta x} \right)$ to find the derivative of $f(x) = \sin x$

Solution:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{f(x_0+\delta x)-f(x_0)}{\delta x} \right) = \lim_{\delta x \to 0} \left(\frac{\sin(x_0+\delta x)-\sin x_0}{\delta x} \right)$$

$$\lim_{\delta x \to 0} \left(\frac{\sin(x_0+\delta x)-\sin x_0}{\delta x} \right) = \lim_{\delta x \to 0} \frac{\sin x_0 \cos \delta x+\sin \delta x \cos x_0-\sin x_0}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x_0 (\cos \delta x-1)+\sin \delta x \cos x_0}{\delta x}$$

Example 3...contd...

$$= \lim_{\delta x \to 0} \frac{\sin x_0(\cos \delta x - 1)}{\delta x} + \lim_{\delta x \to 0} \frac{\sin \delta x \cos x_0}{\delta x}$$

$$= \sin x_0 \lim_{\delta x \to 0} \frac{\cos(\delta x - 1)}{\delta x} + \cos x_0 \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdots \text{(i)}$$

Now $\lim_{\delta x \to 0} \frac{\cos(\delta x - 1)}{\delta x} = 0$ and $\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$

Hence (i) becomes;

$$\frac{dy}{dx}(\sin x) = \cos x$$

Definition 4: Tangent line

Suppose that the function $f(x)$ is differentiable at point a.

Then the tangent to the graph of $y = f(x)$ at say point $\left(x_0, f(x_0)\right)$ is the line through this point of slope $f'(x_0)$.

The equation of the tangent line in point-slope form is;

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Example 1

Determine the equation of the tangent line to the graph of the function;

$f(x) = x^2 + 2x + 1$ at $x = 1$.

Solution: We first compute $f'(1)$ i.e.

$$f'(1) = \lim_{x\to 1}\left(\frac{f(x)-f(1)}{x-1}\right) = \lim_{x\to 1}\left(\frac{x^2+2x+1-4}{x-1}\right)$$

$$\lim_{x\to 1}\left(\frac{x^2+2x-3}{x-1}\right) = \frac{0}{0} - \text{indeterminate}$$

Example 1…contd…

We need to simplify the numerator to get;

$$f'(1) = \lim_{x \to 1}\left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x \to 1}\left(\frac{(x+3)(x-1)}{x-1}\right)$$

$$= \lim_{x \to 1}(x+3) = 4$$

Now we use

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Example 1...contd...

Now we use

$$y - f(x_0) = f'(x_0)(x - x_0)$$

To get;

$y - f(1) = f'(1)(x - 1)$

$y - 4 = 4(x - 1)$

$y = 4x$

This is the equation of the tangent to the graph of $f(x) = x^2 + 2x + 1$ at $x = 1$.

Continuity and differentiation

A function $f(x)$ is said to be differentiable at a point x_0 if the function is defined at least on some open interval, I containing the point x_0 and $f'(x)$exists.

Proof: Assume f is differentiable at point x_0, then $\lim_{x \to x_0} f(x) = l$ if and only if $\lim_{h \to 0} f(x_0 + h) = l$

$f(x)$ will be continuous at x_0 if $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$

Contd...

Since the limit of the product is the product of the limits

$$\lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = \lim_{\Delta x \to 0} [\frac{f(x+\Delta x)-f(x)}{\Delta x} \Delta x]$$

$$= [\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}] \lim_{\Delta x \to 0} \Delta x$$

$$= f'(x) \times 0 = 0$$

Contd…

$$= f'(x) \times 0 = 0$$

Therefore, since the limit of a sum is the sum of the limits, this implies;

$$\lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x) + f(x)]$$

$$= \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] + \lim_{\Delta x \to 0} f(x) = 0 + f(x) = f(x)$$

Alternatively, if $f(x)$ is differentiable at x_0, then $f(x)$ is continuous at x_0.

Definition 6

A function $f(x)$ is differentiable at x_0 with its derivative $f'(x)$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that;

$$|x - x_o| < \delta \Rightarrow \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \varepsilon$$

Example 1

Show that the derivative of $f(x) = 3x^2$ at $x = 1$ is 6.

Proof: Our $x_0 = 1, f(x) = 3x^2, f(x_0) = 3, f'(x_0) = 0$

$$|x - x_o| < \delta \Rightarrow \left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \varepsilon$$

$$|x - 1| < \delta \Rightarrow \left|\frac{3x^2-3}{x-1} - 0\right| < \varepsilon$$

Example 1...contd...

Working with the RHS to get;

$$\left|\frac{3x^2-3}{x-1} - 0\right| < \varepsilon \sim \left|\frac{3(x^2-1)}{x-1}\right| < \varepsilon$$

$$\sim \left|\frac{3(x-1)(x+1)}{x-1}\right| < \varepsilon$$

$$\sim \frac{3|x-1||x+1|}{|x-1|} < \varepsilon$$

$$\sim 3|x-1||x+1| < \varepsilon|x-1|$$

$$\sim |x-1| < \frac{\varepsilon}{3} \cdot \frac{|x-1|}{|x+1|}$$

Example 1...contd...

Let $\delta = \frac{\varepsilon}{3} \cdot \frac{|x-1|}{|x+1|}$

We can let $|x-1| < 1 \cdots$ (i) $\Rightarrow 0 < x < 2$

Hence, $1 < x + 1 < 3$

$|x-1| < \frac{\varepsilon}{3}\frac{|x-1|}{|x+1|} < \frac{\varepsilon}{9} \cdots$ (ii)

$\Rightarrow \delta = \min\{1, \frac{\varepsilon}{9}\}$

Indeed, the function is differentiable at this point.

Application of first principle of differentiation

This will help build on techniques of differentiation, especially in their proofs.

It is directly applicable to rate of change and related changes in the coming lectures.

References

Cowen, R. ., Were, J. ., & Vaz, P. . (1990). *An Introduction to Calculus*. Nairobi University Press.

Kahenya, N. P. (2022). *Mathematics for science*. https://www.hufocw.org/Course/360

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Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

End of Lecture 4

Thank You!