Calculus I

Lecture 12

Differentiation and Kinematics

Lecturer: Kahenya N.P

Introduction to lecture 12

- ❑Lecture 12 will introduce yet another important application of differentiation, that is, kinematics.
- ❑Kinematics deals with the motion of objects without reference to the forces that cause this motion.
- ❑It is a continuation of the lecture on rate of change and optimization.
- ❑Most of the applications in kinematics are cases of determining the critical values.
- ❑It will also build on past definitions and theorems especially Fermat's theorems, first principle of differentiation, and limits, among others.

Intended learning outcomes

Be	• At the end of this lecture, you will be able to;
Show	• Show the relationship between different quantities using derivatives.
Solve	• Solve kinematics problems involving differentiation.

References for further reading

The lecture notes have been adopted from relevant topics from (Briggs et al., 2015; Rogawski et al., 2019; Stewart, 2012; Sullivan & Miranda, 2019).

Definition 1: Displacement (s) vs Time (t)

(i) Velocity v of an object is change in displacement s meters, with respect to time t seconds i.e. $\mathrm{v} = \frac{\mathrm{ds}}{\mathrm{dt}}\ \mathrm{ms}^{-1}$

(ii) If velocity is zero then the object is at rest i.e., $\mathrm{v} = 0 \Rightarrow \frac{\mathrm{ds}}{\mathrm{dt}} = 0$

(iii) If the value of velocity v is negative, then the object is moving in the opposite direction to that in which displacement s is measured i.e. $\frac{\mathrm{ds}}{\mathrm{dt}} < 0$

Note that speed is the absolute value of velocity i.e. $|\mathrm{v}(\mathrm{t})|$.

Definition 2: Velocity (v) vs time (t)

i) Acceleration a refers to the change in velocity$v$ with respect to time t i.e.,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \text{ ms}^{-2}$$

i) If the acceleration of a particle is zero i.e. $a = \frac{d^2s}{dt^2} = 0$ it implies that the velocity of the particle is constant.

ii) If the acceleration of the particle is a positive value i.e. $a = \frac{d^2s}{dt^2} > 0$ then the particle is accelerating, and if the acceleration is a negative value the particle is slowing down or retarding i.e. $a = \frac{d^2s}{dt^2} < 0$.

Definition 3: Motion in a straight line

If a particle is constrained to move on a straight line, we can choose the origin to taken at the point where the particle is situated at $t = 0$, we can get expressions for its position s and velocity v with respect to time t and its acceleration a.

That is,

(i) $v = u + at$

(ii) $s = ut + \frac{1}{2}at^2$

(iii) $v^2 = u^2 + 2as$ where u is velocity when $t = 0$

Example 1

A particle is thrown upwards, and its height after t seconds is s meters, where $s = 12t - 5t^2$.

Find the time when the greatest height is reached, the time when it returns to the original level.

Its velocity after 2 seconds and the acceleration when t = 1.2 s.

Solution:

i) The greatest height reached

If $s = 12t - 5t^2$ then velocity $v = \frac{ds}{dt} = 12 - 10t$.

Note that for the greatest height (A case of maximization) the velocity is zero.

That is

$$\frac{ds}{dt} = 0 = 12 - 10t \therefore t = 1.2 \text{ sec}$$

Example 1...contd...

Therefore, the particle attains the greatest height at 1.2 sec.

This greatest height is;

$s = 12(1.2) - 5(1.2)^2 = 7.2$ m

i) When it returns to the original level, $s = 0 \Rightarrow 12t - 5t^2 = 0$

$t(12 - 5t) = 0$

$\therefore t = 0 \text{ or } t = 2.4 \text{ sec}$

Therefore, it returns to the original position after 2.4 sec.

Example 1...contd...

$v = \frac{ds}{dt} = 0 = 12 - 10t \therefore t = 1.2 \text{ sec}$

when $t = 2s;\ v = 12 - 20 = -8 \text{ m/s}$ i.e. particle is moving in the opposite direction

We have seen that $v = \frac{ds}{dt} = 12 - 10t \therefore a = \frac{d^2s}{dt^2} = -10 \text{ ms}^{-2}$ i.e. the particle is decelerating

Example 2

A missile is shot vertically upwards. It travels a distance m given by $s = 1050t - 15t^2$ in time t seconds. Find the;

(i) velocity with which the missile was shot,

(ii) time taken when it came to rest, and

(iii) distance covered.

Solution:

(i) By definition velocity $v = \frac{ds}{dt} = 1050 - 30t$. Initial time $t = 0 \therefore v = 1050\ m/s$.

Example 2…contd…

A missile is shot vertically upwards. It travels a distance m given by $\mathrm{s} = 1050\mathrm{t} - 15\mathrm{t}^2$ in time t seconds. Find the; time taken when it came to rest, and distance covered.

Solution: The velocity when the missile comes to rest is $\frac{\mathrm{ds}}{\mathrm{dt}} = 0 = 1050 - 30\mathrm{t}$

Therefore, the time it took to come to rest is $\mathrm{t} = 35\mathrm{sec}$

Hence distance $s = 1050(35) - 15(35)^2 = 18375\ m$

Definition 3: Gravity influence on motion

The following equations relate height $\mathrm{s(t)}$ and velocity $\mathrm{v(t)}$ with time of objects projected in the air above the earth's surface. The equations are attributed to Galileo's work.

(i) $\mathrm{s(t)} = \mathrm{s_0} + \mathrm{v_0 t} - \frac{1}{2}\mathrm{gt}^2$. But since $s(0) = 0$ (i.e. when $t = 0$) then $s_0 = 0$ and therefore we have $\mathrm{s(t)} = \mathrm{v_0 t} - \frac{1}{2}\mathrm{gt}^2$

(ii) $\mathrm{v(t)} = \frac{\mathrm{ds}}{\mathrm{dt}} = \mathrm{v_0} - \mathrm{gt}$ where constants s_0 and v_0 are initial values, $\mathrm{g} \approx 9.8\ \mathrm{ms}^{-2}$ is acceleration due to gravity.

Remarks 1

a) We are assuming air resistance.

b) From equations $v(t) = \frac{ds}{dt} = v_0 - gt$ we have $gt = v_0 - v(t) \therefore t = \frac{v_0 - v(t)}{g}$.

Therefore equation $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$ can be written as;

$$s(t) = v_0 \frac{v_0 - v(t)}{g} - \frac{1}{2}g\left(\frac{v_0 - v(t)}{g}\right)^2 = \frac{2v_0^2 - 2v_0 v(t) - v_0^2 - 2v_0 v(t) + v^2(t)}{2g} = \frac{v_0^2 + v^2(t)}{2g}$$

$$\Rightarrow 2gs(t) = v_0^2 + v^2(t)$$

$$\Rightarrow v^2(t) = v_0^2 - 2gs(t)$$

$$|v(t)| = \sqrt{v_0^2 - 2gs(t)}$$

Remark 2

When the particle attains maximum velocity $v = 0 \Rightarrow 0 = v_o^2 - 2gs(t)$

$$\therefore\ 2gs(t) = v_0^2$$

$$\Rightarrow s(t) = \frac{v_0^2}{2g} - \text{maximum height}$$

Example 1

A metal ball is launched upward from ground level with an initial velocity of $67\ \mathrm{m/s}$. Determine;

(i) Velocity of the ball when time $t = 1$ and $t = 10$.

(ii) Time the ball attains the maximum height.

Solution: Note that the initial distance $S_0 = 0$ and the initial velocity $\mathrm{v}_0 = 67$ with $\mathrm{g} = 9.8\mathrm{ms}^{-2}$.

(i) We apply Galileo's equations:

$$\mathrm{s(t)} = 67\mathrm{t} - 4.9\mathrm{t}^2; \mathrm{v(t)} = 67 - 9.8\mathrm{t}$$

When $\mathrm{t} = 1$; $\mathrm{v}(1) = 67 - 9.8(1) = 57.2\ \mathrm{m/s}$

When $\mathrm{t} = 10$, $\mathrm{v}(10) = 67 - 9.8(10) = -31\ \mathrm{m/s}$ – the ball is coming back to the ground.

Example 1...contd...

Example 1: A metal ball is launched upward from ground level with an initial velocity of $67\ \text{m/s}$. Determine;

(i) Velocity of the ball when time $t = 1$ and $t = 10$.

(ii) Time the ball attains the maximum height.

Solution: The greatest height is attained when velocity is zero i.e. $\frac{ds}{dt} = 0 = 67 - 9.8t$

$\Rightarrow 67 = 9.8t \therefore t \approx 6.84$ sec

The greatest height attained at this time is;

$$s\left(\frac{67}{9.8}\right) = 67\left(\frac{67}{9.8}\right) - 4.9\left(\frac{67}{9.8}\right)^2 \approx 229.03\ \text{m}$$

Example 2

A projectile is launched at an angle θ to the horizontal with an initial velocity v_0.

The position of the projectile at time t seconds is given by;

$$p(x) = \begin{cases} x = v_0 t \cos\theta + gt \\ y = v_0 t \sin\theta - 2gt^2 \end{cases} \quad t \geq 0$$

where $g = 9.8\ \mathrm{ms}^{-2}, 0 < \theta < \frac{\pi}{2}$

(i) Express the slope of the tangent line to the motion of the projectile as a function of t..

(ii) Determine the time t when the projectile attains its maximum height.

Example 2...contd...

Solution: The slope of the tangent is by definition $p'(x) = \frac{dy}{dx}$

$\Rightarrow x'(t) = v_0 \cos\theta + g$

$\Rightarrow y'(t) = v_0 \sin\theta - 4gt$

Therefore;

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{v_0 \sin\theta - 4gt}{v_0 \cos\theta + g}$$

Example 2...contd...

The projectile will attain its maximum height when $\frac{dy}{dx} = 0.$

Hence, $\frac{v_0 \sin\theta - 4gt}{v_0 \cos\theta + g} = 0$

$\Rightarrow v_0 \sin\theta - 4gt = 0 \therefore t = \frac{v_0 \sin\theta}{4g}$ sec

The maximum height attained is;

$s(t) = \frac{v_0^2}{2g} = \frac{v_0^2}{19.6}$ m

Definition 4

There are instances where you need to consider the motion of a particle in a plane, say the xy-plane, along a path or trajectory or orbit. The position of the particle at time t seconds is a position vector $\boldsymbol{r}$.

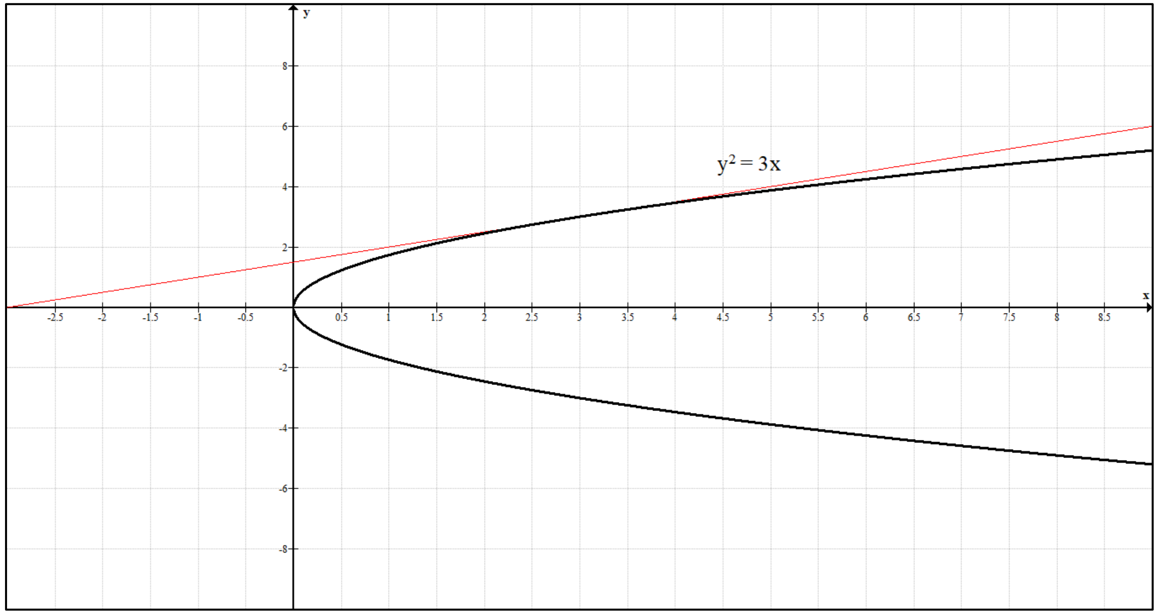
The position vector r is a twice-differentiable vector-valued function. It can be written as;

$$\mathbf{r} = \mathrm{F(t)} - \text{Equation of motion}$$

The velocity of the particle at any instant is the vector

$$\mathbf{v} = \frac{\mathrm{dr}}{\mathrm{dt}} = \mathrm{v}\mathbf{T}$$

Where v is the instantaneous speed of the particle and $\mathbf{T}$ is the unit tangent vector to the path of the particle.

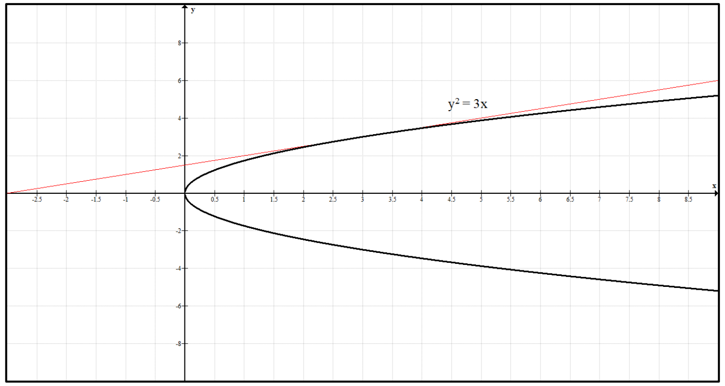


Example 1

A particle moves along the curve of $y^2 = 3x$ from the vertex at a speed of 7 units per second.

Determine the velocity vector **v** as the particle moves through point (3,3).

Example 1...contd...



Solution: At point R(x, y), the position vector $\mathrm{r} = \overline{OR} = x\mathbf{i} + y\mathbf{j}$

$$\Rightarrow \mathbf{r} = \frac{y^2}{3}\mathbf{i} + y\mathrm{j}$$

$$\frac{d\mathbf{r}}{dy} = \frac{2}{3}y\mathbf{i} + \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dy}\right| = \sqrt{\frac{4}{9}y^2 + 1}$$

$$\mathbf{T} = \frac{\frac{d\mathbf{r}}{dy}}{\left|\frac{d\mathbf{r}}{dy}\right|} = \frac{\frac{2}{3}y\mathbf{i}+\mathbf{j}}{\sqrt{\frac{4}{9}y^2+1}} - \text{unit tangent vector}$$

At point (3,3) then vector **T** becomes;

$$\mathbf{T} = \frac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}}$$

Therefore, the velocity of the particle at this point is;

$$\mathbf{v} = v\mathbf{T} = 7 \cdot \frac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}}$$

Remarks

i) This example is an introductory to vectors and differentiation.

ii) It has assumed some basic knowledge of vector projection.

iii) The problem did not calculate velocity as a function of time. To do this one need to use Newton's second law of motion $f = ma$ where f is force, m is mass, and a is acceleration. There is also an assumption that the mass of the particle is constant.

Calculus 1 ILOs

Explain the concepts of limits and continuity of functions.

Explain and apply fundamental theorems in differential calculus.

Explain and apply the rules of differentiation

Demonstrate the application of differential calculus in problem solving

Summary of Calculus 1

Limits and continuity of functions

First principle of differentiation

Techniques of differentiation; power, product, quotient, and chain rules

Basic theorems in differential calculus: Fermat's, Rolle's, and Mean value theorems

Summary of Calculus 1...contd...

- Implicit and partial differentiation
- Logarithmic and parametric differentiation
- Rates of change and related rates
- Optimization
- Differentiation and kinematics

Calculus II

Deals with integration calculus:

Definition of terms

Basic theorems in Integral calculus

Techniques of integrating single real-variable functions

Application of integration

Key References

Briggs, W., Cochran, L., & Bernard, G. (2015). *Calculus* (Global Edi). Pearson Education Limited.

Rogawski, J., Adams, C., & Franzosa, R. (2019). *Calculus: Early Transcendentals* (4th ed.). W.H. Freeman and Company.

Stewart, J. (2012). *Calculus* (7th ed.). Brooks/Cole Cengage Learning.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

End of Calculus 1

Thank You!