Mathematics For Information Technology

Week 2: Algebra : sequences, permutation and combination

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outline

- Intended Learning outcome
- Sequences: arithmetic progression and geometric progression
- Permutation and combination

Intended learning outcomes

- Understanding the basic concept of arithmetic progression, including the formula for finding the n^{th} term and the sum of the first n terms.
- Identifying a geometric sequence and finding the first term, common ratio, and nth term.
- Understanding the concept of permutations and how to calculate them using formulas and techniques.
- Understanding the relationships between permutations, combinations, and other counting techniques.

sequence

- A sequence is a set of numbers, stated in a definite order such that each number can be obtained from the previous number according to some rule i.e.

3,5,7,9,11, … …

1,4,9,16,25, … .

1,2,4,8,16, … .

- Each number of the sequence is called a term

- The "…." at the end of each sequence show that each one could go on indefinitely (infinite).

- However if the sequence is $3,5,7,9,11,\ldots.47.$ such a sequence is called finite

- An expression for n^{th} term (u_n) of a sequence is useful since any specific term of the sequence can be obtained from it.

Example :

- Given a sequence 1,4,9,16,25, … … .

The n^{th} term of the sequence 1,4,9,16,25, … is n^2.

Thus for $n = 1, n = 2, n = 3, n = 4, n = 5$. we obtain the above sequence.

The sum of the terms of a sequence is called a series i.e.

$$3 + 5 + 7 + 9 + 11 + \cdots$$

$$1 + 4 + 9 + 16 + 25 + \cdots$$

Arithmetic progression

- An arithmetic progression (AP) is a sequence of numbers in which each term is obtained by adding a fixed constant number to the previous term.
- This constant number is called the common difference of the arithmetic progression.

- Consider the series $3 + 5 + 7 + 9 + 11 + \cdots$

- The first term of the series is 3 and each subsequent term is obtained by adding a constant value to the previous value by adding .

- Such series are what we call arithmetic progression (AP) with a common difference d and first term a

- The n^{th} term of an arithmetic progression can be calculated using the formula:

$$u_n = a + (n-1)d$$

- where n is the n^{th} term, a is the first term, n is the position of the term in the sequence, and d is the common difference.

Example:

Find the third, tenth, twenty-first terms of the A.P. with first term 6 and common difference 5.

solution

- Using the n^{th} term expression for an A.P

$$u_n = a + (n-1)d$$

$$u_3 = 6 + (3-1)5$$

$$= 6 + 10$$

$$= 16$$

$$u_{10} = 6 + (10 - 1)5$$
$$= 6 + 45$$
$$= 51$$

$$u_{21} = 6 + (21 - 1)5$$
$$= 6 + 100$$
$$= 106$$

Therefore the third, tenth and twenty-first terms are

$16, 51 \; and \; 106$

Sum of the first n terms of an AP

- We write the sum of the n terms of an A.P as s_n, it follows that

$$s_n = a + (a + d) + \cdots + [a + (n - 1)d] \ldots \ldots \ldots \ldots . (1)$$

- Rewriting these terms in the reverse order

$$s_n = [a + (n - 1)d] + \cdots + a \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

Adding (1) and (2)

$$2s_n = [2a + (n-1)d] + \cdots + [2a + (n-1)d]$$

$$2s_n = n[2a + (n-1)d]$$

$$s_n = \frac{n}{2}[2a + (n-1)d]$$

Example: Find s_8 of $2 + 5 + 8 + 11 + \cdots$

$$a = 2, d = 3$$

Using $s_n = \frac{n}{2}(2a + (n-1)d$

$$s_8 = \frac{8}{2}(2 \times 2 + (8-1) \times 3$$

$$= 4(4 + 21)$$

$$= 4 \times 25$$

$$= 100$$

- Find the number of terms in the following A.P.s:

$$50 + 47 + 44 + \ldots + 14,$$

Here we are given

$$a = 50, \qquad d = -3 \; and \; L = 14$$

Therefore to get the number of terms we use the expression for the last term $L = a + (n-1)d$

$$14 = 50 + (n-1) - 3$$

$$-36 = -3(n-1)$$

$$n = 13$$

- Find the sums of the following A.P.s:

$$1 + 3 + 5 + \dots + 101,$$

Here we are given the first term , common difference and the last term

$$a = 1, d = 2 \; and \; L = 101$$

To get the sum, we need to first get the number of terms using last term

$$101 = 1 + (n - 1)2$$

$$100 = 2(n - 1)$$

$$n = 51$$

$$s_{51} = \frac{51}{2}(1 + 101)$$

$$= \frac{51}{2}(102)$$

$$= 51 \times 51$$

$$= 2601$$

- The fifth term of an A.P is 23 and the twelfth term is 37. find the first term, the common difference and the sum of the first eleven terms

$$u_5 = a + (5 - 1)d = 23$$

$$a + 4d = 23 \ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

$$u_{12} = a + (12 - 1)d = 37$$

$$a + 11d = 37 \ldots\ldots\ldots\ldots(ii)$$

Solving the value of a and d using elimination

Since the coefficient of a is the same then we can subtract to eliminate a

$$-7d = -14$$

$$d = 2$$

Substitute d into equation (i)

$$a + 4 \times 2 = 23$$

$$a = 15$$

Therefore the value of the first term is 15 and the common difference is 2

Now to get the sum of the first eleven terms are

$$s_{11} = \frac{11}{2}(2 \times 15 + (11 - 1)2)$$

$$= \frac{11}{2}(30 + 20)$$

$$= 11 \times 25$$

$$= 275$$

- In an arithmetical progression, the thirteenth term is 27, and the seventh term is three times the second term. Find the first term, the common difference and the sum of the first ten terms.

solution

Let the first term be a, and let the common difference be d. Then the thirteenth term is $a + 12d$,

$$a + 12d = 27 \dots\dots\dots (i)$$

The seventh term is $a + 6d$, and the second term is $a + d$, therefore

$$a + 6d = 3(a + d) \dots\dots\dots.(ii)$$

$$\therefore 3d = 2a$$

Substituting in the first equation,

$$a + 8a = 27$$

$$\therefore a = 3$$

and so $d = 2$

To find the sum of the first ten terms, we know that

$$S_n = \frac{n}{2}\{2a + (n - 1)d\} :$$

$$S_{10} = \frac{10}{2}(6 + 9 \times 2)$$

$$= 5 \times 24$$

Therefore the sum of the first ten terms is 120.

Geometric progression

- Another series of common occurrence is the geometrical progression, for example:

$$3 + 6 + 12 + \ldots + 192$$

- In such a progression, the ratio of a term to the previous one is a constant, called the common ratio. Thus, the common ratios of the above progression 2.

- In general if the first term is denoted by a and the ratio of one term to the previous term is r then we can write the series as

$$a + ar + ar^2 + ar^3 + \cdots \ldots$$

- Where the n^{th} term $u_n = ar^{n-1}$ with r the common ratio and a first term

Example

- Find the fifth term of a geometric progression (GP) given below

$$3 + 9 + 27 + 81 + \cdots$$

$$a = 3, \qquad r = 3$$

Using $u_n = ar^{n-1}$

$$u_5 = 3.3^{5-1}$$

$$= 3.3^4$$

$$= 3 \times 81$$

$$= 243$$

Sum of the first n terms of the GP

If the first term of a G.P. is a and the common ratio is r, we may find the sum S_n of the first n terms. The n^{th} term is ar^{n-1}, therefore

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \ldots\ldots\ldots.(i)$$

multiplying equation (i) with r

$$rS_n = ar + ar^2 + \ldots + \mathrm{ar}^n \ldots\ldots\ldots\ldots(ii)$$

Subtracting equation (i) and (ii),

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$s_n = a(\frac{1 - r^n}{1 - r})$$

An alternative formula for the sum of a G.P. is obtained by multiplying numerator and denominator by — 1:

$$s_n = a(\frac{r^n - 1}{r - 1})$$

This is more convenient if r is greater than 1.

- Given a GP below $3 + 6 + 12 + \ldots + 192$.Find the sum of the first 10 terms of a GP.

Solution

Given $a = 3, r = 2$

From $s_n = a(\frac{r^n - 1}{r - 1})$

$$= \frac{3(2^{10} - 1)}{2 - 1}$$

$$= 3(1023)$$

$$s_{10} = 3069$$

- In a geometrical progression, the sum of the second and third terms is 6, and the sum of the third and fourth terms is — 12. Find the first term and the common ratio.

solution

Let the first term be a, and let the common ratio be r.

Then the second term is ar, and the third term is ar^2,

$$\therefore \; ar + ar^2 = 6 \ldots\ldots\ldots\ldots (i)$$

The third term is ar^2, and the fourth term is ar^3, therefore

$$ar^2 + ar^3 = -12 \ldots\ldots\ldots\ldots\ldots (ii)$$

Factorizing the left-hand sides of the equations,

$$ar(1 + r) = 6$$

$$ar^2(1 + r) = -12$$

- We may eliminate a by dividing:

$$\frac{ar(1 + r)}{ar^2(1 + r)} = -\frac{6}{12}$$

$$\frac{1}{r} = -\frac{1}{2}$$

$$r = -2$$

- Substituting $r = — 2$ in $ar(1 + r) = 6$,

$$a(-2)(-1) = 6$$

$$a = 3$$

Permutation and combination

The factorial notation

Let n be an integer then the continued product of the 1st n natural numbers is called n factorial which is denoted by $n!$

$$n! = n(n-1)(n-2)\ldots\ldots 3 \times 2 \times 1$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

$$0! = 1$$

- Evaluate

$$\frac{9!}{2!\,7!} = \frac{9 \times 8 \times 7!}{2 \times 1 \times 7!}$$

$$\frac{9 \times 8}{2}$$

$$36$$

- $8! - 3!$

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1$$

$$40320 - 6$$

$$40314$$

- Find the values of n given that $\frac{n!}{(n-2)!} = 6$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 6$$

$$n(n-1) = 6$$

$$n^2 - n - 6 = 0$$

$$(n-3)(n+2) = 0$$

$$n = 3, n = -2$$

Therefore we consider a positive value. Hence $n = 3$

Permutation

- A permutation is a way of arranging a set of distinct objects in a particular order
- Consider the three letters A, B and C. if these letters are written in a row one after the other, then there are six different possible arrangements

$$\boldsymbol{ABC\ , ACB, BAC, BCA, CAB, CBA}$$

- Each arrangement is a possible permutation of the letters A, B and C and so there are six permutations all together.

- Generally given n objects to be arranged in groups of r chosen from n unlike objects and if r is less than n, then

$$n_{p_r} = P(n, r) = \frac{n!}{(n-r)!},$$

- which is the number of permutations of n different objects taken r at a time.

- **Example**
- Find how many different numbers can be made by using 4 out of the 9 digits.

$$Given\ n = 9, r = 4$$

$$9_{p_4} = \frac{9!}{(9-4)!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{5!}$$

$$= 3024\ ways$$

- In how many ways can letters of the word ATIM be arranged when all letters are taken at a time

$$n = 4, r = 4$$

$$n_{p_r} = \frac{4!}{(4-4)!}$$

$$24 \; ways$$

Permutations of n objects that are not all unlike (repetition)

$$n_{p_r} = \frac{n!}{p!\,q!\,r!}..$$

- How many permutations can be made from the letters of the name SSEKAMATTE taken all at a time

$$n = 10, s = 2, E = 2, k = 1, A = 2, M = 1, T = 2$$

$$= \frac{10!}{2! \times 2! \times 2! \times 2!}$$

$$= 226800\ ways$$

Permutations of like and unlike objects but with some restrictions

- Find in how many ways can the letters of the name ABUBAKARI be arranged in a row when all letters are taken at a time
- In how many of these arrangements when the 2B's are not together

$$ABUBAKARI$$

$$n = 9, A = 3, B = 2$$

$$n_{p_r} = \frac{9!}{3!\,2!}$$

$$= 30240\ ways$$

- For $2B$'s together there will be 8 letters effectively with A appearing thrice, K,R,I,U appearing once

$$2B's\ are\ not\ together$$

$$= total\ no\ of\ arrangements$$

$$- 2B's\ are\ together$$

$$= 30240 - 6720$$

$$= 23520\ ways$$

combinations

- In the last section, attention was given to permutations, where the order of a set of objects was of importance; but in other circumstances, the order of selection is irrelevant.
- When a selection of objects is made with no regard being paid to order, it is referred to as a combination.
- Thus, ABC, ACB, CBA, are different permutations, but they are the same combination of letters.

Example:

suppose four digits 1,2,3,4 are selected in order of 2 digits. Then

$$12, 13, 14, 23, 24, 34$$

Hence we have 6 combinations

In how many ways can r objects be chosen from n unlike objects?

it was shown that there are

$$n!/(n - r)!$$

permutations of r objects chosen from n unlike objects.

Now each combination of r

- Objects can be arranged in $r!$ ways, therefore the number of permutations $= r! \times (the\ number\ of\ combinations)$

$\frac{n!}{(n-r)!} = r! \times$(the number of combinations)

- Hence the number of combinations of r objects chosen from n unlike objects is

$$n_{c_r} = \frac{n!}{(n-r)!\,r!}$$

Example: A mixed hockey team containing 5 men and 6 women is to be chosen from 7men and 9 women. In how many ways can this be done?

solution

Five men can be selected from 7 men in 7_{C_5} ways,

6 women can be selected from 9 women in 9_{C_6} ways.

Now for each of the 7_{C_5} ways of selecting the men, there are 9_{C_6} ways of selecting the women, Therefore there are $7_{C_5} \times 9_{C_6}$ ways of selecting the team.

$$= 7_{C_5} \times 9_{C_6}$$

$$= \frac{7!}{2!\,5!} \times \frac{9!}{3!\,6!}$$

$$= 21 \times 84$$

$$= 1764\ ways$$

Therefore the team can be chosen in 1764 ways.

References

- Sadler, A.J.& Thorning, D.W.S. (2004).Understanding pure mathematics. Oxford university press.
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Next lecture we shall talk about systems of linear equations

Thank you