12. TRANSPORT OF PARTICLES

This chapter provides an introduction to the transport of particles that are either more dense (e.g. mineral sediment) or less dense (e.g. bubbles) than the fluid. A method of estimating the settling velocity of particles is explained, and then the loss of settling particles from a laminar flow and from a turbulent flow are contrasted. A simple scaling analysis tells us that if the settling velocity of the particles is much less than the friction velocity of the flow, then the turbulence will be sufficiently vigorous to keep the particles in suspension.

Sample problems require the user to gauge the settling (or rise) velocity of suspended particles and determine the effect that it has on downstream particle concentration.

Introduction to the Transport of Particles

Small, neutrally buoyant particles exactly follow the fluid flow, (u, v, w), such that their transport is described by the same equation used for dissolved chemicals. Particles whose density deviates from that of the fluid, either more (e.g. mineral grains) or less (e.g. gas bubbles), will have a vertical velocity relative to the fluid, w_P, which constitutes an additional component of advection for the particle. The particle velocity is proportional to the density difference between the particle and the fluid and to the particle diameter, d. With z taken as the vertical coordinate, the additional velocity component appears in the third term of advective flux.

$$(1a) \qquad \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + (w + w_P)\frac{\partial C}{\partial z} = D_X\frac{\partial^2 C}{\partial x^2} + D_Y\frac{\partial^2 C}{\partial y^2} + D_Z\frac{\partial^2 C}{\partial z^2}$$

In (1a), we have assumed that the particles have a narrow range of size and density, such that w_P is the same for all particles in the flow. If the range of particle size and density is large, then multiple transport equations must be evaluated, each for a different sub-set of particles with comparable w_p. In (1a) the concentration is given in C[kg m^{-3}]. In some cases it is more convenient to consider the particle concentration as n [particles m^{-3}]. If the distribution of particle diameter, d, and density, ρ_P, is narrow, then $C = \rho_P(\pi/6)d^3 n$. Dividing through by $\rho_P(\pi/6)d^3$, (1a) becomes,

(1b) $$\frac{\partial n}{\partial t} + u\frac{\partial n}{\partial x} + v\frac{\partial n}{\partial y} + (w + w_P)\frac{\partial n}{\partial z} = D_X\frac{\partial^2 n}{\partial x^2} + D_Y\frac{\partial^2 n}{\partial y^2} + D_Z\frac{\partial^2 n}{\partial z^2}.$$

In turbulent flow, the diffusion coefficients for particles can be assumed to be the same as those for dissolved species. Under laminar flow conditions, particle diffusion will be a function of particle size, as shown below.

Particle Velocity

To determine the particle velocity, we apply conservation of vertical momentum to a particle considering the forces of weight, buoyancy and drag. For simplicity, we consider a solid, spherical particle of diameter, d, and density, ρ_P. The fluid density is ρ_F. In a coordinate system with z positive upward, we have

Weight = $-\rho_P g\,(\pi/6)\,d^3$
Buoyancy = $\rho_F\, g\,(\pi/6)\,d^3$
Drag = $-(1/2)\,\rho_F C_D\,(\pi/4)\,d^2 w_P|w_P|$ [sign of drag will be opposite to velocity]

C_D, the drag coefficient for a sphere, depends on the Reynolds number defined by the particle velocity, i.e. $Re_P = d\,w_P/\nu$. The conservation of vertical momentum is then,

(2) $$\rho_P(\pi/6)d^3\frac{\partial w_P}{\partial t} = (\rho_F - \rho_P)g(\pi/6)d^3 - \frac{1}{2}\rho_F C_D(\pi/4)d^2 w_P|w_P|$$

Consider a particle starting from rest ($w_P = 0$). If $\rho_P > \rho_F$ the particle begins to accelerate downward ($\partial w_P/\partial t < 0$). As w_P increases, the drag on the particle increases, and acts in the opposite direction to w_P. Eventually the drag grows large enough to exactly balance the particle weight and buoyancy, making the right-hand side of (2) zero. At this point the particle acceleration becomes zero and w_P becomes constant. This is called the terminal velocity. Typically the time to reach the terminal velocity is very short compared to the time scale of interest, so that terminal velocity is assumed for all time. When terminal velocity is reached, $\partial w_P/\partial t = 0$, and (2) can be solved for w_P.

(3) $$|w_P| = \left[\frac{4}{3}\frac{gd\,(\rho_P - \rho_F)}{\rho_F\, C_D}\right]^{1/2}$$

where $C_D = f(Re_P)$ is described by the following empirical approximation for $Re_P < 10^4$.

(4) $$C_D = \frac{24}{Re_P} + \frac{3}{\sqrt{Re_P}} + 0.34 \qquad \text{for } Re_P < 10^4$$

For Re < 1 (called creeping flow), an analytical solution exists for the drag, which yields:

(5) $$C_D = \frac{24}{Re_P} \qquad \text{for Re} < 1 \text{ [creeping flow]}$$

Using (5), we can simplify (3)

(6) $$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} \qquad \text{for Re<1 [creeping flow]} \qquad \mu_F = \rho_F \nu_F$$

(6) is called the Stokes velocity. Because Re_P depends on w_P, we do not know *apriori* whether (6) will apply. One can assume creeping flow, find w_P using (6), and then confirm the assumption of creeping flow. If creeping flow is not confirmed, then an iterative solution is needed (see example 2 below)

Example 1: Find the settling (particle) velocity for 0.01mm diameter quartz sand. The density of quartz is 2600 kgm^{-3}. The density and kinematic viscosity (ν) of water are $\approx$ 1000 kgm^{-3} and $10^{-6}m^2s^{-1}$. Assume $Re_P < 1$, then

$w_P = (9.8\ ms^{-2})\ (10^{-5})^2\ (2600\text{-}1000\ kgm^{-3})/(18 \times 1000\ kgm^{-3} \times 10^{-6}m^2s^{-1}) = 9 \times 10^{-5}ms^{-1}$.

Check assumption of creeping flow

$Re_P = w_P d/\nu = (9\text{x}10^{-5}\ ms^{-1})(10^{-5}m)/(10^{-6}m^2s^{-1}) = 9 \times 10^{-4} << 1$ check

Example 2: Find the settling (particle) velocity for 1mm diameter quartz sand.
Assume $Re_P < 1$, then from (6)
$w_P = (9.8\ ms^{-2})\ (10^{-3})^2\ (2600\text{-}1000\ kgm^{-3})/(18 \times 1000\ kgm^{-3} \times 10^{-6}m^2s^{-1}) = 9 \times 10^{-1}ms^{-1}$.

Check assumption of creeping flow

$Re_P = w_P d/\nu = (9\text{x}10^{-1}\ ms^{-1})(10^{-3}m)/(10^{-6}m^2s^{-1}) = 900 >> 1$ not creeping flow

Use estimated $Re_P = 900$ to estimate C_D
From (4) C_D ($Re_P = 900$) = 0.47
Then from (4) $w_P = 0.2\ ms^{-1}$
New $Re_P = (0.001m)(0.2ms^{-1})/10^{-6}m^2s^{-1} = 200$

Guess of $Re_P = 900$ does not match resulting $Re_P = 200$. Use new Re_P to repeat process.

Guess $Re_P = 200$; then $C_D = 0.67$; and w_P=0.17 ms^{-1}; yielding $Re_P = 170$.

Guess of $Re_P = 200$ does not match resulting $Re_P = 170$. Repeat once more.

Guess $Re_P = 170$; then $C_D = 0.71$; and $w_P = 0.17\ ms^{-1}$; **yielding $Re_P = 170$**

When the resulting Re_P matches the guessed Re_P within 10%, you can stop.

Effects of Particle Shape

Most quartz grains (common beach sand) are roughly spherical and solid, such that (6) and (3) work well. Bubbles also fit the assumptions of (6) and (3) very well. However, many mineral grains and clays have flat or flake-like structure, *i.e.* not spherical. These particles do not fall straight down, but tend to waft is a zig-zag pattern, like a leaf falling. So (3) and (6) may only be taken as ball-park values, with the deviation in velocity between flake and spherical morphology increasing as Re_P increases. Finally, many particles are not solid, but are aggregates (flocs) of smaller particles, which can be quite porous. When flocs are very porous, their effective density is reduced to a value closer to the water, and w_P is decreased.

Particle Diffusion

If the fluid flow is laminar, the diffusion of particles, like the diffusion of dissolved molecules, depends on the Brownian motion of the fluid molecules. The particle diffusion coefficient, D_P, is described by the

Stokes-Einstein Equation:

$$D_P = \frac{kT}{6\pi\mu\, r}, \qquad (7)$$

where r is the particle radius, T [°K] is the absolute temperature, and k is the Boltzmann constant, $k = 1.381 \times 10^{-23}$ J/°K. This equation is based on a random walk model, in which each step executed by a suspended particle is caused by the impact of a fluid molecule. The impact transfers kinetic energy from the fluid molecule to the particle, such that immediately following the impact the particle has kinetic energy,

$$(1/2)m\, u_o^2 = (1/2)\, k\, T, \qquad (8)$$

where u_o is the initial velocity of the particle after the collision and m is the particle mass. The subsequent motion of the particle is described by the momentum equation,

$$\frac{d}{dt}(mu) = -\frac{1}{2}\rho_F C_D \pi\, r^2 u^2. \qquad (9)$$

Assuming that $Re_P < 1$, $C_D = 24/Re_P$ and (9) becomes

$$m\frac{du}{dt} = -6\pi\mu r u \qquad (10)$$

From which the particle motion can be represented as

$$u(t) = u_o \exp(-t/\tau), \qquad (11)$$

where $\tau = m/(6\pi\mu r)$ is a time-scale describing the duration of motion before the particle returns to rest relative to the mean flow. Using this time scale and the initial velocity, u_o, the distance traveled after one collision is

$$\ell = u_o \tau = \frac{\sqrt{mkT}}{6\pi\mu r}. \tag{12}$$

A suspended particle experiences a continuous sequence of collisions, and after each collision it takes time τ to move a distance ℓ along the line of impact. The net result is a random walk with step size $\Delta x = \ell$ and step time $\Delta t = \tau$ that results in a Fickian Diffusion. The coefficient of diffusion, as defined in Chapter 1, is

$$D_P = \frac{\Delta x^2}{\Delta t} = \frac{\ell^2}{\tau} = \frac{kT}{6\pi\mu r}.$$

Instantaneous Point Source of Particles

In previous chapters we derived solutions for the concentration field created by point sources. With the above considerations, these solutions can be used to describe particle concentration as well. As an example we consider a cloud of N particles released from a height, $z = h$, and $x = y = 0$ into a domain unbounded in x and y. The cloud is advected by a mean velocity, u, and diffused by an isotropic turbulent diffusivity, D. We will assume that any particle that touches the ground ($z = 0$) settles out and cannot be resuspended, so the ground is a perfect absorber. The center of mass for the cloud will be at ($x = ut$, $y = 0$, $z = h-w_P t$), and for the negative image at ($x = ut$, $y = 0$, $z = -h+w_P t$). The particle concentration, n [particles m^{-3}], is

$$\begin{aligned} n = {} & \frac{N}{(4\pi Dt)^{3/2}} \exp\left(-\frac{(x-ut)^2 + y^2 + (z-(h-w_P t))^2}{4Dt} \right) \\ & - \frac{N}{(4\pi Dt)^{3/2}} \exp\left(-\frac{(x-ut)^2 + y^2 + (z+(h-w_P t))^2}{4Dt} \right) \end{aligned} \tag{13}$$

Settling - A sink for suspended particle concentration.

As in the above example, the settling of particles onto a boundary represents a flux of particles out of the fluid domain (a sink). With the positive z-axis pointing upward, the flux at the boundary is $\dot{m} = -w_P C A_H$, where A_H is the horizontal projection of the boundary and C is the concentration in the fluid next to the boundary. In some systems, particles that have settled can be resuspended. Resuspension of particles creates a flux into the fluid domain (a source). The ability of a flow to resuspend particles from the bed depends on the shear stress exerted at the boundary and the physical characteristics of the particles. The relative magnitude of the settling and resuspension determines whether there is a net loss or gain particles from the fluid. In this chapter we ignore resuspension and consider only the settling flux.

Settling in a System with Laminar Flow or Slow Mixing

Consider a simple rectangular system with horizontal area A and depth h and (u, v, w,) = (0, 0, 0). Let z be vertically upward and z = 0 at the bottom. The system has an initial concentration C_o that is uniform throughout the fluid domain. If diffusion is slow compared to settling, then we can neglect diffusion (mixing), and assume that the concentration within the particle cloud is unchanged as settling progresses. Assuming a cloud of uniform particle size and density, the particles will all settle at the same velocity w_P. The particle flux at the bed is due to vertical advection, $\dot{m}(z = 0) = -w_P C_o A$. This flux continues until the entire water depth is cleared of particles. Under these conditions the loss of particle mass, M, follows zeroth-order decay.

(14) $\partial M/\partial t = -w_P C_o A$ = constant and not a function of mass remaining, M.

If we define a depth-averaged concentration as C = M/hA,

(15) $\partial C/\partial t = -(w_P/h)\, Co$,

From which we get,

(16) $C(t) = C_o\,(1 - \frac{w_P}{h}\, t)$, for $t < h/w_P$.

All particles are lost from the system in exactly $T_{settle} = h/w_P$.

Settling in a System with Turbulent Flow or Rapid Mixing

For the same system described above now suppose that mixing is sufficiently rapid to maintain a uniform concentration, C, throughout the system, even as particles are lost to the bed through settling. The flux at the bed is now $\dot{m}(z = 0) = -w_P C A$. Although diffusion (mixing) cannot be neglected in this system, we have assumed that C is uniform, so we may neglect the diffusion terms because $\partial C/\partial z = \partial C/\partial y = \partial C/\partial x = 0$. The mass conservation equation for this system is,

(17) $\partial M/\partial t = Ah\, \partial C/\partial t = -w_P C A$,

from which

(18) $\partial C/\partial t = -(w_P/h)\, C$,

which is a first-order decay, with a rate constant $k[\text{time}^{-1}] = w_P/h$. The particle concentration decays exponentially, with 95% of the initial mass lost in time $3h/w_P$.

Choosing a settling model

The two models described above differ in the relative importance of mixing and settling. We compare these two processes by comparing the time-scale for settling over the depth, $T_{settle} \sim h/w_P$, and the time scale for mixing over the depth, $T_D \sim h^2/D$. If $T_D >> T_{settle}$, the slow-mixing model will apply. If $T_D << T_{settle}$, the fast-mixing model will apply.

Using the scale $D \sim u_* h$, for turbulent channel flow, we find the ratio of time-scales,

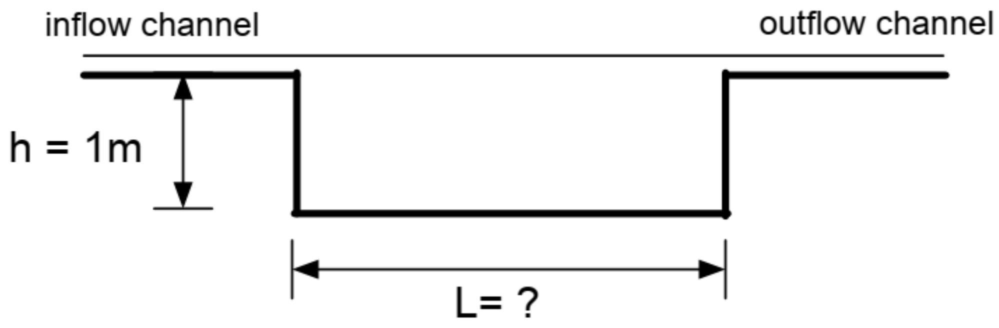
$$(19) \qquad \frac{\text{time - scale for settling over h}}{\text{time - scale for mixing over h}} = \frac{h/w_p}{h^2/D} = \frac{w_p}{u_*}.$$

Then, for turbulent channel flow, if $w_P << u_*$, the turbulence in the water column is strong enough to keep the particles mixed, the fast-mixing model applies, and the suspended sediment load decays exponentially. If $w_P >> u_*$, the turbulence is too weak to mix sediment vertically, the slow-mixing model applies, and the suspended sediment load decays linearly.

Exercise with solutions

Problem 1

You are an environmental manager for a major construction site. The local regulatory authority requires that 85% of all suspended solids be removed from any water that is released off site. To remove the suspended solids you plan to pass the water through a settling tank. Assume that the majority of the suspended solids are silt with a mean diameter d = 62 micron and density $\rho_P = 2.6 \text{gcm}^{-3}$. During typical operations you need to treat 100 m^3 of water per hour, or approximately $Q = 0.03\ m^3s^{-1}$. The available area for digging the settling pond is 2 m wide and can be 1 m deep. How long should the pond be?



Problem 2

Consider a 20-cm-deep water channel with a mean flow of 0.1 cm/s. The flow conditions are laminar. N particles are released instantaneously from a point source located h = 5 cm above the bed. The particles are spherical with diameter d =10 µm and density $\rho_P = 2000\ \text{kgm}^{-3}$. Describe the downstream position and length of the particle patch left on the bed after all particles have settled. For particles of this size, the particle diffusion rate is $10^{-13}\ m^2s^{-1}$.

Problem 3

A tree releases its seeds once per year from pods located at H = 10m above the ground. At this height, the horizontal extent of branches is 4 m^2. The opening of the pods is triggered by a shift in air temperature, and once triggered all seeds are released within five seconds. The seeds are spherical with diameter d = 50 μm and have density $\rho_s = 600\ kgm^{-3}$. During the release a wind is blowing at U = 2 m/s, which you may assume is constant over height. Assume the constant, uniform, turbulent diffusivities in the vertical ($D_Z = 0.1\ m^2s^{-1}$) and horizontal ($D_X = D_Y = 1\ m^2s^{-1}$). The air density is $\rho_a = 1.2\ kgm^{-3}$, and absolute viscosity is $\mu_a = 1.8\text{x}10^{-5}$ N s m^{-2}. Assuming that a total of N seeds are released, write an expression for the seed concentration (seeds m^{-3}) in the air after the release. Describe the longitudinal extent of the seeds on the ground.

Exercise – solution

Answer 1.

First, we find the particle settling velocity, w_P. Assuming $Re_P < 1$, then

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8ms^{-2}\left(62 \text{ x } 10^{-6}m\right)^2(2600 - 1000)kgm^{-3}}{18(1000\ kgm^{-3})(10^{-6}m^2s^{-1})} = 0.0033ms^{-1}.$$

We check our assumption: $Re_P = (62 \text{ x } 10^{-6}m)(0.0033ms^{-1})/(10^{-6}m^2s^{-1}) = 0.2 < 1$. OK.

Second, we determine if the system is laminar or turbulent in order to pick an appropriate settling model. The mean velocity is $u = Q/(b\ h) = 0.03m^3s^{-1}/(2m \text{ x } 1m) = 0.015\ ms^{-1}$. From this we estimate the friction velocity $u_* \approx 0.1u = 0.0015ms^{-1}$. Since $u_* < w_P$, settling will more closely follow the laminar or slow-mixing model. Specifically, with weak mixing the concentration in the water column remains constant even as mass is lost to the bed through settling. We may additionally assume that longitudinal mixing is weak, and thus use a plug-flow type model (see Chapter 2). The flow is considered to be a series of thin fluid slabs, each of longitudinal thickness dx. As each slab enters it contains suspended particles at concentration C_O. As the slab moves downstream at speed u, mass is lost from the slab through settling at the rate $\partial M/\partial t = -w_PC_obdx$. If the slab enters the pond at t = 0 with initial mass $M_o = C_ohbdx$, the mass remaining in the slab for t > 0 is

$M(t) = C_ohbdx - (w_PC_obdx)t$,

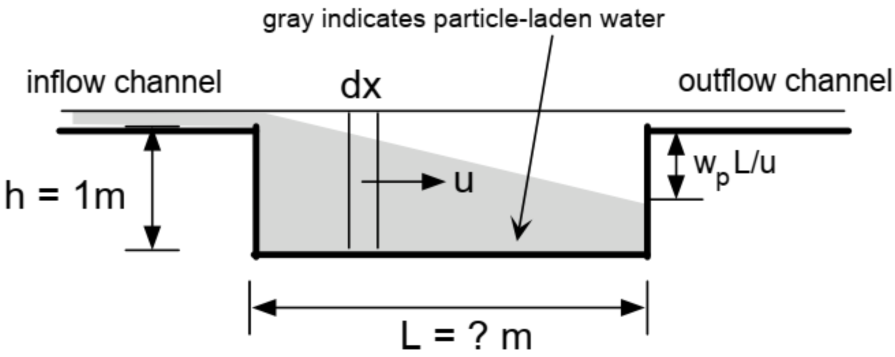
which can also be written,

$(M/M_o) = 1 - (w_P/h)t$.

The slab reaches the outflow at $t = L/u$, at which time we would like $M/M_O = 0.15$. Using these values we can solve for the required L.

$$\frac{M}{M_o} = 1 - \frac{w_P}{h}\frac{L}{u}$$

which gives $L = \dfrac{h\,u\,(1 - M/M_o)}{w_P} = \dfrac{(1\text{m})\,(0.015\text{ms}^{-1})(1 - 0.15)}{0.0033\text{ms}^{-1}} = 3.9\text{ m}$



The above picture shows how the region of particle-laden water evolves across the pond. The picture suggests that if outflow is drawn from the surface, it will be free of particles, *i.e.* the effective removal can be 100%.

Answer 2.

Start by estimating the settling velocity of the particles assuming creeping flow.

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8\text{ms}^{-2}(10^{-5}\text{m})^2(2000 - 998\text{ kgm}^{-3})}{18(10^{-3}\text{kgm}^{-1}\text{s}^{-1})} = 5.5 \times 10^{-5}\text{ms}^{-1}$$

Check assumption of creeping flow. $Re = \rho_F w_P d/\mu_F = 0.0005 < 1$, so creeping flow assumption is OK. Next, find the time-scale at which the center of mass of the particle cloud will settle.

$T_{settle} = h/w_s = 0.05\text{ m} / (5.5 \times 10^{-5}\text{ ms}^{-1}) = 910\text{ s}$.

With this time scale and the mean advection, $u = 0.1$ cm/s (assumed constant over channel depth), the center of the cloud will settle at a distance $x = uT_{settle} = 91$ cm downstream from the source. The footprint of the particle cloud, *i.e.* its length and width, depends on the size of the cloud as it meets the bed. While in suspension, the particles spread in all directions by isotropic diffusion, $D = 10^{-13}\ m^2s^{-1}$. Neglecting the boundary for a moment, at $t = T_{settle}$ the particle cloud will be spherical with diameter $\approx 4\sigma = 4\sqrt{(2 \times 10^{-13} m^2s^{-1} \times 910\ s)} = 5.4 \times 10^{-5}$ m. Because the cloud is distributed vertically over a distance $\approx 4\sigma$, individual particles may settle slightly before or after the mean settling time given above. The range of settling times is $\Delta T = 4\sigma/w_s = 1$ s. With advection u = 0.1cm/s, the range of settling times will spread the particle patch longitudinally over a distance, $\Delta T\ u = 0.1$ cm. In addition, the cloud is already distributed over a longitudinal distance 4σ as it settles. But, the longitudinal spread due to diffusion while in suspension, $4\sigma = 0.005$ cm, is small compared to the longitudinal spread accomplished by the differential settling times (0.1 cm), so we ignore the former. Thus we expect the footprint of the particle cloud to be 0.1 cm long, 0.005 cm wide, and centered 91 cm downstream of the release. Given that the longitudinal distance traveled (91 cm) is much greater than the distribution caused by diffusion (0.1 cm), to first order the particles' fate is determined solely by the settling velocity and current speed, i.e. by advection. We could have predicted this apriori by considering the Peclet number based on the particle settling speed. $Pe = w_P h/D = 3x10^7 >> 1$, indicating that advective transport dominates in this case.

Answer 3.

Start by estimating the fall velocity of the seeds assuming creeping flow

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8ms^{-2}(5x10^{-5}m)^2(600 - 1.4\ kgm^{-3})}{18(1.8x10^{-5}kgm^{-1}s^{-1})} = 0.045\ m/s$$

Checking the assumption of creeping flow for the seed, $Re = \rho_a w_P d/\mu_a = 0.15 < 1$, so the creeping flow assumption is OK. If we assume that the release is instantaneous and occurs at a point, then we can write the following equation for the concentration downstream of the release. The settling velocity contributes an effective advection speed, w_P, such that the seed cloud's center of mass follows the trajectory ($x = Ut$, $y = 0$, $z = H - w_P t$). A negative image is required to make the ground a perfect absorber, such that any seed that touches the ground settles and is removed from the air. The center of mass of the image source follows the trajectory ($x = Ut$, $y = 0$, $z = -H + w_P t$). The first and second lines of the equation are the real and image source, respectively.

$$C(x,y,z,t) = \frac{N}{(4\pi t)^{3/2}\sqrt{D_x D_y D_z}} \exp\left(-\frac{(x-ut)^2}{4D_x t} - \frac{(y)^2}{4D_y t} - \frac{(z-(H-w_P t))^2}{4D_z t}\right)$$

$$- \frac{N}{(4\pi t)^{3/2}\sqrt{D_x D_y D_z}} \exp\left(-\frac{(x-ut)^2}{4D_x t} - \frac{(y)^2}{4D_y t} - \frac{(z-(-H+w_P t))^2}{4D_z t}\right)$$

This equation is valid at distances downstream of the tree for which 1) the travel time is long compared to the release time; and 2) the cloud's horizontal cross-section is large compared to the release area. The first condition is met at $x >> UT_{release} = 10$ m. For $x >> 10$ m, the concentration in the air appears as if released from an instantaneous source. The second condition is met when the horizontal size of the cloud is much greater than the cloud size at release. The tree area is 4m^2. Assuming a round tree, this area can be approximated as a circle of diameter 2 m. Thus the lateral dimension of the release is 2m. The longitudinal dimension of the release is set by the tree scale (2m) and by advection during release (10-m). The release is thus 2 m wide by 12 m long. Since $D_x = D_y$, the more restrictive condition will be on the longitudinal extent of the cloud. The release will appear as a point at distances for which $4\sigma_x = 4\sqrt{(2D_X x/U)} >> 12$m, or $x >> 9$ m.

When the seeds begin to settle, two processes contribute to their longitudinal spread along the ground, 1) the longitudinal extent of the cloud as it settles ($4\sigma_x$) and 2) differences in settling time that result in differences in advected distance. The latter can be estimated by considering the vertical extent of the cloud ($4\sigma_z$). The resulting spread in settling times, $\Delta T = 4\sigma_z/w_s$, results in longitudinal spread of $\Delta x = U\ \Delta T$. We evaluate the relative contribution of these processes at the mean settling time (H/w_P).

$$\frac{\text{longitudinal diffusion}}{\text{differential settling and advection}} = \frac{\sigma_X}{U\,(\sigma_z/w_s)} = \frac{w_s\sqrt{2D_x H/w_s}}{U\sqrt{2D_z H/w_s}} = \frac{w_s}{U}\sqrt{\frac{D_X}{D_z}} = 0.07$$

That this ratio is so small indicates that the longitudinal distribution of seeds on the ground is predominantly determined by the differential settling times coupled to advection, and longitudinal diffusion contributes little additional spread. Now, to estimate the longitudinal distribution on the ground we need to know the time (and thus distance) at which the first and last seed settles. Considering the sketch below.

First seed settles when: $w_P t + 2\sqrt{2\,D_z t} = H$

Last seed settles when: $w_P t - 2\sqrt{2\,D_z t} = H$

Note that by using the 2σ contour to define the edge of the cloud we describe the fate of 95% of the seeds. Solving these two equations via a spreadsheet, the seeds are found to settle between $t = 64$ and 776 s. The seeds are thus distributed over the distances $x = 128$m to 1552 m, a spread of 1424 m. This is an overestimate, because in fact the velocity approaches zero near the ground due to the no-slip condition there. The neglected velocity shear would also contribute shear-dispersion that would augment longitudinally spreading.

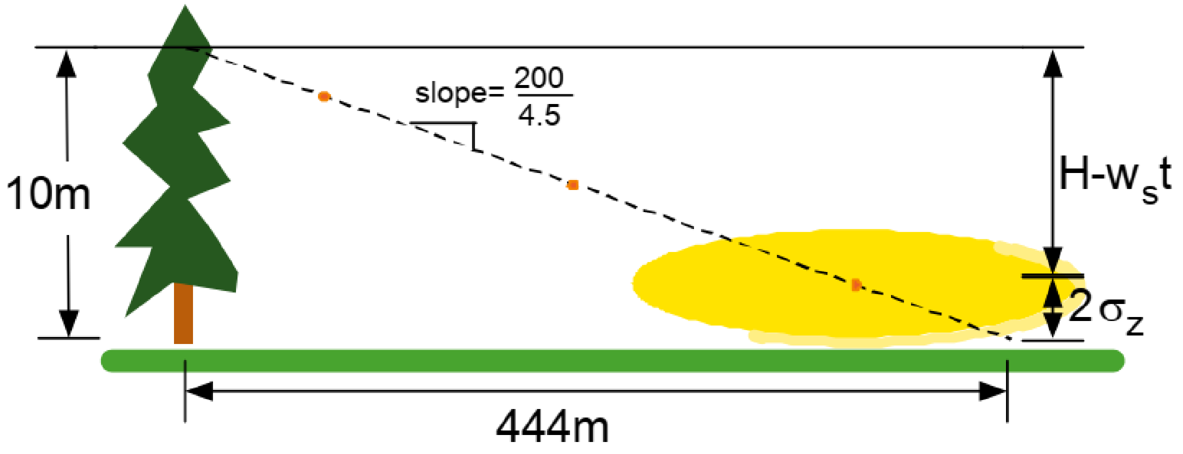


Figure is not to scale

Finally, for comparison, we estimate the longitudinal spread due only to longitudinal diffusion, i.e. neglecting. The extent of the cloud as the center of mass settles, *i.e.* at H/w_P, is $4\sqrt{(2 \times 1 m^2 s^{-1} \times 222 s)} = 84$ m, which is much less than the spread accomplished by advection and differential settling (1.4 km), confirming the scaling analysis above.