Calculus I

Lecture 8

Implicit and Partial Differentiation

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Introduction to lecture 8

Lecture 8 will introduce the techniques of differentiating Implicit functions and partial differentiation.

The functions discussed in this lecture involve more than one variable but ideally the differentiation is similar with that for differentiating explicit functions.

Intended learning outcomes

Be	At the end of this lecture, you will be able to;
Differentiate	Differentiate implicit functions.
Apply	Apply the technique of partial differentiation to given functions.

References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Stewart, 2012; Sullivan & Miranda, 2019).

Implicit Functions

So far, we have dealt with functions that involving expressing one variable explicitly in terms of another variable

e.g., $y = f(x)$.

There are situations where functions can be defined implicitly by a relation between two or more variables

e.g., $z = f(x, y), x^2 + y^2 = 9; x^2 + y = xy$.

Implicit Functions...contd...

Sometimes it is possible to write an implicit function as two or more independent explicit functions e.g., the equation of the circle $x^2 + y^2 = 9$ can be written explicitly as $y = \pm\sqrt{9 - x^2}$.

These are the upper and lower semicircles that form the circle.

A function $f(x)$ is a function defined implicitly by equation $z = f(x, y)$ if the equation is true for all values of x in the domain of $f(x)$.

Definition 1

A continuous function f defined on an open interval I is said to be implicit in an equation involving the variables x and y if when y is replaced by f(x), the resultant is true for all values of x in the domain of f(x).

Example 1: Equation $x^2 + y^2 = 9 \Rightarrow x^2 + [f(x)]^2 = 9$ is true for all values of x in f(x).

Example 2: Equation $x^2 + y = xy \Rightarrow x^2 + f(x) = x[f(x)]$.

Definition 2

The technique of implicit differentiation consists of differentiation both sides of the equation with respect to x and then solving the resulting equation for y'.

Example 1

Given $x^3 + y^2 = 10$ determine $\frac{dy}{dx}$. Hence find the equation of the tangent to the circle at point $x = 1$.

Solution: We first differentiate both sides of the equation to get;

$$\frac{d}{dx}\left(x^3 + y^2\right) = \frac{d}{dx}(10)$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^2\right) = 0 \cdots \text{(i)}$$

Example 1...contd...

Note that y is a function of x. We can apply the chain rule for the second term of (i) to get;

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$3x^2 + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -3x^2 \therefore \frac{dy}{dx} = -\frac{3x^2}{2y}$$

Example 1…contd…

At point $x = 1$ we have $1 + y^2 = 10 \therefore y^2 = 9$

$\Rightarrow y = \pm 3$

Therefore, $\frac{dy}{dx} = -\frac{3}{2(3)} = -\frac{1}{2}$ or $\frac{dy}{dx} = -\frac{3}{2(-3)} = \frac{1}{2}$

The equation of the tangent line 1 is;

$y - f(1) = f'(1)(x - 1)$

$y - 3 = -\frac{1}{2}(x - 1) \therefore 2y + x - 4 = 0$

The equation of the tangent line 2 is.

$y + 3 = \frac{1}{2}(x - 1) \Rightarrow 2y - x + 4 = 0$

Example 2

Determine $\frac{d^2y}{dx^2}$ given $x^3 + y^3 = 3$

Solution: $3x^2 + 3y^2 \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \cdots (i)$

Example 2...contd...

We use the Quotient rule to find the derivative of (i) i.e.

$$\frac{d^2y}{dx^2} = -\left(\frac{2xy^2 - x^2 \cdot 2y\frac{dy}{dx}}{y^4}\right)$$

$$= -\left(\frac{2xy^2 - x^2 \cdot 2y\left(-\frac{x^2}{y^2}\right)}{y^4}\right)$$

$$= -\left(\frac{2xy^2 + \frac{2x^4}{y}}{y^4}\right) = -\frac{2xy^3 + 2x^4}{y^5} = -\frac{2x\left(y^3 + x^3\right)}{y^5}$$

Example 3

Given the equation of a circle centre $O(0,0)$, radius r, and passing through point $P(x, y)$ i.e., $x^2 + y^2 = r^2$ then the tangent to the circle at point P is perpendicular to line OP.

Proof: We first determine the gradient of the tangent by finding $\frac{dy}{dx}$.

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \cdots \text{(i)}$$

Example 3…contd…

The equation of the line OP is

$y = mx + c$

But $c = 0 \Rightarrow y = mx \therefore m = \frac{y}{x} \cdots \text{(ii)} - \text{Gradient of OP.}$

If OP and the tangent are perpendicular then the product of their gradients is -1 i.e.,$\text{(i)} \times \text{(ii)} = -1$

$\Rightarrow -\frac{x}{y} \cdot \frac{y}{x} = -1$

Example 4

Determine $\frac{dy}{dx}$ given $\sin^2 y = 9e^x$.

Solution: $\frac{d}{dx}\left(\sin^2 y\right) = \frac{d}{dx}(9e^x)$

$\sin 2y \frac{dy}{dx} = 9e^x$

$\therefore \frac{dy}{dx} = \frac{9e^x}{\sin 2y}$

Example 5

Given $\cot\, y = 9e^x$ determine $\frac{dy}{dx}$ in terms of x alone.

Solution: $\frac{d}{dx}(\cot y) = \frac{d}{dx}(9e^x)$

$-\csc^2 y \frac{dy}{dx} = 9e^x$

$\therefore \frac{dy}{dx} = -\frac{9e^x}{\csc^2 y}$

But; $\csc^2 y = 1 + \cot^2 y = 1 + 81e^{2x}$

Therefore;

$\frac{dy}{dx} = -\frac{9e^x}{1+81e^{2x}}$

Partial Differentiation

If a function $z = f(x, y)$ is a function of two variables (representing a surface), then its partial derivatives are the functions f_x and f_y defined by

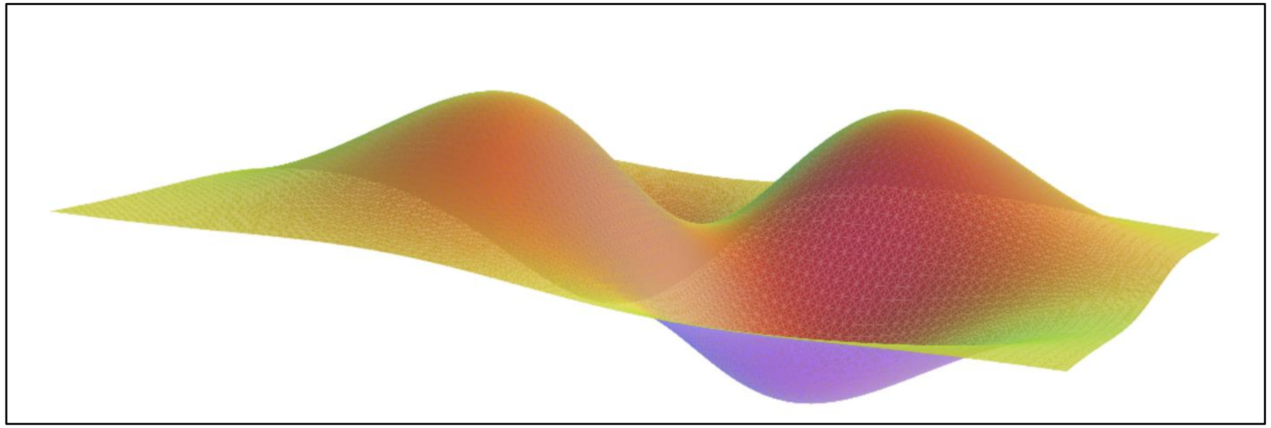
$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h};$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Provided the limits on the RHS exist.

Alternative we can denote it as;

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x};\ f_y(x, y) = f_y = \frac{\partial f}{\partial y}$$



Rules for finding the partial derivates of f(x, y)

Rule 1: To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

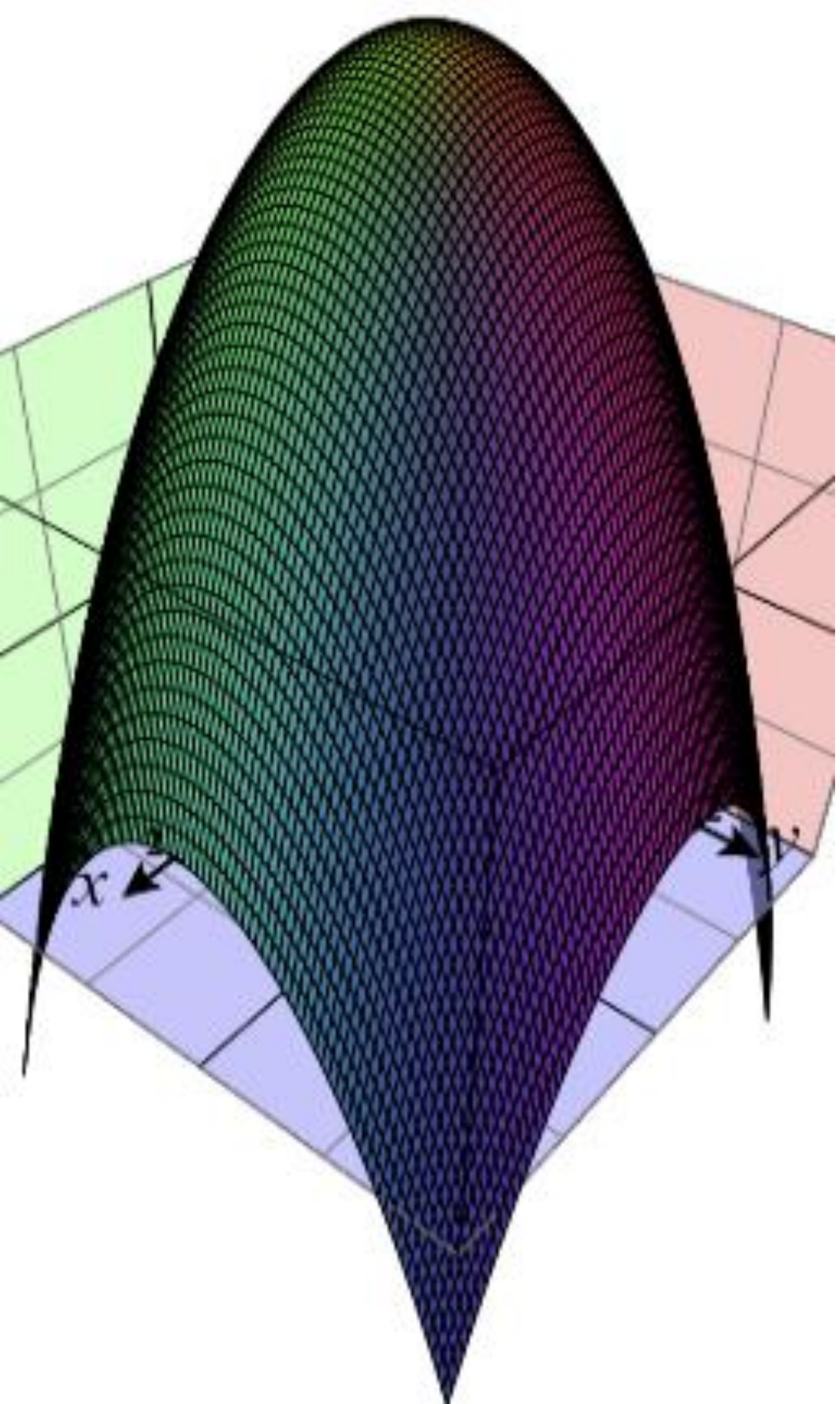
Rule 2: To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Remarks

Remark 1: Partial derivate if similar with the normal derivative involving one variable.

We only need to differentiate one variable as you fixed the rest.

Remark 2: Partial derivative is applicable in understanding changes of parameters in a 3-D perspective e.g., change temperatures in a space, slope changes on a hill etc.



Example 1

Consider the function $z = f(x, y) = 4 - x^2 - y^2$ (see the paraboloid in Figure to the right).

Determine $f_x(1,2)$ and $f_y(1,2)$.

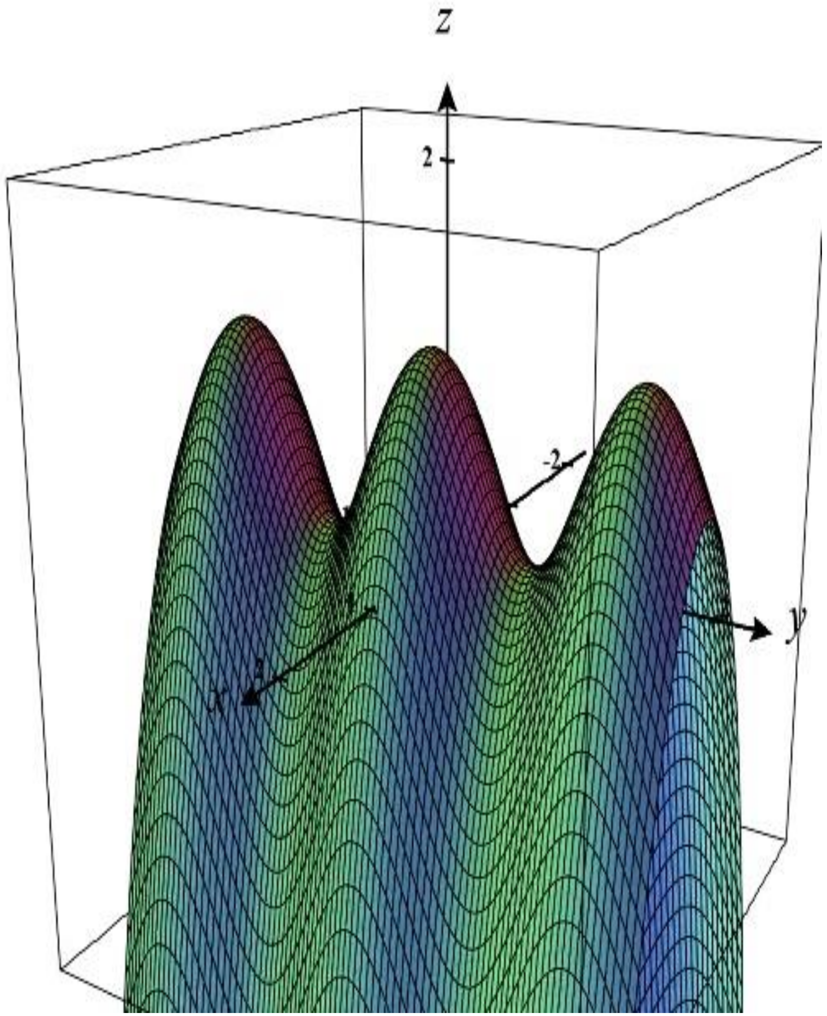
Solution: $f_x = \frac{\partial f}{\partial x} = -2x \therefore \left.\frac{\partial f}{\partial x}\right|_{x=1} = -2$; $f_y = \left.\frac{\partial f}{\partial y}\right|_{y=2} = -2y = -4$

Definition 1

Given $z = f(x, y)$ then $\frac{dy}{dx} = -\frac{f_x}{f_y}$. This method is easier for implicit functions.

Example 1: Determine $\frac{dy}{dx}$ given $x^3 + y^2 = 10$ using partial derivatives.

Solution: $f_x = 3x^2; f_y = 2y \quad \therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2}{2y}$



Example 2

Given the function $\cos^2 2y - 2x^2 + \sin x$ (see figure on the right). Determine $\frac{dy}{dx}$.

Solution: $f_x = -4x + \cos x$; Let $t = 2y \Rightarrow \frac{dt}{dy} = 2$ but we have

$\cos^2 t = \cos t \cos t$

$\therefore \frac{d}{dt}(\cos t \cos t) = \cos t \cdot (-\sin t) + (-\sin t) \cdot \cos t = -2 \sin t \cos t$

But $-2 \sin t \cos t = -\sin 2t$. Hence;

$\frac{d}{dy}(\cos^2 2y) = \frac{dt}{dy} \cdot \frac{d}{dt} = -2 \sin 2t = -2 \sin 4y$

$\therefore f_y = -2 \sin 4y$. Finally, $\frac{dy}{dx} = -\left(\frac{-4x+\cos x}{-2 \sin 4y}\right)$

Example 3

Find the gradient of the tangent line to the cross section cut from the surface of

$z = 6x^2y - xy^3$ by the plane $y = 3$ at the point

$p = (1, 3, -9)$

Solution: We need to hold y constant and find f_x i.e.

$$\frac{\partial z}{\partial x} = f_x = 12xy - y^3 \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{x=1, y=3}$$

$$= 12(1)(3) - 3^3 = 9$$

Example 4

Use the definition of partial differentiation to find f_x and f_y given $f(x, y) = xy^2$

Solution: We have;

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}; \; f_y(x, y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Therefore;

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{(x+h)y^2 - xy^2}{h} = \lim_{h \to 0} \frac{xy^2 + y^2h - xy^2}{h}$$

$$= \lim_{h \to 0} \frac{y^2 h}{h} = y^2$$

Example 4...contd...

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{x(y+h)^2-xy^2}{h} = \lim_{h\to 0} \frac{xy^2+2xyh+xh^2-xy^2}{h}$$

$$= \lim_{h\to 0} \frac{2xyh+xh^2}{h}$$

$$= \lim_{h\to 0}(2xy + xh) = 2xy$$

Example 5

Use the definition of partial differentiation to find $f_x + f_y + f_z$ when $x = y = z = 1$ given $f(x, y, z) = 2x^2y + yz^2$

Solution: We have;

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h};$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z(x, y, z) = \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

Example 5...contd...

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(x+h)^2 y + yz^2\right] - \left[2x^2 y + yz^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[2x^2 y + 4xhy + 2yh^2 + yz^2\right] - \left[2x^2 y + yz^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{4xhy + 2yh^2}{h} = \lim_{h \to 0} (4xy + 2yh) = 4xy$$

Example 5...contd...

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h,z)-f(x,y,z)}{h}$$

$$= \lim_{h\to 0} \frac{\left[2x^2(y+h)+(y+h)z^2\right]-[2x^2y+yz^2]}{h}$$

$$= \lim_{h\to 0} \frac{2x^2y+2x^2h+z^2y+z^2h-2x^2y-yz^2}{h}$$

$$= \lim_{h\to 0} \frac{2x^2h+z^2h}{h} = 2x^2 + z^2$$

Example 5…contd…

$$f_z(x,y,z) = \lim_{h\to 0} \frac{f(x,y,z+h)-f(x,y,z)}{h}$$

$$= \lim_{h\to 0} \frac{[2x^2y+y(z+h)^2]-[2x^2y+yz^2]}{h}$$

$$= \lim_{h\to 0} \frac{[2x^2y+yz^2+2yzh+yh^2]-[2x^2y+yz^2]}{h}$$

$$\lim_{h\to 0} \frac{[2yzh+yh^2]}{h} = \lim_{h\to 0}(2yz+yh) = 2yz$$

Finally;

$$f_x + f_y + f_z = 4xy + 2x^2 + z^2 + 2yz\Big|_{x=y=z=1}$$

$$= 4 + 2 + 1 + 2 = 8$$

Definition 2: Second Partial Derivatives

Suppose f is a function of two variables x and y, (or more variables) then its partial derivatives f_x and f_y are also functions of two (or more) variables.

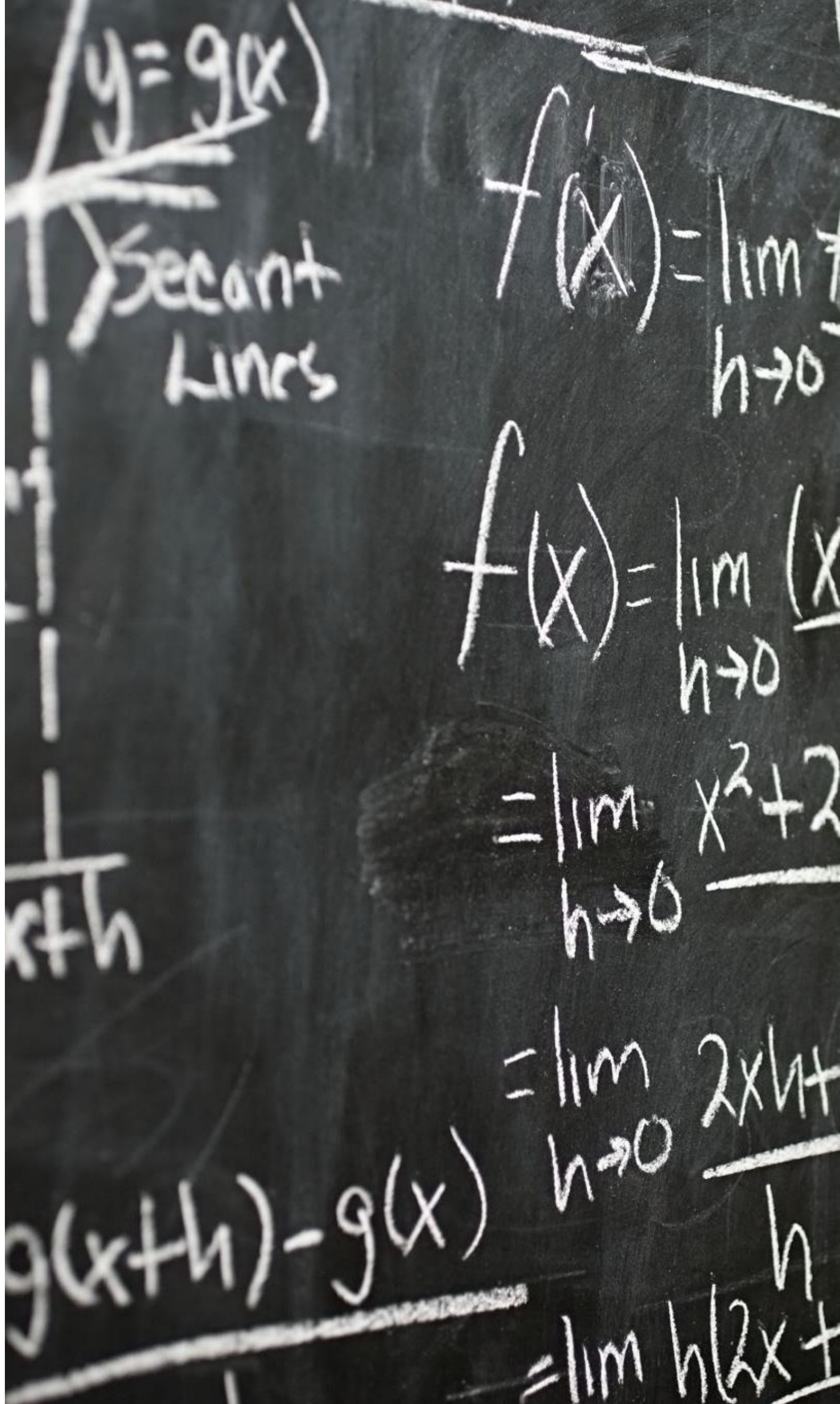
One can get their second partial derivatives namely;

$(f_x)_x = f_{xx}$,

$(f_x)_y = f_{xy}$,

$(f_y)_x = f_{yx}$, and

$(f_y)_y = f_{yy}$



Alternative notations for second partial derivatives

Given the function $f(x, y)$ then we have the following notations for the second partial derivatives;

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Note that given $f_{x_1 x_2}$ it implies that one first differentiates with respect to x_1 and then with respect to x_2.

Example 1

Determine the second derivatives of the following function;

$f(x, y) = x^2 + 3y^3 + 4xy^3$

Solution: $f_x = 2x + 4y^3$

$f_y = 9y^2 + 12xy^2$

$f_{xx} = 2$

$f_{xy} = 12y^2$

$f_{yy} = 18y + 24xy$

$f_{yx} = 12y^2$

Remark: You can plot the graph of $x^2 + 3y^3 + 4xy^3$ here: https://math.libretexts.org/Learning_Objects/CalcPlot3D_Interactive_Figures/CalcPlot3D

References

Cowen, R. ., Were, J. ., & Vaz, P. . (1990). *An Introduction to Calculus.* Nairobi University Press.

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

CalcPlot3D (libretexts.org)

End of Lecture 8

Thank You!