Calculus I

Lecture 9

Logarithmic and Parametric Differentiation

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Introduction to lecture 9

Lecture 9 will introduce the techniques of differentiating parametric functions and use of logarithm to differentiate functions.

Differentiating parametric functions may involve multiple application of different techniques that we have discussed so far.

Intended learning outcomes

Be

At the end of this lecture, you will be able to;

Differentiate

Differentiate function using logarithms.

Differentiate

Differentiate parametric functions.

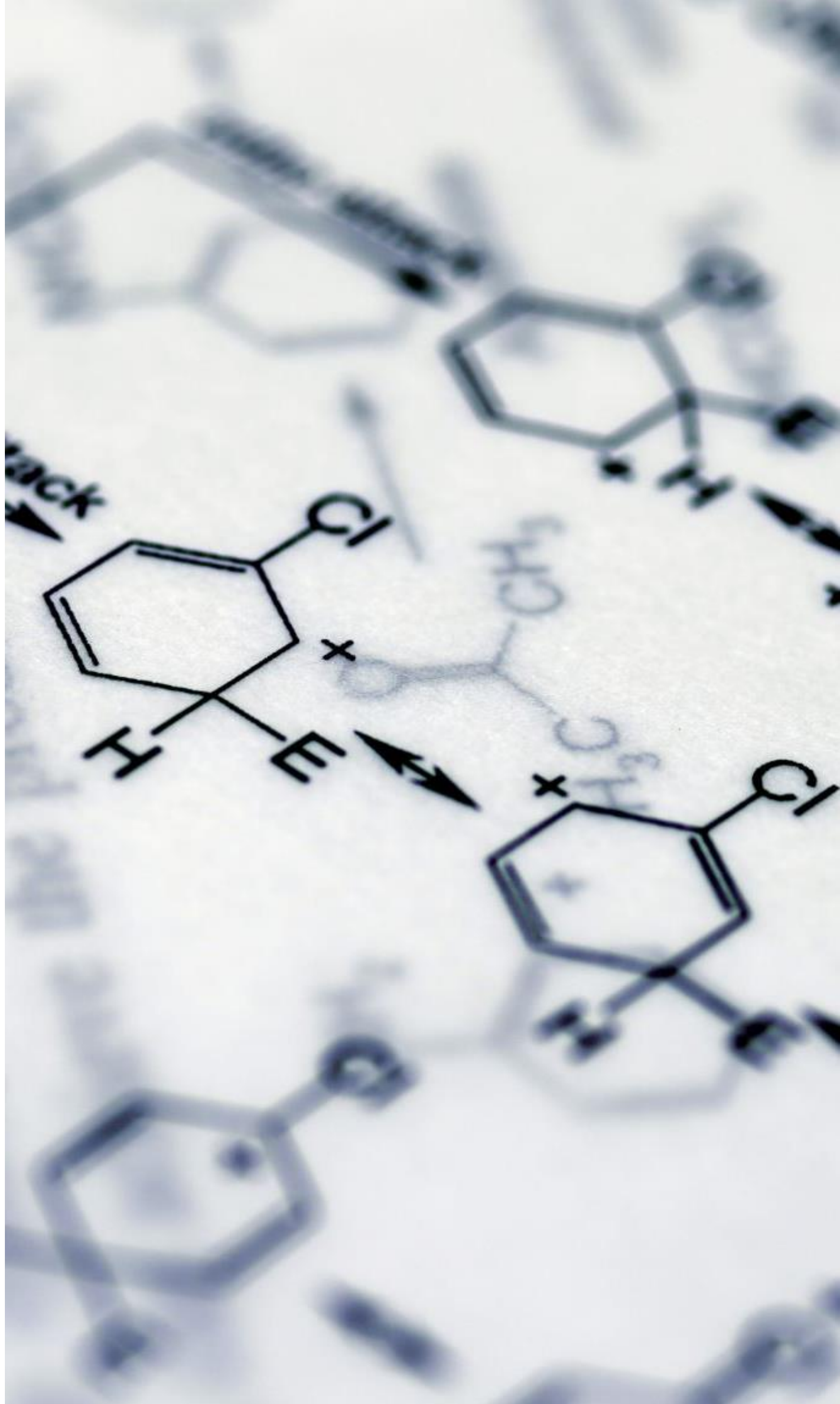
References for further reading

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Stewart, 2012; Sullivan & Miranda, 2019).

Logarithmic differentiation

This is the case where derivative of complicated functions is simplified by using logarithms.

These are situations where the techniques of product, quotient or power rules is tedious and complicated.



Definition 1: Properties of logarithms

To successfully use logarithmic differentiation, knowledge in some basic properties is important.

1) $\ln(xy) = \ln x + \ln y$ e.g $\ln(3\times4) = \ln 3 + \ln 4$

2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ e.g $\ln\left(\frac{18}{3}\right) = \ln 18 - \ln 3$

3) $\ln(x^n) = n \ln x$ for some $x > 0, y > 0, n \in \mathbb{R}$ e.g $\ln 20^9 = 9 \ln 20$

Remarks

Remark 1: In case $f(x) \leq 0 \ \forall x$, then $\ln f(x)$ is not defined.

Hence, we determine $|f(x)|$.

$\ln 0 = \infty$

$\ln(-2) = \ln|-2|$

$= \ln 2$

Remark 2: $\ln\left(\frac{x}{y}\right) \neq \frac{\ln x}{\ln y}$

$\ln\left(\frac{18}{9}\right) = \underline{\underline{\ln 18 - \ln 9}}$ ✓

$\neq \frac{\ln 18}{\ln 9} \leftarrow$ ✗

Example 1

Consider the function

$$y = \frac{(x^2-1)^5\sqrt{x+1}}{x^3+3}$$

to compute $\frac{df}{dx}$ one need to apply the quotient, product, and chain rules.

This may complicate the operation.

We need to apply logarithms to simplify the operation.

Example 1…contd…

Solution: We take the logarithm of both sides to get;

$$\ln y = \ln\left(\frac{(x^2-1)^5\sqrt{x+1}}{x^3+3}\right)$$

Next, we apply the rules/properties of logarithms to expand the RHS i.e.

$\ln y = \ln\left((x^2-1)^5\sqrt{x+1}\right) - \ln(x^3+3)$ by property; $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln y = \ln(x^2-1)^5 + \ln(x+1)^{\frac{1}{2}} - \ln(x^3+3)$ by property; $\ln(xy) = \ln x + \ln y$

$\ln y = 5\ln(x^2-1) + \frac{1}{2}\ln(x+1) - \ln(x^3+3)$by property; $\ln x^n = n\ln x$

Example 1...contd...

Differentiate both sides with respect to x i.e.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(5\ln\left(x^2-1\right) + \frac{1}{2}\ln(x+1) - \ln(x^3+3)\right)$$

$$\frac{d}{dx}(\ln y) = 5\frac{d}{dx}\ln\left(x^2-1\right) + \frac{1}{2}\frac{d}{dx}(\ln|x+1|) - \frac{d}{dx}(\ln|x^3+3|) \cdots \text{(i)}$$

Note that for $\frac{d}{dx}(5\ln(x^2-1)$ let $x^2-1=u \Rightarrow \frac{du}{dx} = 2x \therefore \frac{d}{du}(\ln u) = \frac{1}{u}$

Example 1...contd...

Hence $\frac{d}{dx}(5\ln(x^2-1)) = 5 \cdot \frac{1}{u} \cdot 2x = \frac{10x}{x^2-1}$

Again for $\frac{d}{dx}(\ln|x^3+3|)$ let $x^3+3=u$

$\Rightarrow \frac{du}{dx} = 3x^2$ and

$\frac{d}{du}(\ln u) = \frac{1}{u}$

Hence $\frac{d}{dx}(\ln|x^3+3|) = 3x^2 \cdot \frac{1}{u} = \frac{3x^2}{x^3+3}$

Example 1...contd...

Therefore equation (i) becomes;

$$\frac{1}{y}\frac{dy}{dx}=\frac{10x}{x^2-1}+\frac{1}{2(x+1)}-\frac{3x^2}{x^3+3}$$

$$\frac{dy}{dx}=y\left(\frac{10x}{x^2-1}+\frac{1}{2(x+1)}-\frac{3x^2}{x^3+3}\right)$$

$$\therefore\ \frac{dy}{dx}=\left(\frac{\left(x^2-1\right)^5\sqrt{x+1}}{x^3+3}\right)\left(\frac{10x}{x^2-1}+\frac{1}{2(x+1)}-\frac{3x^2}{x^3+3}\right)$$

Example 2

Differentiatiate with respect to x the function $\boxed{y = x^{\cos x}}$

$$\ln y = \ln\left(x^{\cos x}\right)$$

$$\ln y = \cos x \ln x \qquad \left(\ln A^n = n \ln A\right)$$

$$\frac{d}{dx}\left(\ln \underline{y}\right) = \frac{d}{dx}\left(\underline{\cos x}\,\underline{\ln x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln x + \frac{\cos x}{x}$$

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

$$= x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

Example 3

Find $\frac{dy}{dx}$ given $y = x^{(3x+7)}$

$$\ln y = \ln\left(x^{3x+7}\right) \quad \left(\ln x^n = n\ln x\right)$$

$$\ln y = (3x+7)\ln x$$

differentiate both sides $\bar{c}$ respect to x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\underline{(3x+7)\ln x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\ln x + \frac{1}{x}\cdot(3x+7)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\ln x + \frac{3x+7}{x}$$

$$\therefore \frac{dy}{dx} = y\left(3\ln x + \frac{3x+7}{x}\right)$$

$$\frac{dy}{dx} = x^{3x+7}\left(3\ln x + \frac{3x+7}{x}\right)$$

Remarks

(i) $\frac{d}{dx}(\ln x) = \frac{1}{x}$ ✓

(ii) $\frac{d}{dx}(\ln|x^a + b|) = \frac{ax^{a-1}}{x^a+b}$

$\frac{d}{dx}\left(\ln|x^3+3|\right)$

$= \frac{3x^2}{x^3+3}$

(iii) $\frac{d}{dx}(a^b) = \underline{0}$ where a and b are constants.

(iv) $\frac{d}{dx}\left(f(x)\right)^a = a\,\underline{[f(x)]^{a-1}f'(x)}$ ✓

(v) $\frac{d}{dx}\left(b^{f(x)}\right) = \underline{b^{f(x)} \cdot \ln(b) \cdot f'(x)}$

Example 4

Given $y = x^x$ determine $\frac{dy}{dx}$

Solution: $\ln y = \ln x^x$

$\ln y = x \ln x$

$$\frac{dy}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x}$$

Example 4…contd…

$$\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x}$$

$$\frac{dy}{dx} = y(\ln|x+1|) = \underline{x^x \ln|x+1|}$$

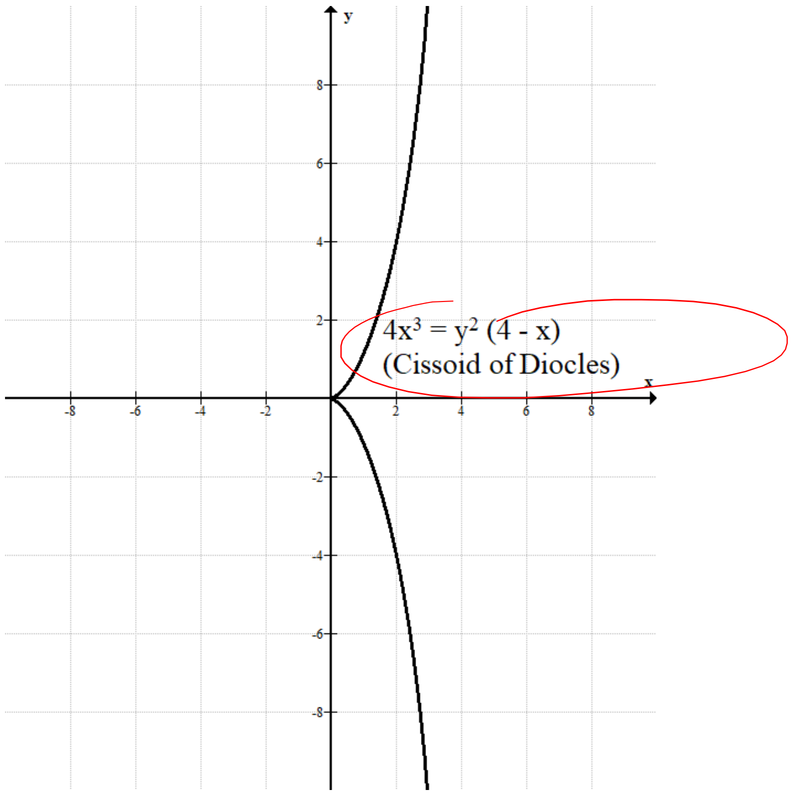
$\boxed{x \neq 0}$

$$\therefore \underline{\frac{dy}{dx} = x^x \ln|x+1|}$$

Remarks

- ❑Graphs of implicit functions are quite interesting.
- ❑Sometimes you need to express them as parametric equations for easier plotting and differentiation.
- ❑Examples of graphs of implicit functions e.g., polar graphs.

Examples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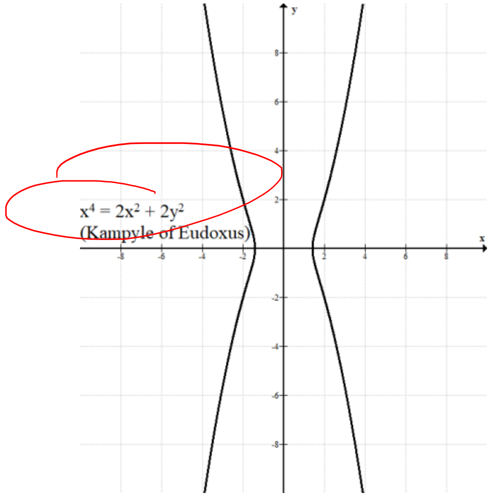
4x³ = y² (4 - x)
(Cissoid of Diocles)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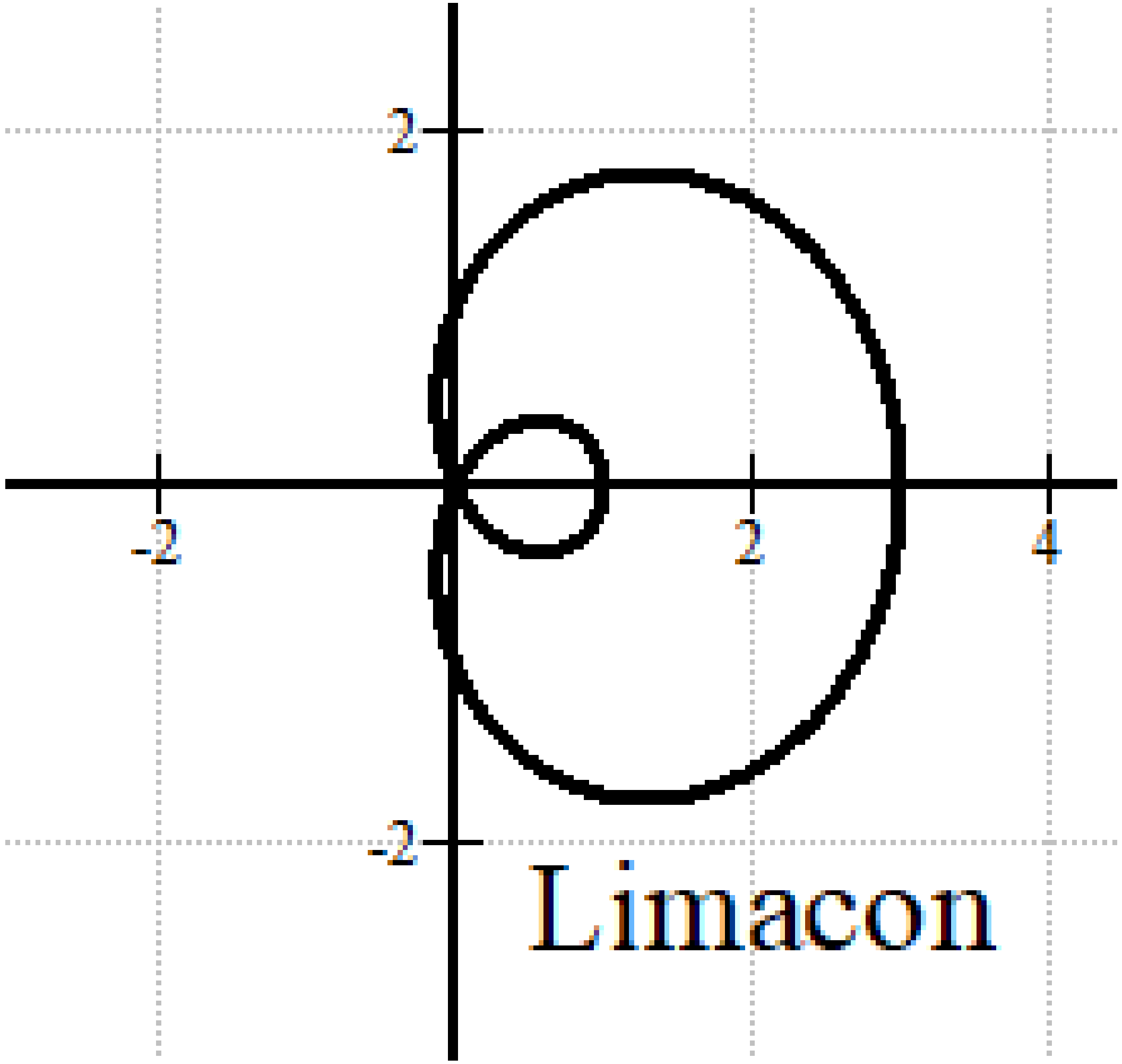
x⁴ = 2x² + 2y²
(Kampyle of Eudoxus)

Examples…contd…



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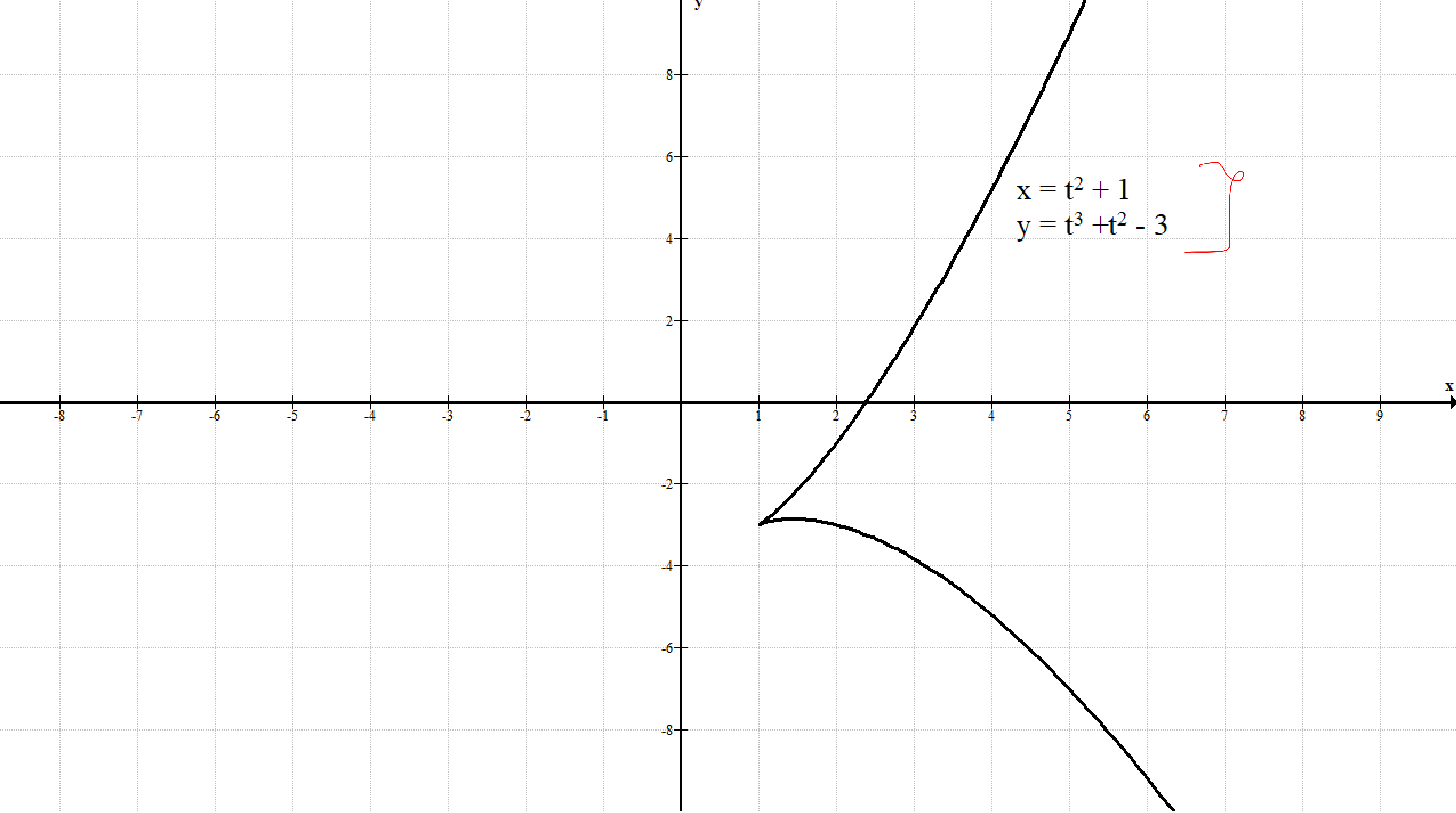
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$x = t^2 + 1$
$y = t^3 + t^2 - 3$

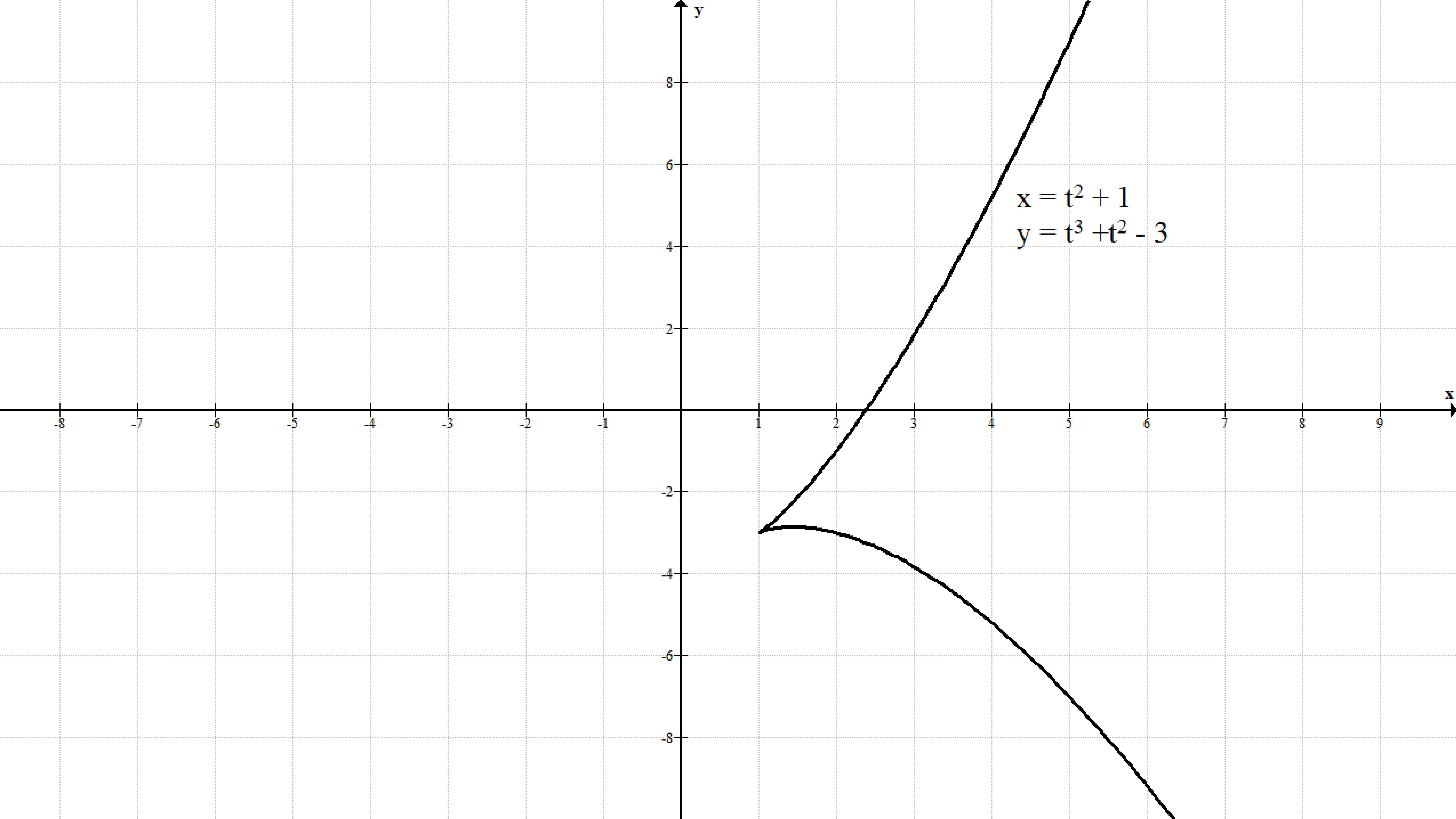
Differentiating Parametric functions

Explicit functions show relationship between two quantities.

However there exist situations e.g. implicit functions where this relationship is complicated and hence requires the introduction of another variable referred to as a parameter.

The two quantities are then independently expressed in terms of the new parameter.

See the graph of the parametric function $f(x) = \begin{cases} x = t^2 + 1 \\ y = t^3 + t^2 - 3 \end{cases}$


y
x
$x = t^2 + 1$
$y = t^3 + t^2 - 3$
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9

Example 1

The equation of a circle centre the origin, radius r and passing through a point (x, y) can be expressed as an implicit function $\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{r}^2$.

This is the rectangular equation of a circle.

However, it can be expressed as a polar equation i.e.

$\mathrm{x} = \mathrm{r}\cos\theta, \mathrm{y} = \mathrm{r}\sin\theta\,, 0 \leq \theta \leq 2\pi$

This is in parametric form.

A new variable is θ.

Example 2

Find $\frac{dy}{dx}$ given the parametric equations $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Solution: We need to differentiate each of the equation with respect to the parameter. In this case parameter θ.

$$\frac{dx}{d\theta} = -r\sin\theta; \frac{dy}{d\theta} = r\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = r\cos\theta \cdot \frac{1}{-r\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$$

In general, if given the parametric equation $\begin{cases} x = f(t) \\ y = f(t) \end{cases}$

then $\frac{df}{dx} = \frac{y'(t)}{x'(t)}$ provided $x' \neq 0$.

$$f = \begin{cases} x = 4t^3 \\ y = 1 - t^2 \end{cases}$$

then $x'(t) = 12t^2$

$y'(t) = -2t$

$$\therefore \frac{df}{dx} = \frac{-2t}{12t^2}$$

$$= \frac{-1}{6t}$$

$\square$

Example 3

Consider the parametric equation; $\begin{cases} x = t^2 + 1 \\ y = t^3 - t \end{cases}$

Determine $\frac{dy}{dx}$ at $x = 1, x = 2, x = 5$

Solution: $x'(t) = 2t; y'(t) = 3t^2 - 1 \therefore \frac{dy}{dx} = \frac{3t^2-1}{2t}$

(i) At $x = 1$ we have $1 = t^2 + 1 \therefore t = 0 \Rightarrow \frac{dy}{dx} = \infty.$

It is clear from the graph that the tangent will be a vertical line at x = 1.

At $x = 0$ we have $0 = t^2 + 1$

$$\therefore t^2 = -1, t \notin \mathbb{R}.$$

The curve does not exist at x=0

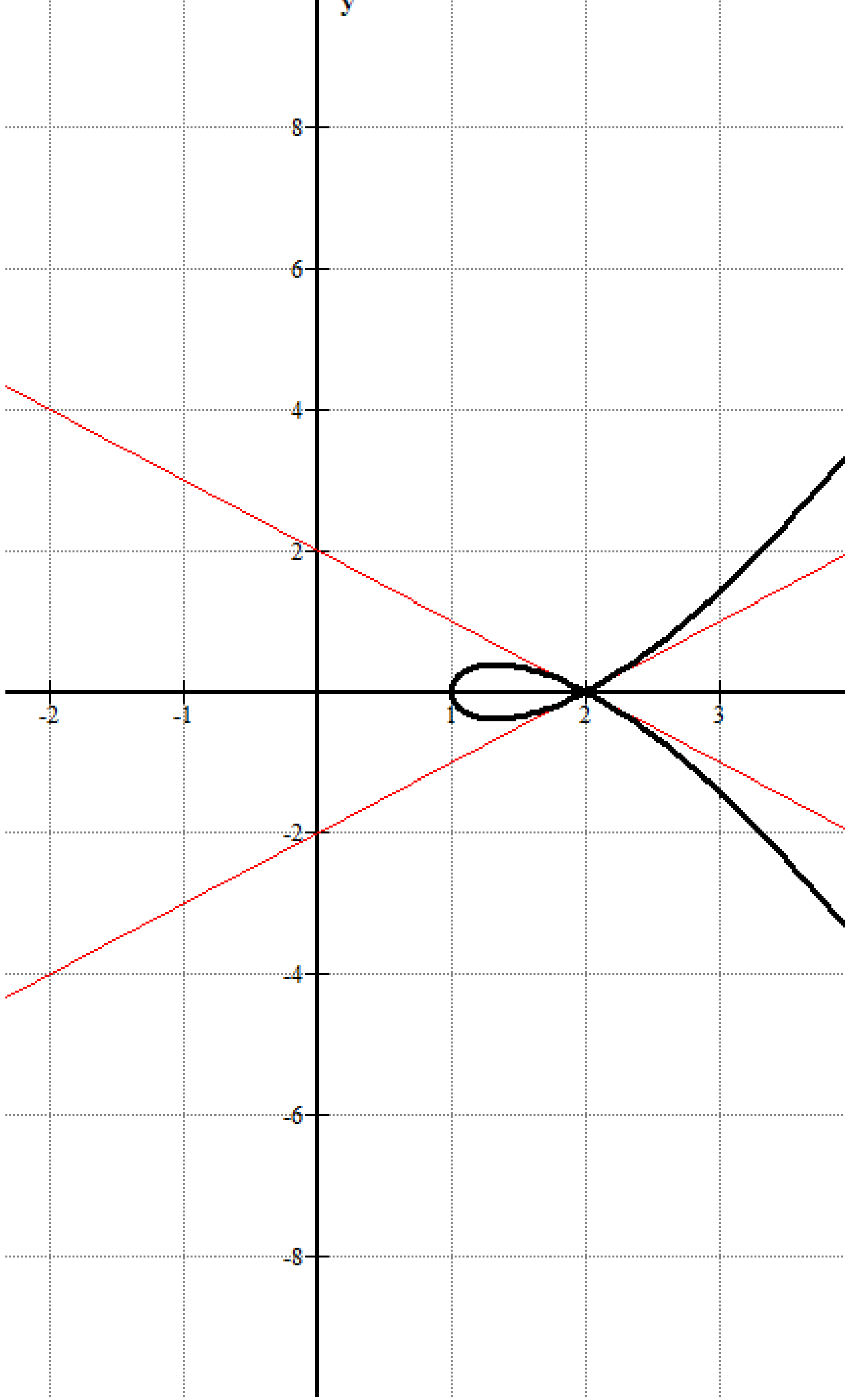
Example 3…contd…

(i) At x $= 2$ we have $2 = t^2 + 1 \therefore t = \pm 1$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{2} = 1 \text{ and}$$

$$\left.\frac{\mathrm{dy}}{\mathrm{dx}}\right|_{=t-1} = \frac{2}{-2} = -1$$

See the tangent lines l_1 and l_2 on the right.



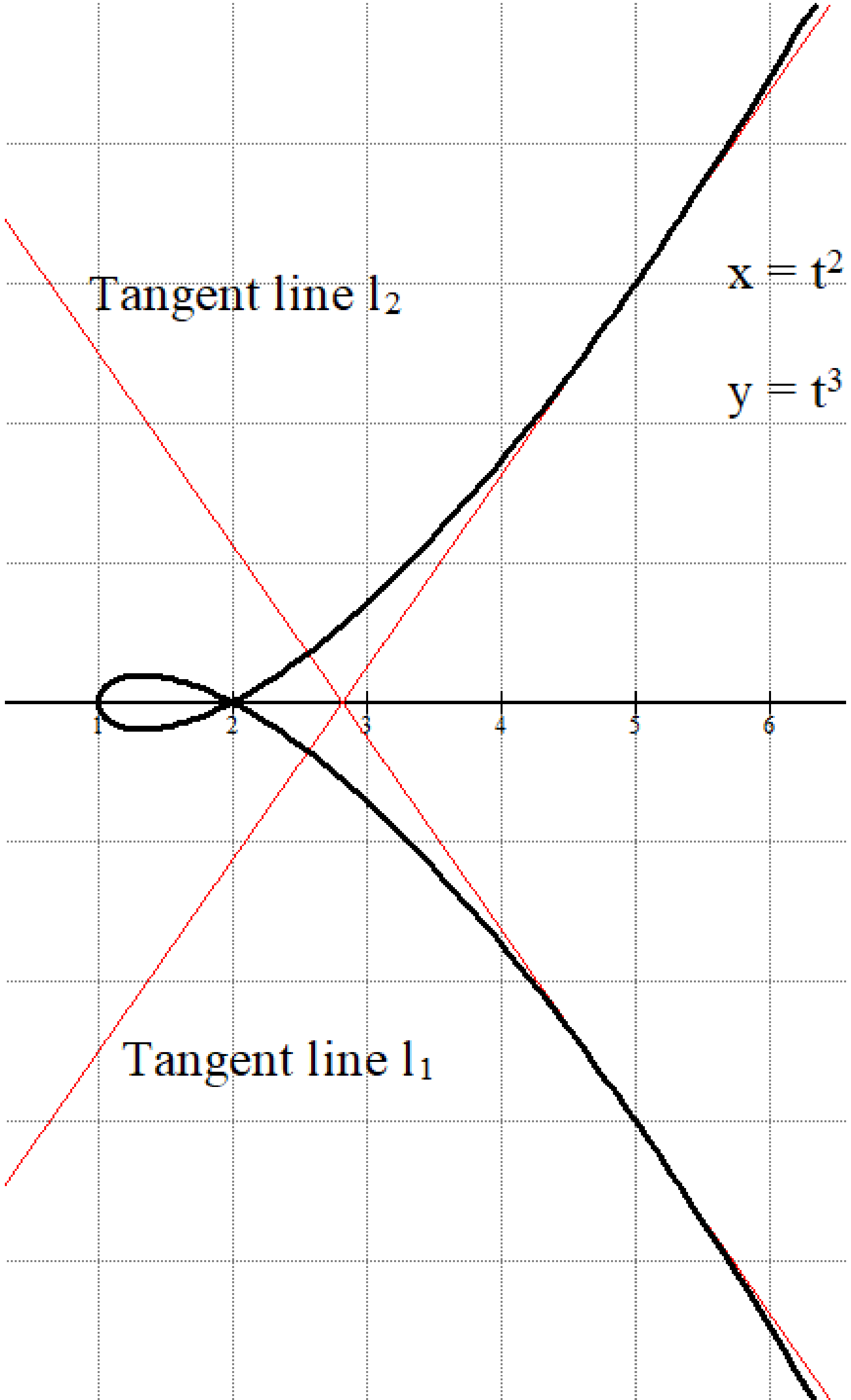
Example 3…contd…

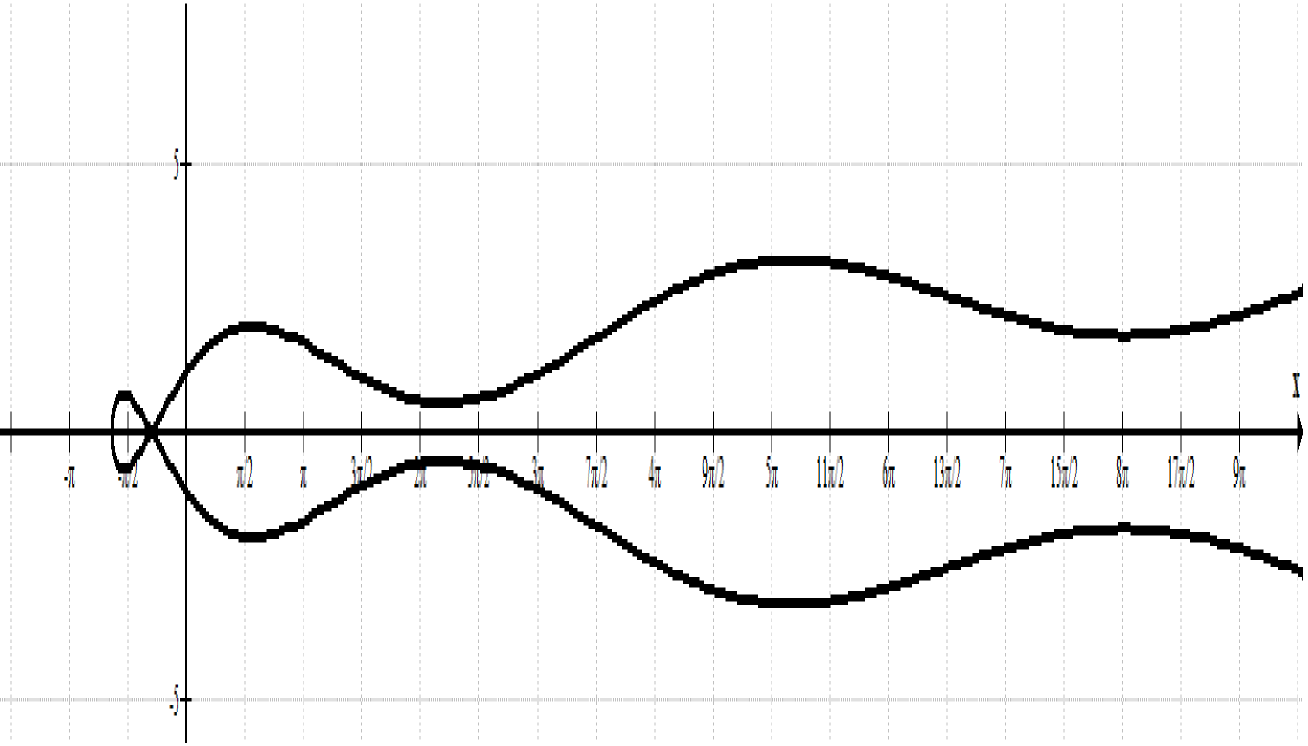
(i) At x = 5 we have $5 = t^2 + 1 \therefore t = \pm 2$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{3(2)^2 - 1}{2(2)} = \frac{13}{4}$$

$$\text{and } \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{13}{4}$$

See the tangent lines in the figure on the right.





Example 4

Find $\frac{dy}{dx}$ given

$$f(x) = \begin{cases} x = 3t^2 + \cos t \\ y = t - \sin t \end{cases}$$

(see figure)

Example 4 ...contd...

Solution: $x'(t) = 6t - \sin t;$

$\boxed{y'(t) = 1 - \cos t}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Therefore

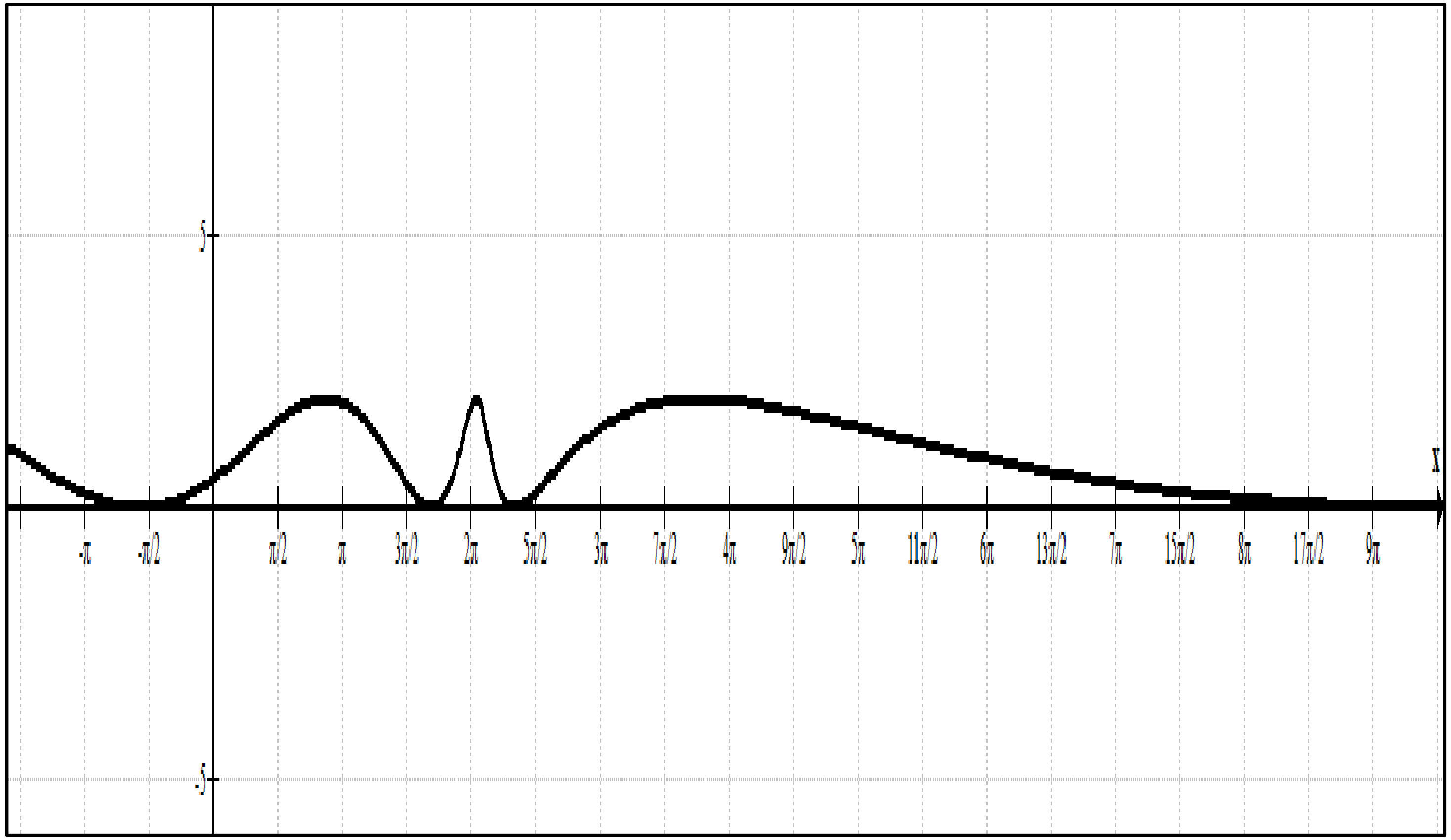
$$\boxed{\frac{dy}{dx} = \frac{1-\cos t}{6t-\sin t}}$$

Example 5

Find $\frac{dy}{dx}$ given

$$f(x) = \begin{cases} x = 5 - t^2 + e^t \\ y = 1 + \sin 3t \end{cases}$$

(see figure below).



Example 5...contd...

Solution: $y'(t) = \underline{3\cos 3t}$; $y = 1 + \sin 3t$

$x'(t) = \underline{-2t + e^t}$ $x = 5 - t^2 + e^t$

$$\therefore \frac{dy}{dx} = \frac{3\cos 3t}{-2t + e^t}$$

$=$

Application of parametric functions

Parametric functions are applicable in expressing the relationship between two or more variables. A key area where parametric functions are applied include but not limited to;

a) Circular motion

b) Projectile motion

Example 1 (circular motion)

A boy is on a Ferris wheel. His position at time t is expressed by the parametric equations;

$$\begin{cases} x = 12\cos\frac{\pi}{4}t \\ y = 12\sin\frac{\pi}{4}t \end{cases}$$

Example 2

A ball is thrown from a point (a, b) at an angle θ to the right of an initial velocity of v_0 m/s.

The position of the ball over time can be modeled using the parametric equations. It horizontal and vertical component can be modeled as;

$$\begin{cases} x = vt\cos\theta + a \\ y = -\frac{1}{2}gt^2 + vt\sin\theta + b \end{cases}$$

References

Cowen, R. ., Were, J. ., & Vaz, P. . (1990). *An Introduction to Calculus.* Nairobi University Press.

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

End of Lecture 9

Thank you