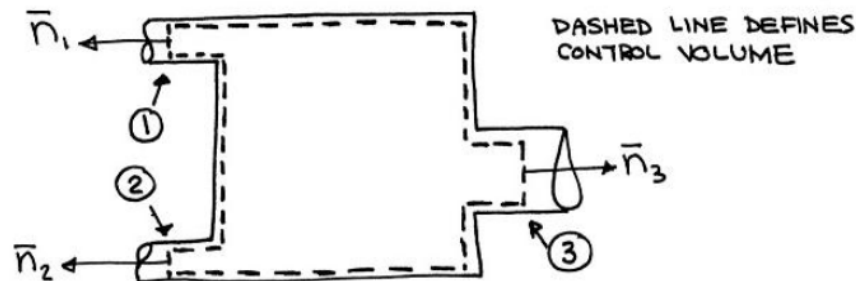


## Solution 1

A convenient control volume can be drawn around the interior volume of the tank, and extending into pipes 1 and 2 to positions of uniform concentration, i.e.  $\frac{\partial C}{\partial n} = 0$  along pipe.



Now evaluate Eq.4 for this control volume.

(A)

$$\frac{\partial}{\partial t} \int_{CV} C dV = - \int_{CS} C \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

Because we assume steady state,  $\frac{\partial}{\partial t} = 0$ , the first term is zero. No source or sink is mentioned,  $\therefore$  set  $S = 0$ . We evaluate the two surface integrals,  $\int_{CS}$ , at the three indicated areas of flux. Note that we placed the surface 1, 2, 3 far enough into the pipes that  $\frac{\partial C}{\partial n} = 0$  at each surface.  $\therefore$  there is no diffusive flux,  $\int_{CS} D_n \frac{\partial C}{\partial n} dA = 0$ .

(B) Evaluating  $\int_{CS} C \vec{V} \cdot \vec{n} dA$  at each flux area,

$$0 = +u_1 A_1 C_1 + u_2 A_2 C_2 - u_3 A_3 C_3.$$

From conservation of fluid mass (continuity), we also have  $u_1 A_1 + u_2 A_2 = u_3 A_3$  for incompressible flow.

(C) Using this to replace  $u_3 A_3$  in (B) and solving for  $C_3$ ,

$$C_3 = \frac{u_1 A_1 C_1 + u_2 A_2 C_2}{(u_1 A_1 + u_2 A_2)}$$

or,

$$C_3 = \frac{(20 \text{ cm/s})(10 \text{ cm}^2)(9 \text{ mg/l}) + (10 \text{ cm/s})(10 \text{ cm}^2)(0 \text{ mg/l})}{(20 \text{ cm/s})(10 \text{ cm}^2) + (10 \text{ cm/s})(10 \text{ cm}^2)}$$

$$C_3 = \frac{20}{30} * 9 \text{ mg/l} = 6 \text{ mg/l}$$

## Solution 2

Apply the integral form of mass conservation to the control volume indicated by dashes.

$$\frac{\partial}{\partial t} \int_{CV} C dV = - \int_{CS} C \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA + S$$

(A)  $\emptyset$ , b/c we assume steady state evaluate at surface sections 1,2  $\emptyset$ , b/c we place surfaces 1 & 2 given where  $\frac{\partial C}{\partial n} = 0$

(B)

$$0 = u_1 C_1 A_1 - u_2 C_2 A_2 + S$$

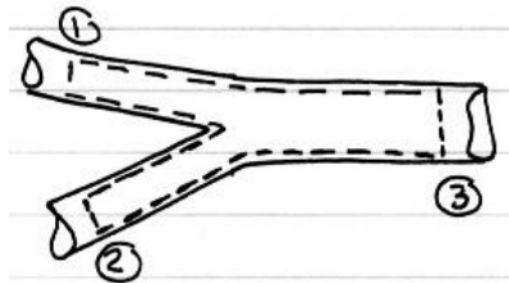
Note: From continuity,  $u_2 = u_1$ , because  $A_2 = A_1$

(C)

$$C_2 = \frac{S}{u_2 A_2} = \frac{S}{u_1 A_1} = \frac{5 \text{ g/s}}{(10 \text{ cm/s})(10 \text{ cm}^2)} = 50 \text{ mg/cm}^3$$

## Solution 3

Choose a control volume (dash) far enough away from juncture such that  $\frac{\partial T}{\partial n} = 0$  at each flux surface.



(A)

$$\frac{\partial}{\partial t} \int_{CV} C dV = - \int_{CS} C \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

The concentration of heat energy is,

$$C [Jm^{-3}] = \rho c_p T$$

fluid density  $[kgm^{-3}]$       specific heat  $[Jkg^{-1}K^{-1}]$       temp  $[K]$

$C_p = 4200 Jkg^{-1}K^{-1}$  for water

For simplicity, assume  $\rho, c_p \neq f(T)$ . If we assume steady state, then the first term in (A) is zero. Because we position surfaces 1, 2, 3 where  $\frac{\partial T}{\partial n} = 0$ , the diffusive flux term is zero. Because the pipes are insulated,  $S = 0$ . So, finally (A) becomes,

(B)

$$0 = \rho c_p T_1 u_1 A_1 + \rho c_p T_2 u_2 A_2 - \rho c_p T_3 u_3 A_3$$

Dropping  $\rho c_p$ , and solving for  $T_3$ ,

(C)

$$T_3 = \frac{u_1 A_1 T_1 + u_2 A_2 T_2}{u_3 A_3}$$

Note from statement  $u_1 A_1 = u_2 A_2$ . And from fluid mass conservation  $(u_1 A_1 + u_2 A_2) = u_3 A_3$ .

(D)

$$T_3 = \frac{u_1 A_1 (T_1 + T_2)}{2u_1 A_1} = \frac{1}{2} (T_1 + T_2)$$

$$\therefore T_3 = 15 \text{ deg C } [288K]$$

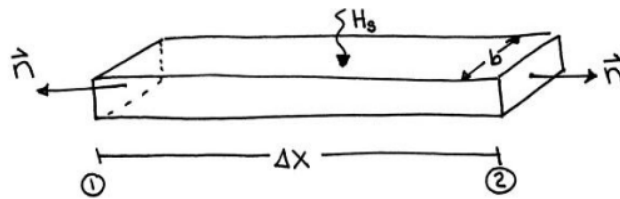
## Solution 4

### Control Volume Approach:

Select a short length of river,  $\Delta X$ , and evaluate the control volume (integral) form of the conservation equation. For conservation of heat energy, replace  $C = \rho c_p T$  in Eq.4.

(A)

$$\frac{\partial}{\partial t} \int_{CV} \rho c_p T dV = - \int_{CS} \rho c_p T \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA + H_s \Delta x b$$



As the problem statement does not indicate any unsteadiness, we assume steady flow, i.e.  $\frac{\partial}{\partial t} = 0$ .

(B) Evaluating the flux terms in (A),

$$0 = -\rho c_p U b h (T_2 - T_1) + \rho c_p D b h \left( \left. \frac{\partial T}{\partial X} \right|_2 - \left. \frac{\partial T}{\partial X} \right|_1 \right) + H_3 \Delta x b$$

Using a Taylor expansion, assuming  $T$  is continuous in  $X$ ,

$$T_2 = T_1 + \frac{\partial T}{\partial X} \Delta X$$

$$\left. \frac{\partial T}{\partial X} \right|_2 = \left. \frac{\partial T}{\partial X} \right|_1 + \frac{\partial}{\partial X} \left( \frac{\partial T}{\partial X} \right) \Delta X$$

Plug these expansions in (B), and divide out the common terms,  $\Delta x b$ .

(C)

$$0 = \rho c_p h \left( -U \frac{\partial T}{\partial X} + D \frac{\partial^2 T}{\partial X^2} \right) + H_s$$

From which, one could solve for  $\frac{\partial T}{\partial X}$ .

It is useful to consider the relative importance of the advective and diffusive fluxes. Here, specifically the relative magnitudes of  $U \frac{\partial T}{\partial X}$  and  $D \frac{\partial^2 T}{\partial X^2}$ . The scale of each term can be estimated from this system. Consider the control volume length,  $\Delta x$ , as an appropriate length-scale, then

$$U \frac{\partial T}{\partial X} \sim U \frac{\Delta T}{\Delta X}$$

$$D \frac{\partial^2 T}{\partial X^2} \sim D \frac{\Delta T}{\Delta X^2}$$

Where  $\Delta T$  is the temperature change across  $\Delta X$ . The relative magnitude of these terms is then,

$$\frac{\text{advective flux}}{\text{diffusive flux}} = \frac{U \frac{\Delta T}{\Delta X}}{D \frac{\Delta T}{\Delta X^2}} = \frac{U \Delta X}{D}$$

This dimensionless parameter is called the pecelet number. It is discussed in detail in Chapter 5.

If  $\frac{U \Delta X}{D} \gg 1$ , then advective fluxes dominate diffusive fluxes, and we can drop the term  $D \frac{\partial^2 T}{\partial X^2} \ll U \frac{\partial T}{\partial X}$ .

If  $\frac{U \Delta X}{D} \ll 1$ , diffusive fluxes  $\left( D \frac{\partial^2 T}{\partial X^2} \right)$  are much larger than advective fluxes  $\left( U \frac{\partial T}{\partial X} \right)$ , and we can drop  $U \frac{\partial T}{\partial X}$ .

Since  $\Delta X$  is not specifically defined, we ask, e.g., for what length-scale will advection dominate transport?

$$\frac{U \Delta X}{D} \gg 1 \text{ iff } \Delta X \gg \frac{D}{U} = \frac{0.1 \text{ m}^2 \text{ s}^{-1}}{1 \text{ m}^2 \text{ s}^{-1}} = 0.1 \text{ m}$$

∴ over any length-scale,  $\Delta X \gg 0.1 \text{ m}$ , we may neglect the impact of diffusion in (A) for this system. The problem asks for a description of  $\frac{\partial T}{\partial X}$  along a river channel. In such a system, the length scales of interest are much larger than 10 cm, and are more like 100 m to km's. Therefore, for this system, we can safely drop the diffusive transport term. Then, (C) reduces to,

(D)

$$0 = -\rho c_p h U \frac{\partial T}{\partial X} + H_s$$

from which,

(E)

$$\frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h U} = \frac{J s^{-1} m^{-2}}{(kg m^{-3})(J kg^{-1} K^{-1})(m)(m s^{-1})} = \frac{K}{m}$$

Using the stated parameters,

(F)

$$\frac{\partial T}{\partial X} = \frac{800 \text{ W m}^{-2}}{(1000 \text{ kg/m}^3)(4200 \text{ J/kgK})(1 \text{ m})(1 \text{ m/s})} = 2 \times 10^{-4} \text{ K/m} = 0.2 \text{ C/km}$$

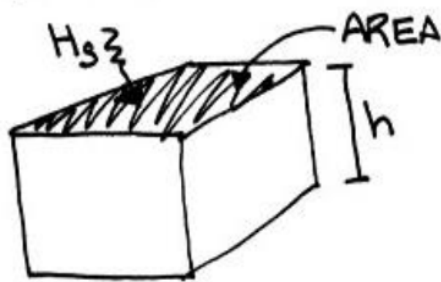
### Differential Approach:

The conservation of mass equation can be applied to the transport of heat energy by noting the concentration of heat energy,  $C[J/m^3] = \rho c_p T$ . Then, the differential form of the conservation equation is, for incompressible flow,

(G)

$$\begin{aligned} & \frac{\partial}{\partial t} (\rho c_p T) + U \frac{\partial}{\partial x} (\rho c_p T) + V \frac{\partial}{\partial y} (\rho c_p T) + W \frac{\partial}{\partial z} (\rho c_p T) \\ & = \frac{\partial}{\partial x} D_x \frac{\partial}{\partial x} (\rho c_p T) + \frac{\partial}{\partial y} D_y \frac{\partial}{\partial y} (\rho c_p T) + \frac{\partial}{\partial z} D_z \frac{\partial}{\partial z} (\rho c_p T) \pm S \end{aligned}$$

- If we neglect  $\rho = f(T)$ , then  $\rho \neq f(x, y, z, t)$ .
- The problem statement gives us  $V = W = 0$ , and isotropic, homogeneous  $D = D_x = D_y = D_z \neq f(x, y, z)$ .
- If we assume the system is uniform (well-mixed) in y and z, then  $T \neq f(y, z)$ .
- The source term is given as a surface flux,  $H_s = [J s^{-1} m^{-2}]$ . Since the equation deals in volume concentration, we must divide by depth to put the source term in consistent units.



$$H_s = \frac{J/m^2}{s} = \frac{\text{energy per surface area}}{s}$$

$$\frac{H_s}{h} = \frac{J/m^3}{s} = \frac{\text{energy per volume}}{s}$$

Applying the above points, (G) reduces to

(H)

$$\rho c_p \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial X} \right] = D \frac{\partial^2 T}{\partial X^2} + \frac{H_s}{h}$$

For typical length scales of interest along a river channel,  $\Delta X \sim 100m$  to km's, it is easy to show that the diffusion term,  $D \frac{\partial^2 T}{\partial X^2}$ , is small compared to  $U \frac{\partial T}{\partial X}$ , the advection term. Thus we will drop  $D \frac{\partial^2 T}{\partial X^2} \ll U \frac{\partial T}{\partial X}$ .

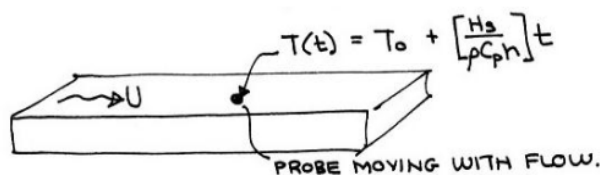
See scaling arguments given above.

Finally, note that the bracketed term in (H) is the total derivative.

(I)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h} = \left[ \frac{^\circ C}{T} \right]$$

This equation may be read in the Lagrangian context as, following a particular fluid particle, we would observe its temperature to increase at the rate  $\left[ \frac{H_s}{\rho c_p h} \right]^\circ C/s$ .



If the flow/thermal conditions are steady,  $\frac{\partial T}{\partial t} = 0$ , then (I) also provides a simple description of spatial gradient.

(J)

$$\frac{\partial T}{\partial X} = \frac{H_s}{U \rho c_p h} = \left[ \frac{^\circ C}{L} \right]$$