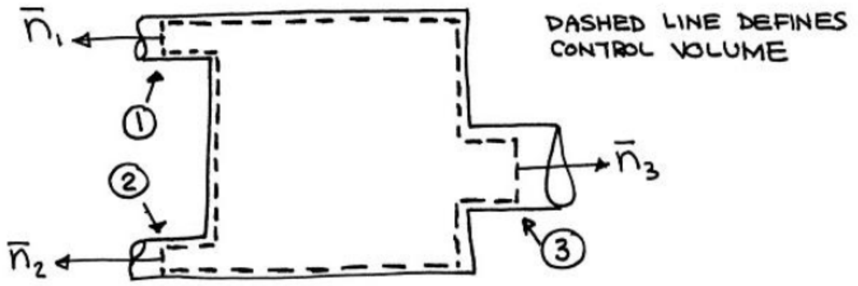
Solution 1

A convenient control volume can be drawn around the interior volume of the tank, and extending into pipes 1 and 2 to positions of uniform concentration, i.e. $\frac{\partial C}{\partial n} = 0$ along pipe.



Now evaluate Eq.4 for this control volume.

(A)

$$\frac{\partial}{\partial t}\int_{C\forall} C d\forall = -\int_{CS} C\vec{V}\cdot\vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

Because we assume steady state, $\frac{\partial}{\partial t} = 0$, the first term is zero. No source or sink is mentioned, $\therefore$ set $S = 0$. We evaluate the two surface integrals, $\int_{CS}$, at the three indicated areas of flux. Note that we placed the surface $1, 2, 3$ far enough into the pipes that $\frac{\partial C}{\partial n} = 0$ at each surface. $\therefore$ there is no diffusive flux, $\int_{CS} D_n \frac{\partial C}{\partial n} dA = 0$.

(B) Evaluating $\int_{CS} C\vec{V}\cdot\vec{n} dA$ at each flux area,

$$0 = +u_1A_1C_1 + u_2A_2C_2 - u_3A_3C_3.$$

From conservation of fluid mass (continuity), we also have $u_1A_1 + u_2A_2 = u_3A_3$ for incompressible flow.

(C) Using this to replace u_3A_3 in (B) and solving for C_3,

$$C_3 = \frac{u_1A_1C_1 + u_2A_2C_2}{(u_1A_1 + u_2A_2)}$$

or,

$$C_3 = \frac{(20\,cm/s)(10\,cm^2)(9\,mg/l) + (10\,cm/s)(10\,cm^2)(0\,mg/l)}{(20\,cm/s)(10\,cm^2) + (10\,cm/s)(10\,cm^2)}$$

$$C_3 = \frac{20}{30} * 9\,mg/l = 6\,mg/l$$

Solution 2

Apply the integral form of mass conservation to the control volume indicated by dashes.

	$\frac{\partial}{\partial t}\int_{C\forall} C d\forall$	$= -\int_{CS} C\vec{V}\cdot\vec{n}dA$	$+\int_{CS} D_n \frac{\partial C}{\partial n} dA$	$+S$
(A)	∅, b/c we assume steady state	evaluate at surface sections 1,2	∅, b/c we place surfaces 1 & 2 where $\frac{\partial C}{\partial n} = 0$	given

(B)

$$0 = u_1 C_1 A_1 - u_2 C_2 A_2 + S$$

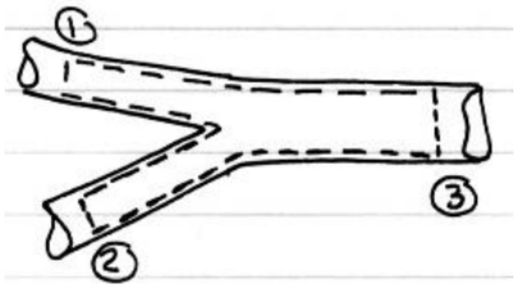
Note: From continuity, $u_2 = u_1$, because $A_2 = A_1$

(C)

$$C_2 = \frac{S}{u_2 A_2} = \frac{S}{u_1 A_1} = \frac{5\, g/s}{(10\, cm/s)(10\, cm^2)} = 50\, mg/cm^3$$

Solution 3

Choose a control volume (dash) far enough away from juncture such that $\frac{\partial T}{\partial n} = 0$ at each flux surface.



(A)

$$\frac{\partial}{\partial t}\int_{CV} C d\forall = -\int_{CS} C\vec{V}\cdot\vec{n}dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

The concentration of heat energy is,

$$C[Jm^{-3}] = \rho c_p T$$

fluid density $[kgm^{-3}]$ — specific heat $[Jkg^{-1}K^{-1}]$ — temp $[K]$

$C_p = 4200\, Jkg^{-1}K^{-1}$ for water

For simplicity, assume $\rho, c_p \neq f(T)$. If we assume steady state, then the first term in (A) is zero. Because we position surfaces 1, 2, 3 where $\frac{\partial T}{\partial n} = 0$, the diffusive flux term is zero. Because the pipes are insulated, $S = 0$. So, finally (A) becomes,

(B)

$$0 = \rho c_p T_1 u_1 A_1 + \rho c_p T_2 u_2 A_2 - \rho c_p T_3 u_3 A_3$$

Dropping ρc_p, and solving for T_3,

(C)

$$T_3 = \frac{u_1 A_1 T_1 + u_2 A_2 T_2}{u_3 A_3}$$

Note from statement $u_1 A_1 = u_2 A_2$. And from fluid mass conservation $(u_1 A_1 + u_2 A_2) = u_3 A_3$.

(D)

$$T_3 = \frac{u_1 A_1 (T_1 + T_2)}{2 u_1 A_1} = \frac{1}{2}(T_1 + T_2)$$

$\therefore T_3 = 15 \deg C \quad [288K]$

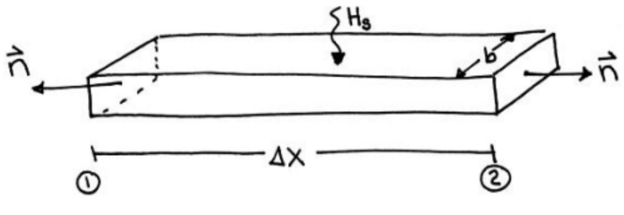
Solution 4

Control Volume Approach:

Select a short length of river, ΔX, and evaluate the control volume (integral) form of the conservation equation. For conservation of heat energy, replace $C = \rho c_p T$ in Eq.4.

(A)

$$\frac{\partial}{\partial t} \int_{CV} \rho c_p T d\forall = - \int_{CS} \rho c_p T \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA + H_s \Delta x b$$



As the problem statement does not indicate any unsteadiness, we assume steady flow, i.e. $\frac{\partial}{\partial t} = 0$.

(B) Evaluating the flux terms in (A),

$$0 = -\rho c_p U b h (T_2 - T_1) + \rho c_p D b h \left(\left.\frac{\partial T}{\partial X}\right|_2 - \left.\frac{\partial T}{\partial X}\right|_1 \right) + H_3 \Delta x b$$

Using a Taylor expansion, assuming T is continuous in X,

$$T_2 = T_1 + \frac{\partial T}{\partial X}\Delta X$$
$$\left.\frac{\partial T}{\partial X}\right|_2 = \left.\frac{\partial T}{\partial X}\right|_1 + \frac{\partial}{\partial X}\left(\frac{\partial T}{\partial X}\right)\Delta X$$

Plug these expansions in (B), and divide out the common terms, $\Delta x b$.

(C)

$$0 = \rho c_p h\left(-U\frac{\partial T}{\partial X} + D\frac{\partial^2 T}{\partial X^2}\right) + H_s$$

From which, one could solve for $\frac{\partial T}{\partial X}$.

It is useful to consider the relative importance of the advective and diffusive fluxes. Here, specifically the relative magnitudes of $U\frac{\partial T}{\partial X}$ and $D\frac{\partial^2 T}{\partial X^2}$. The scale of each term can be estimated from this system. Consider the control volume length, Δx, as an appropriate length-scale, then

$$U\frac{\partial T}{\partial X} \sim U\frac{\Delta T}{\Delta X}$$
$$D\frac{\partial^2 T}{\partial X^2} \sim D\frac{\Delta T}{\Delta X^2}$$

Where ΔT is the temperature change across ΔX. The relative magnitude of these terms is then,

$$\frac{\text{advective flux}}{\text{diffusive flux}} = \frac{U\frac{\Delta T}{\Delta X}}{D\frac{\Delta T}{\Delta X^2}} = \frac{U\Delta X}{X}$$

This dimensionless parameter is called the peclet number. It is discussed in detail in Chapter 5.

If $\frac{U\Delta X}{D} >> 1$, then advective fluxes dominate diffusive fluxes, and we can drop the term $D\frac{\partial^2 T}{\partial X^2} << U\frac{\partial T}{\partial X}$.

If $\frac{U\Delta X}{D} << 1$, diffusive fluxes $\left(D\frac{\partial^2 T}{\partial X^2}\right)$ are much larger than advective fluxes $\left(U\frac{\partial T}{\partial X}\right)$, and we can drop $U\frac{\partial T}{\partial X}$.

Since ΔX is not specifically defined, we ask, e.g., for what length-scale will advection dominate transport?

$$\frac{U\Delta X}{D} >> 1 \text{ iff } \Delta X >> \frac{D}{U} = \frac{0.1\, m^2 s^{-1}}{1\, m^2 s^{-1}} = 0.1\, m$$

$\therefore$ over any length-scale, $\Delta X >> 0.1\,m$, we may neglect the impact of diffusion in (A) for this system. The problem asks for a description of $\dfrac{\partial T}{\partial X}$ along a river channel. In such a system, the length scales of interest are much larger than 10 cm, and are more like 100 m to km's. Therefore, for this system, we can safely drop the diffusive transport term. Then, (C) reduces to,

(D)

$$0 = -\rho c_p h U \frac{\partial T}{\partial X} + H_s$$

from which,

(E)

$$\frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h U} = \frac{J s^{-1} m^{-2}}{(kg m^{-3})(J kg^{-1} K^{-1})(m)(m s^{-1})} = \frac{K}{m}$$

Using the stated parameters,

(F)

$$\frac{\partial T}{\partial X} = \frac{800\,W m^{-2}}{(1000\,kg/m^3)(4200\,J/kgK)(1\,m)(1\,m/s)} = 2 \times 10^{-4} K/m = 0.2\,C/km$$

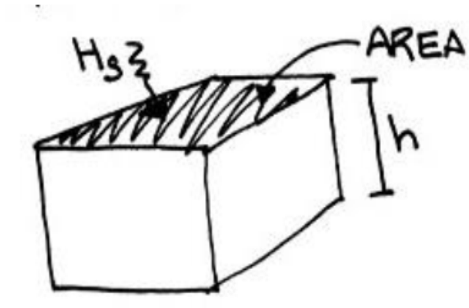
Differential Approach:

The conservation of mass equation can be applied to the transport of heat energy by noting the concentration of heat energy, $C[J/m^3] = \rho c_p T$. Then, the differential form of the conservation equation is, for incompressible flow,

(G)

$$\frac{\partial}{\partial t}(\rho c_p T) + U \frac{\partial}{\partial x}(\rho c_p T) + V \frac{\partial}{\partial y}(\rho c_p T) + W \frac{\partial}{\partial z}(\rho c_p T)$$
$$= \frac{\partial}{\partial x} D_x \frac{\partial}{\partial x}(\rho c_p T) + \frac{\partial}{\partial y} D_y \frac{\partial}{\partial y}(\rho c_p T) + \frac{\partial}{\partial z} D_z \frac{\partial}{\partial z}(\rho c_p T) \pm S$$

- If we neglect $\rho = f(T)$, then $\rho \neq f(x, y, z, t)$.
- The problem statement gives us $V = W = 0$, and isotropic, homogeneous $D = D_x = D_y = D_z \neq f(x, y, z)$.
- If we assume the system is uniform (well-mixed) in y and z, then $T \neq f(y, z)$.
- The source term is given as a surface flux, $H_s = [J s^{-1} m^{-2}]$. Since the equation deals in volume concentration, we must divide by depth to put the source term in consistent units.



$$H_s = \frac{J/m^2}{s} = \frac{\text{energy per surface area}}{\text{s}}$$
$$\frac{H_s}{h} = \frac{J/m^3}{s} = \frac{\text{energy per volume}}{\text{s}}$$

Applying the above points, (G) reduces to

(H)

$$\rho c_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial X} \right] = D \frac{\partial^2 T}{\partial X^2} + \frac{H_s}{h}$$

For typical length scales of interest along a river channel, $\Delta X \sim 100m$ to km's, it is easy to show that the diffusion term, $D\frac{\partial^2 T}{\partial X^2}$, is small compared to $U\frac{\partial T}{\partial X}$, the advection term. Thus we will drop $D\frac{\partial^2 T}{\partial X^2} << U\frac{\partial T}{\partial X}$.

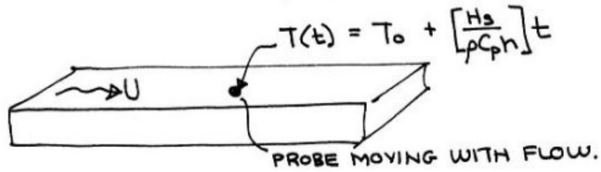
See scaling arguments given above.

Finally, note that the bracketed term in (H) is the total derivative.

(I)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h} = \left[\frac{°\text{C}}{T} \right]$$

This equation may be read in the Lagrangian context as, following a particular fluid particle, we would observed its temperature to increase at the rate $\left[\frac{H_s}{\rho c_p h}\right]^{\circ}C/s$.



If the flow/thermal conditions are steady, $\frac{\partial T}{\partial t} = 0$, then (I) also provides a simple description of spatial gradient.

(J)

$$\frac{\partial T}{\partial X} = \frac{H_s}{U \rho c_p h} = \left[\frac{°\text{C}}{L} \right]$$

SOLUTION 5

a) Estimate the time at which the concentration at A and A' begin to diverge?

The concentrations at A and A' diverge when the boundary impacts the solution in System 1. This occurs when the diffusing cloud reaches the boundary. Estimate this time by equating the edge of the cloud with the length scale 3σ. That is, the cloud will touch the boundary when $3\sigma = 3\sqrt{2Dt}$ = 50cm, such that $t = (50\text{cm})^2 / 18 \bullet 2\text{cm}^2\text{s}^{-1}$ = 70 seconds.

b) What is the final concentration at A (A'), and when is this concentration achieved?

The final concentration in **System 1** will be C = (100g)/ (1m x 1m x 0.1m) = 1000 ppm.
It is achieved when the mass is fully mixed across the domain. Using the largest dimension to estimate this time-scale, $t = (L)^2/(4\,D)$ = 1250 s. As noted in the text, this is a conservative estimate, and $t = (L)^2/(8\,D)$ = 625 s, is also a reasonable estimate. Because **System 2** is unbounded, infinite dilution is possible and the final concentration is C = 0 ppm, but theoretically this will take infinite time. Because the probe has a detection limit of 10 ppm, zero concentration will be recorded for any concentration C < 10 ppm, which occurs in a finite time.

c) Describe the evolution of the concentration field in each system, i.e. C(x,y,z,t).

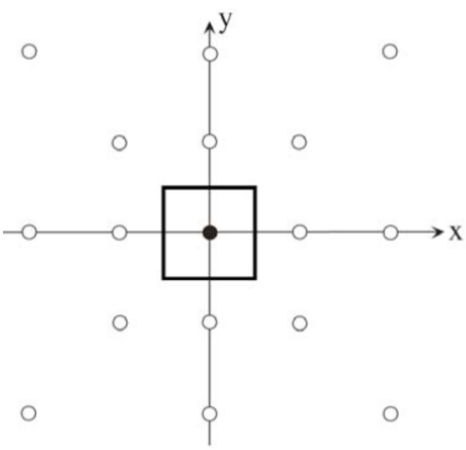
In both systems the concentration is uniform in z by $t = (10\text{cm})^2/(4\text{x}2\text{cm}^2\text{s}^{-1})$=12.5 s after release. Because this is short relative to other time scales of interest (1250 sec), we neglect the 3-D phase and use a two-dimensional solution. For System 1, an infinite number of image sources is needed to satisfy the no-flux boundaries.

$$C1(x,y,t) = \frac{M}{L_z\, 4\pi\, Dt}\left[\exp\left(-\frac{x^2+y^2}{4Dt}\right) + \sum_{n=1}^{\infty} \exp\left(-\frac{x^2 + (y \pm 2nL)^2}{4Dt}\right) + \sum_{n=1}^{\infty} \exp\left(-\frac{(x \pm 2nL)^2 + y^2}{4Dt}\right)\right]$$

real source images along y axis images along x axis

$$+ \frac{M}{L_z\, 4\pi\, Dt}\left[\sum_{n=1}^{\infty} \exp\left(-\frac{(x \pm 2nL)^2 + (y \pm 2nL)^2}{4Dt}\right)\right]$$

corner images



The images in the second line are located at the corners (see sketch), to offset the losses of y-axis images across the boundaries at x ± L/2; and the loss of x-axis images across the boundaries at y = ± L/2. In practice an infinite number of images is not needed. For time less than required to reach a well-mixed condition between the boundaries ($t < L^2/4D$), three images per boundary is sufficient to approximate the full solution with infinite images. Beyond this time, the concentration is steady and uniform, and the detailed solution above is no longer needed. System 2 is described by a simple two-dimensional slug-release,

$$C2(x,y,t) = \frac{M}{L_z\, 4\pi D\, t}\exp\left(-\frac{x^2+y^2}{4Dt}\right).$$

SOLUTION 6

a) Since the system is unbounded in each coordinate direction (x, y, z), the concentration can never become uniform in any direction and a three-dimensional solution will apply. Assuming the release occurs at t = 0 s, the trajectory of the chemical cloud's center of mass is (x = Ut, y = H, z = 0). To satisfy the no-flux boundary at y = 0, we must add an image source at y = -H. The center of mass of the image source follows the trajectory (x = Ut, y = -H, z = 0). With these conditions, and isotropic diffusivity the concentration solution is

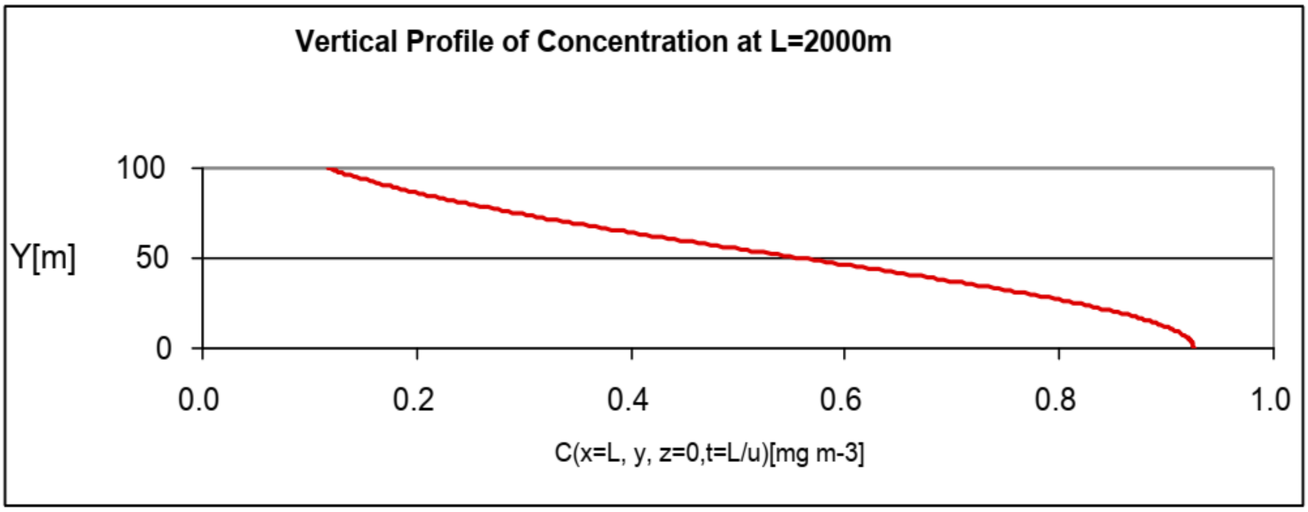
C (x, y, z, t) =

$$\frac{M}{(4\pi Dt)^{3/2}}\left[\exp\left(-\frac{(x-ut)^2+(y-H)^2+z^2}{4Dt}\right)+\exp\left(-\frac{(x-ut)^2+(y+H)^2+z^2}{4Dt}\right)\right].$$

b) Given the length scale x = L = 500 m, Pe# = (2 ms^{-1})(500m)/(1m^2s^{-1}) = 1000 >> 1. This tells us that transport from the smokestack to this position (x = 500 m) is dominated by advection.

c) The release will appear to be instantaneous if the time scale for release, T_R, is much shorter than the time scale of transport. At distances for which Pe# = Ux/D >> 1, the transport is dominated by advection. This is true for x > > D/u = 0.5 m, essentially the entire flow domain. Then, the release will appear to be instantaneous at distances for which $T_U >> T_R$. Or, x/U >> T_R. This is true for x >> (300 s)(2 ms^{-1}) = 600 m.

d) At x = 2000 m the peak concentrations should arrive at T_u = 2000m/(2ms^{-1})=1000 s. The vertical profile at x = L, t = L/u is shown below for 2000m. Note that the peak concentration occurs at the no-flux boundary and not at the height of the release.



SOLUTION 7

a. Assume the flow fills the channel uniformly, then U =Q/A = 2.5 x 10^{-4} ms^{-1}. Using L = 75, Pe = 0.19. Alternatively, for L = (75-25) = 50 m, Pe = 0.13. In either case the Peclet number indicates that the system is dominated by diffusion.

b. According to the Peclet number, transport is dictated by diffusion. The time-scale for the contaminant to reach the harbor can be estimated as the time-scale for diffusive transport over the 50-m between the spill and the harbor entrance. T_D = $(50m)^2/(8 \times 0.1\ m^2s^{-1})$ = 3125 s ≈ 1hr. So, I have one hour to put up a contaminant-absorbing barrier and protect the harbor

c. In the vertical, $t_{mix}=(2\ m)^2/(4 \times 0.1\ m^2s^{-1}) = 10$ s
In the lateral, $t_{mix}=(10\ m)^2/(4 \times 0.1\ m^2s^{-1}) = 250$ s

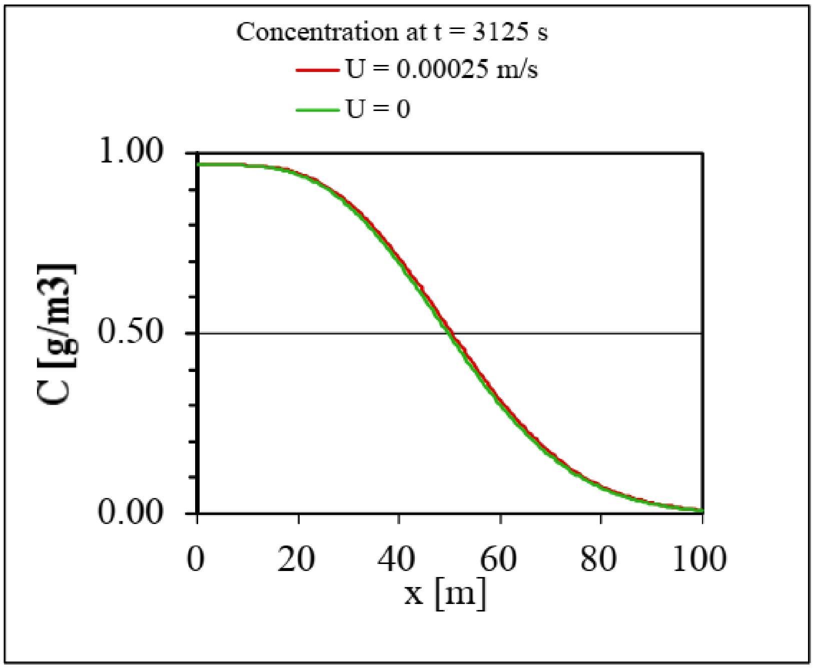
d. Since the mixing time scales in both the lateral and vertical are much shorter than the time required for the contaminant to reach the harbor, we can assume the contaminant is mixed across the channel area when it reaches x = 75 m. If concentration is uniform (well-mixed) in the lateral and vertical, we can drop these two dimensions, and use a one-dimensional solution.

e.
$$C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}}\left[\underbrace{\exp\left(-\frac{(x-25\ -\ Ut)^2}{4Dt}\right)}_{\text{real source}} + \underbrace{\exp\left(-\frac{(x+25\ -\ Ut)^2}{4Dt}\right)}_{\text{image source}}\right]$$

The image source is needed to satisfy the no-flux boundary at x = 0. Since Pe<<1, we could also neglect U entirely and still get a good representation of C(x,t).

$$C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}}\left[\exp\left(-\frac{(x-25)^2}{4Dt}\right) + \exp\left(-\frac{(x+25)^2}{4Dt}\right)\right]$$

A comparison of the full solution (U = 0.00025 m/s) and the solution that neglects U is given below. One can quickly see that U is indeed negligible, as implied by Pe.

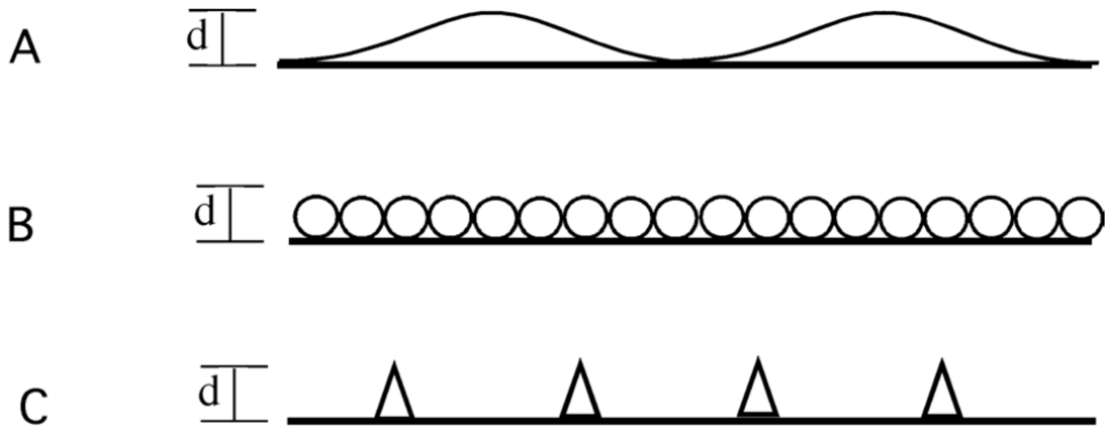


f. U and D are assumed to be uniform, i.e. not functions of (x, y, z), but in fact the no-slip condition at the channel boundaries will make U = f(y,z).
We assume that D is isotropic. In fact, turbulent diffusion in the longtudinal direction will be much more rapid than in the vertical or horizontal. This is discussed further in Chapter 9. We neglect losses to the atmosphere, when in fact the gasoline is volatile. We assume that no gasoline absorbs to the sediment. We assume that the flow and thus velocity are not functions of time.

SOLUTION 8.

If $d \ll \delta_s$, every shape will be buried within the laminar sub-layer. The outer flow will not feel any of the shapes and the flow will be Smooth Turbulent. Under these conditions all cases have the same $y_o = \nu/9u_*$.

If $d \gg \delta_s$ the flow is Rough Turbulent and the flow feels each shape. The shape that disturbs the flow the most, *i.e.* creates the most separation and flow distortion, will create the greatest momentum sink and will feel the roughest. This is shape C. Shape A is the most streamlined, and thus produces the least disturbance and separation. Based on this we expect $y_{oC} > y_{oB} > y_{oA}$.

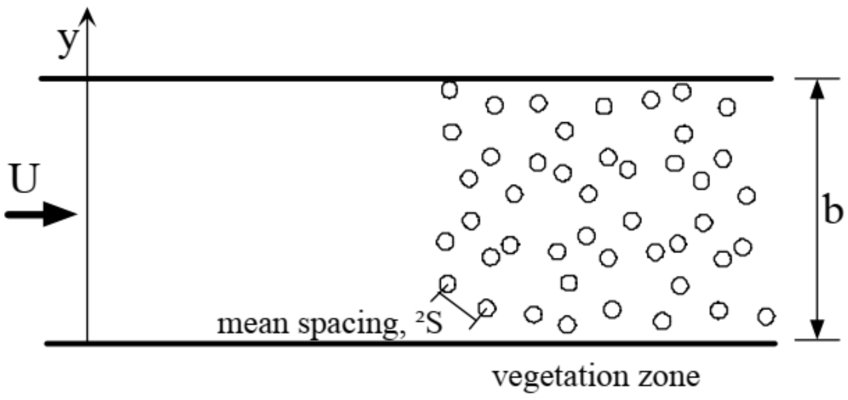


SOLUTION 9.

a) In the unvegetated zone the lateral eddy scales are set by the channel width, ly ~ b. But, in the vegetated zone eddies of scale b are destroyed by the physical obstruction of the plant stems. In this zone the lateral eddy scale is set by the vegetation spacing and diameter, such that ly ~ d or ΔS. Because of the confining walls, the mean channel velocity is approximately the same in both regions. Because the eddy scale is so much reduced in the vegetated zone (ΔS, d << b) we expect the lateral diffusivity to be reduced in this region, relative to the unvegetated zone.

b) If $Ud/\nu = 1$, the wakes behind each stem will be laminar and will not contribute additional eddies to the flow. If $Ud/\nu = 1000$, the stems contribute stem-scale eddies to the flow through separation and the generation of Kelvin-Helmholtz vortices. So, for the later case we expect that v' will be augmented by the wake-vortices, and that $D_{t,y}$ will be greater with the higher Reynolds number.

c) If the flow is unconfined the vegetative drag will lead to a reduction of velocity within the vegetated zone as flow is redirected around the patch of vegetation. Then, the velocity in the vegetated zone, U_V, will be less than the velocity in the free stream, U. If $U_V < U$ and the stem wakes are not contributing additional turbulence ($U_V\, d/\nu \lessapprox 100$) we would expect that the turbulence intensity in the vegetated zone is also less, $v'_V < v'$. In addition, the turbulence scales will be smaller in the vegetated zone, as discussed in a). With both v' and ly smaller in the vegetated zone, then $D_{t,y}$ will also be smaller in the vegetated zone. Even if the stem Reynolds' number, $U_V d/\nu$, is sufficiently high that the stems contribute additional eddies, their scale (ly ~ d) is too small compared to those in the free stream, ly > b, and still the diffusivity in the vegetated zone will remain small compared to the unvegetated zone.



SOLUTION 10.

a) The logarithmic region is fitted with a red line. From this line we estimate the value of u_*.

$$u_* = \frac{0.4}{2.3}\frac{(15-0 \ \ \text{cm/s})}{\log_{10}(4/0.2)} = 2.0 \text{ cm/s}$$

b) The thickness of the laminar sub-layer is $5\nu/u_* = 5\,(0.01\ \text{cm}^2\text{s}^{-1})/(2.0\text{cms}^{-1}) = 0.025$ cm.

c) Considering that the gravel diameter, 1 cm, is much larger than the laminar sub-layer, it is most likely that the gravel does contribute to the bed resistance, i.e. the flow feels the gravel roughness. More formally, we must determine if the flow is smooth or rough turbulent. From the graph, $y_o = 0.2$ cm. For smooth turbulent flow $y_o = \nu/9u_* = 5.6 \times 10^{-4}$ cm, which is smaller than the observed y_o, so the flow is not smooth. If the flow were rough turbulent, the equivalent roughness, $\varepsilon = 30y_o = 6$ cm, and the roughness Reynolds' Number would be, $\varepsilon u_*/\nu = 1200$, which is much higher than the limit for rough turbulent flow (70-100). So we conclude the flow is rough turbulent, and the gravel contributes to the roughness of the flow.

d) A covering of uniform sand grains of 6-cm diameter would be needed to produce the same flow resistance.