Course Title

Engineering Economic Analysis

Chapter 2

Interest and Time Value of Money

Lecture 3 (Week 3)

Uniform/Equal Series Cash flow, Linear Gradient Series & Geometric Gradient Series Cash Flow

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Learning Objective

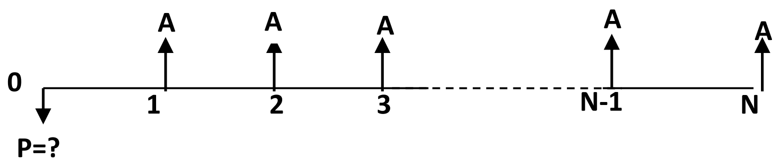
From studying this chapter the students will be able to understand on the topics:

- The concept of uniform/equal cash flow series formula.
- The concept of linear gradient series cash flow formula.
- The concept of geometric gradient series cash flow formula.

3.1 Uniform/Equal series cash flow formula

When a cash flow diagram involves the series of uniform (equal) magnitude of cash flow occurring at equal interval of time over the length of the project is called annuity. While making the economic analysis of project, normally the single cash flow is not being faced, instead of it cash flow on an annuity basis is faced like insurance payment, home loan payment etc. [1]

It should be noted that the formulas are derived and presented such that annuity 'A' occurs at the end of each period. [1]. P (present equivalent) occurs one interest period before the First A (uniform amount). F (Future equivalent) occurs at the same time as the last A, and N periods after P as shown in figure 2.6.



(Fig 2.6: Uniform series cash flow)

(a) To find F when A is given

Here we find the future equivalent value (F) if the uniform amount of 'A' is occurred uniformly from period 1 to N at interest rate of i%. (Figure: 2.7)

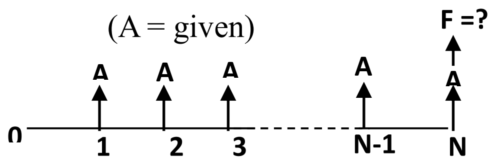


Fig: 2.7 Annuity to Future

Summing the future equivalents of each of the cash flows.

$F = A\,(1+i)^{N-1} + A\,(1+i)^{N-2} + \ldots\ldots.. + A\,(1+i)^{N-(N+1)} + A\,(1+i)^{N-N}$

$F = A\,[(1+i)^{N-1} + (1+i)^{N-2} + \ldots\ldots.. + (1+i)^{N-(N+1)} + (1+i)^{N-N]} \ldots.\ (1)$

Multiplying Both Sides by (1+i) gives

$F\,(1+i) = A\,[(1+i)^{N} + (1+i)^{N-1} + \ldots\ldots.. + (1+i)^{N-(N+2)} + (1+i)^{1]} \ldots\ (2)$

Now,

Subtracting equation 1 from equation 2, we get

$F\,(1+i-1) = A\,[(1+i)^{N} - 1]$

$Fi = A\,[(1+i)^{N} - 1]$

$$\mathbf{F = A\,\{((1+i)^{N} - 1)/i\}}$$

The quantity in a bracket is called the *uniform series compound amount factor.*

Functionally, **F = A (F/A, i%, N)**

Note: F (future worth) coincides with last annuity 'A'

A (annuity) occurs at the end of each period

P (present worth) occurs at on the interest period before the first 'A'

(b) To find P when A is given

Here we find the present equivalent value (P) if the uniform amount of 'A' is occurred uniformly from period 1 to N at interest rate of i%. (Figure: 2.8)

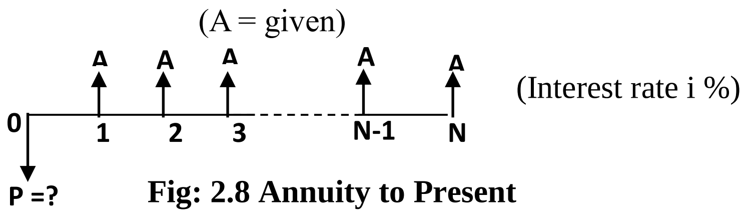


Fig: 2.8 Annuity to Present

We know, the single payment compound factor is

$$F = P (1+ i)^{N} \ldots\ldots\ldots\ldots \quad (3)$$

Again, uniform series Compound amount factor is

$$F = A \{((1+i)^{N} - 1)/i\} \ldots\ldots\ldots\ldots (4)$$

Equating the equation (3) and (4), we get:

$$\mathbf{P = A [(1+i)^{N} - 1] / [i * (1+i)^{N}]}$$

The quantity in a bracket is called the *uniform series present worth factor*

Functionally, P = A (P/A, i%, N)

(c) To find A when F is given

Here we find the Annuity equivalent value (A) if the Future amount (F) is occurred at the end of the period N at interest rate of i%. (Figure: 2.9)

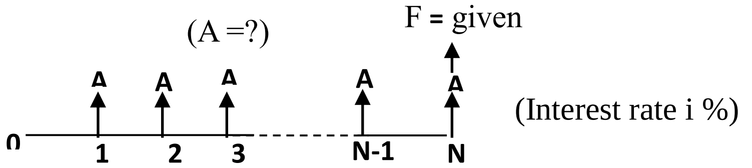


Fig 2.9: Future to Annuity

We know from the Uniform series compound amount factor

$$F = A \{[(1+i)^{N} - 1]/i\} \ldots\ldots\ldots\ldots (5)$$

If we reverse the equation 5, we get

$$A = F [i / ((1+i)^N - 1)]$$ the factor in a bracket is called the *sinking fund factor*

Functionally, A = F (A/F, i%, N)

Sinking Fund

A fund established to accumulate a given future amount through the collection of a uniform series of payments is called sinking fund. [2]. It is commonly established for the purpose of replacing the fixed assets. [3]. Example of sinking fund is the premium for life insurance. After the maturity of the insurance period, the wholesome money can be withdrawn including interest (bonus). But until the maturity period, the fixed sum of money is deposited into the account which is sinking fund.

(d) To find A when P is given

Here we find the Annuity equivalent value (A) if the Present amount (P) is occurred at the end of the period 0 at interest rate of i%. (Figure: 2.10)

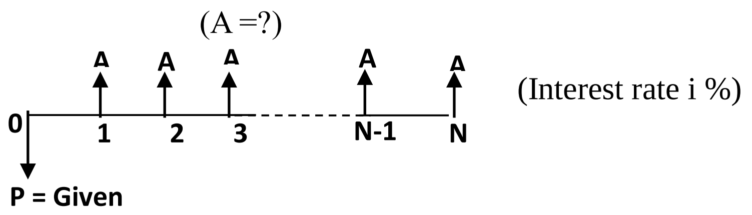


Fig 2.10: Present to Annuity

We know the uniform series present amount factor is

$P = A [(1+i)^N - 1] / [i * (1+i)^N]$ ……………. (6)

If we reverse the equation 6, we get

$$A = P [i*(1+i)^N] / [(1+i)^N - 1]$$

The quantity in a bracket is called the *capital recovery factor*

Functionally, **A= P (A/P, i%, N)**

Capital recovery is the annual equivalent of capital cost

3.2 Gradient Cash Flow Series Formula.

Sometimes the cash flow that occurs in consecutive interest periods are not same amount (not on A value). If the cash flow changes in a predictable way then these cash flows are known as gradients. [4].

Two types of gradient series cash flow

(a) Linear or Arithmetic gradient series.

(b) Geometric gradient series.

(a) Linear or Arithmetic gradient series.

Formulas previously developed for uniform cash flow series have earned amounts of equal value. In case of a gradient series, each year end cash flow is different, so new formulas must be derived. Engineers frequently encounter situations involving periodic payments that increases or decreases by a constant amount from period to period. [3] If the cash flow from one period to another period increases or decreases by constant amount then it is called linear gradient series cash flow.

The constant amount is generally represented by G. The gradient G can be either positive or negative. The equivalent value of present, future and annual is calculated using the linear gradient series formulas.

- If $G > 0$, the series is referred to as an *increasing* gradient series.
- If $G < 0$, the series is referred to as a decreasing gradient series.

The cash flow at the end of year 1 is not part of the gradient series, but is rather a base amount. (First cash flow is always zero) [As shown in figure: 2.11 (a) and (b)]

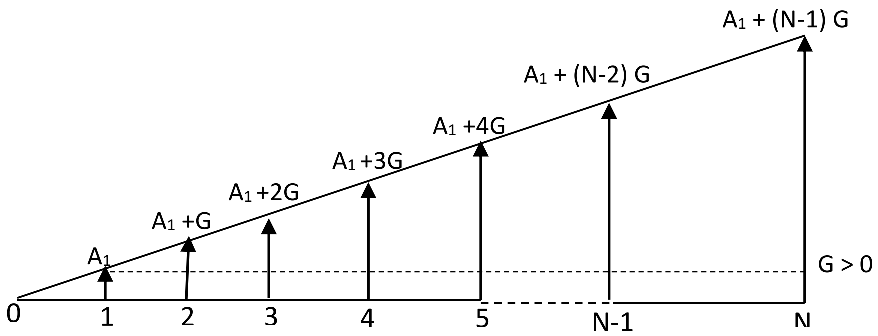


Fig 2.11 (a): Increasing Linear Gradient Series

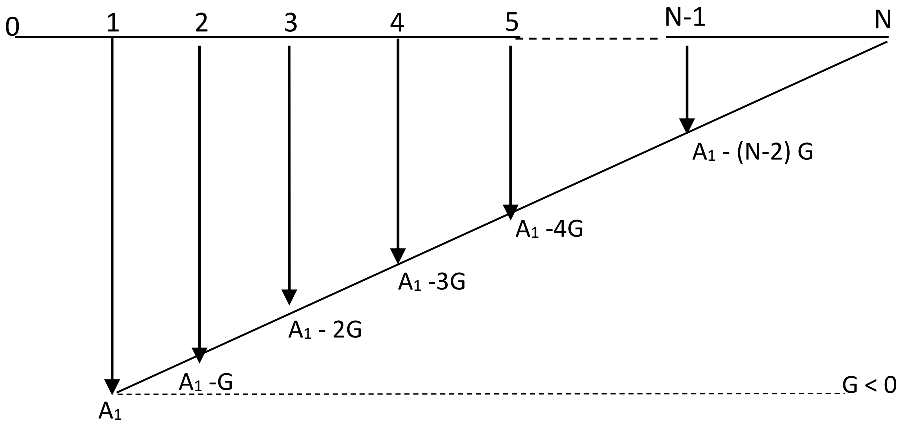
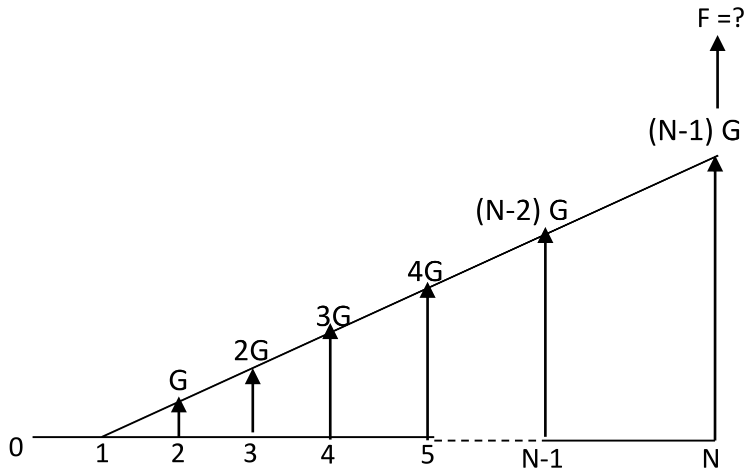


Fig 2.11 (b): Decreasing Linear Gradient Series [3]

Future worth factor

To find F when G is given

The future worth of the arithmetic sequence of cash flows is shown in the figure 2.12. Here the cash flow occurrence is in the increasing gradient where the cash flow from one period to another period is increased by the constant amount G. The first cash flow is always zero, that is, in case of application of arithmetic gradient series formulas the cash flow begins from year one. The future value of such cash flow can be derived as:



First cash flow occurs at the end of period 1

Figure 2.12: Future worth of arithmetic gradient series. [3]

F = G (F/A, i%, N-1) + G (F/A, i%, N-2) + ………..+ G (F/A, i%, 2) + G (F/A, i%, 1)

$F = G \{[(1+i)^{N-1} - 1]/i\} + [(1+i)^{N-2} - 1]/i\} + \ldots\ldots\ldots + \{[(1+i)^{2} - 1]/i\} + \{[(1+i)^{1} - 1]/i\}$

$F = G/i \{(1+i)^{N-1} + (1+i)^{N-2} + \ldots\ldots + (1+i)^{2} + (1+i)^{1}\}$

$$\mathbf{F} = \frac{G}{i}\left[\frac{(1+i)^N - 1}{i}\right] - \frac{NG}{i}$$

The factor in the bracket is called the *Gradient to Future equivalent factor*
Functionally, **(F/G, i%, N)**

Annual worth Factor

To find A, When given G, i, N

From gradient to future equivalent factor

$$F = \frac{G}{i}\left\{\frac{(1+i)^N - 1}{i}\right\} - \frac{NG}{i} \quad \ldots\ldots\ldots (7)$$

Again from Uniform series compound amount factor

$$F = A\,\frac{(1+i)^N - 1}{i} \quad \ldots\ldots\ldots\ldots\ldots\ldots.. (8)$$

Equating equation 7 and 8, we get

$$\mathbf{A} = G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]$$

The factor in the bracket is called the *Gradient to Uniform Series Factor,*

Functionally, A =G (A/G,i%,N)

Present worth Factor

<u>To Find P, When given G, i, N</u>

We know gradient to future equivalent factor is

$$F = \frac{G}{i}\left[\frac{(1+i)^N - 1}{i}\right] - \frac{NG}{i} \text{ .... (9)}$$

Again we know single payment compound amount factor is

$$F = P(1+i)^N \text{ .... (10)}$$

Equating equation 9 and 10, we get

$$\mathbf{P} = \frac{G}{i^2}\left[\frac{(1+i)^N - Ni - 1}{(1+i)^N}\right]$$

The factor in the bracket is called the *Gradient to present equivalent factor*

Functionally, G (P/G, i%, N)

(b) <u>Geometric gradient series.</u>

Many Economic problems involve cash flows that increase or decrease over time, not by a constant amount (linear gradient) but rather by a constant percentage (geometric), which is called compound growth. A geometric series is a non-uniform progressions that grows or declines at a constant percentage rate per period. [2] Many engineering problems particularly related to construction costs involve the geometric gradient series. Price change due to inflation and deflation is the good example of such geometric gradient series. [3] We use g to designate the percentage change in payment from one period to the next. The cash flow for increasing and decreasing geometric gradient series cash flow is presented in the figure 2.13 (a) and (b) respectively. The magnitude of n^{th} payment, A_n, is related to the first payment A_1, is expressed by:

$A_n = A_1 (1+g)^{n-1}$, where n=1, 2, 3…N

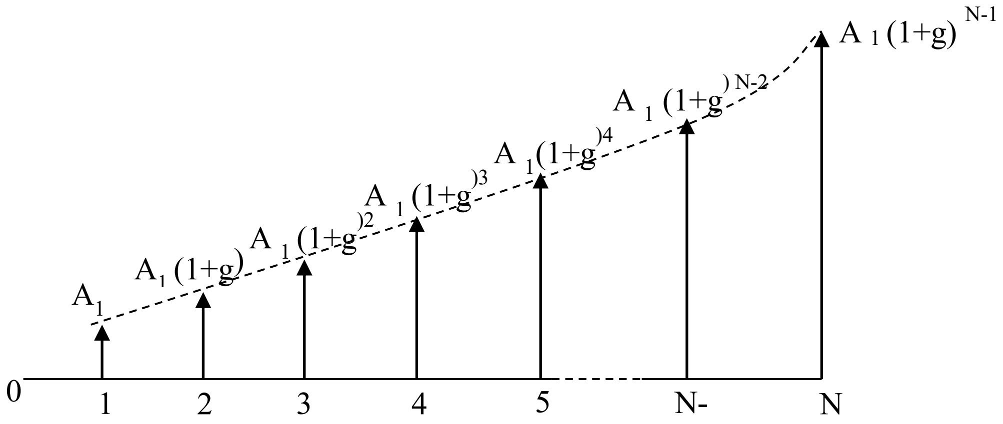


Figure 2.13 (a): Increasing geometric gradient series. [3]

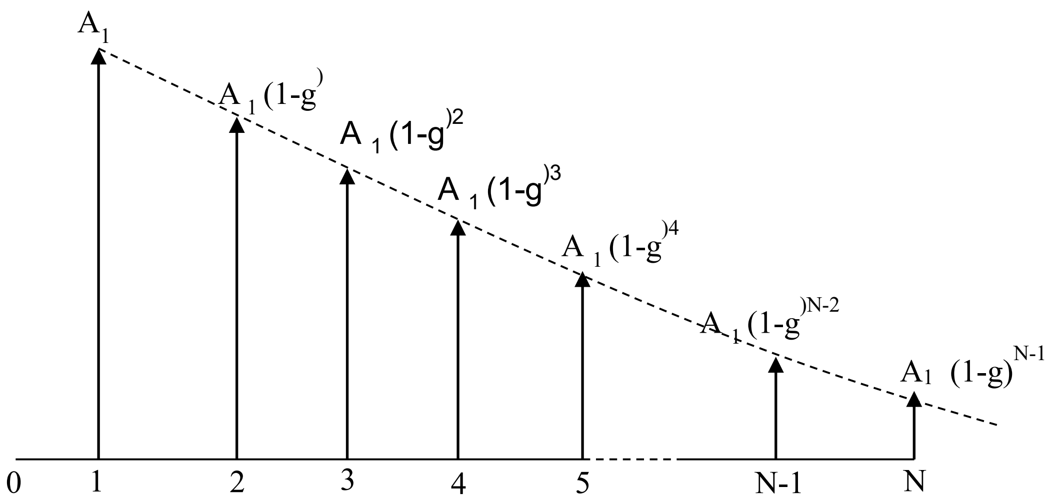


Figure 2.13 (b): Decreasing geometric gradient series. [3]

<u>Present worth Factor</u>

<u>Find P, given A_1, g, i, N</u>

If $i \neq g$

$$\mathbf{P = A_1 \frac{[1-(1+g)^{N}(1+i)^{-N}]}{i-g}}$$

<u>If i = g</u>

$$\mathbf{P = [NA_1/(1+i)^{-1}]}$$

The factor in the bracket is *Geometric gradient series present worth factor*

Functionally,
$P = A_1 (P/A_1, g, i, N)$

Future worth Factor

Find F, given A_1, g, i, N

If $i \neq g$

$$F = A_1 \frac{[(1+i)^{-N} - (1+g)^{N}]}{i - g}$$

If $i = g$

$$F = NA_1 (1+i)^{N-1}$$

The factor in the bracket is called *Geometric gradient series future worth factor*

Functionally,
$F = A_1 (F/A_1, g, i, N)$

References:

[1] *Engineering Economy:* William G. Sullivan, James A. Bontadelli & Elin M. Wicks, Eleventh Edition, Pearson Educations, Inc. 2000.

[2] *Engineering Economics:* James L. Riggs, David D. Bedworth and Sabah U. Randhawa, Fourth Edition, Tata McGraw Hill Education Private Limited, New Delhi, India, 2004.

[3] *Contemporary Engineering Economics*: Chan S. Park Second Edition, Addison-Wesley Publishing Company, 1997.

[4] *Basics of Engineering Economy*: Leland Blank and Anthony Tarquin, Indian Edition, Tata McGraw Hill Education Private Limited, New Delhi, India, 2013.