Mathematics For Information Technology

Week 4: Systems of linear Equations : crammer's rule

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outline

- Intended learning outcome
- Matrix operation i.e. addition, subtraction and multiplication
- Determinants
- Crammer's rule
- Application of crammer's rule

Intended learning outcome

- Use determinants to solve systems of linear equations.
- Understand the concept of a system of linear equations and how to represent it in matrix form.
- Understand Cramer's rule and how to use it to solve a system of linear equations.
- Be able to apply Cramer's rule to solve systems of linear equations with two or three variables.

Matrices

- A matrix is a rectangular array of numbers, variables, or functions arranged in rows and columns.
- Matrices can be used to represent a variety of mathematical objects such as systems of linear equations, transformations, and graphs.
- Matrices can be added, subtracted, and multiplied in a variety of ways, depending on their dimensions and properties

Addition, subtraction and multiplication of matrices

- Find A+B and A-B, given that $A = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$

Solution

$$A + B = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 0-6 \\ 3+6 & -6+7 \\ 5+3 & 1+0 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & -6 \\ 9 & 1 \\ 8 & 1 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2-0 & 0--6 \\ 3-6 & -6-7 \\ 5-3 & 1-0 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & 6 \\ -3 & -13 \\ 2 & 1 \end{pmatrix}$$

- If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ $and\ B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$. Find AB

solution

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 7 + 2 \times 8 \\ 3 \times 7 + 4 \times 8 \\ 5 \times 7 + 6 \times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 16 \\ 21 + 32 \\ 35 + 48 \end{pmatrix}$$

$$AB = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Determinants

- A determinant is a number that is assigned to a square array of numbers in a certain way. i.e. The determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by the formula:

$$\det(A) = ad - bc$$

The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is given by the formula:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

- Find the determinant of the matrix

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

solution

$$= \begin{vmatrix} 2 & 5 & 1 \\ 0 & 3 & 4 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} - 5 \begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 2(3 \times (-2) - 4 \times 2) - 5(0 \times -2 - 4 \times 1)$$
$$+ 1(0 \times 2 - 3 \times 1)$$

$$= -28 + 20 - 3$$
$$= -11$$

- Therefore, the determinant of the matrix is -11.

Crammer's rule

- Cramer's Rule is a method used to solve systems of linear equations that involves finding the determinants of matrices.
- Given a system of linear equations with n variables, we can write it as:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b_1$$
$$b_1x_1 + b_2x_2 + \dots + b_nx_n = b_2$$
$$\dots$$
$$z_1x_1 + z_2x_2 + \dots + z_nx_n = z_n$$

where $a_i, b_i, and\ z_i$ are constants.

- We can write this system of linear equations in matrix form as:

$$A \times = \ b$$

where A is the matrix of coefficients:

$$A = \begin{bmatrix} a_1 & a_2 \ldots\ldots & a_n \\ b_1 & b_2 \ldots\ldots & b_n \\ z_1 & z_2 \ldots & z_n \end{bmatrix}$$

- × is the column vector of variables:

$$\times = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

and b is the column vector of constants:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}$$

- Cramer's Rule states that the solution for x_i can be obtained by dividing the determinant of the matrix obtained by replacing the i^{th} column of A with b, by the determinant of A.

- In other words:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

- where A_i is the matrix obtained by replacing the i^{th} column of A with b.
- For example, if we have the following system of linear equations:

$$2x + y = 5$$

$$x - y = 1$$

- We can write it in matrix form as:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

- The determinant of A is:

$$\det(A) = \big(2 \times (-1)\big) - (1 \times 1)$$

$$= -3$$

- To solve for x, we first find the determinant of the matrix obtained by replacing the first column of A with b:

$$A_i = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\det(A_i) = \left(5 \times (-1)\right) - (1 \times 1)$$

$$= -6$$

- Therefore,

$$x = \frac{\det(A_i)}{\det(A)}$$

$$= \frac{(-6)}{(-3)}$$

$$x = 2$$

- To solve for y, we find the determinant of the matrix obtained by replacing the second column of A with b:

$$A_i = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$$

$$\det(A_i) = (2 \times 1) - (5 \times 1)$$

$$= -3$$

- Therefore,

$$y = \frac{\det(A_i)}{\det(A)}$$

$$= \frac{(-3)}{(-3)}$$

$$y = 1$$

- Solve the following system of linear equations using Cramer's Rule:

$$x + 2y - 3z = 1$$

$$2x - y + z = -1$$

$$3x + 4y - 2z = 3$$

Solution

To solve the system of linear equations using Cramer's Rule, we need to compute the determinants of the coefficient matrix and the matrices obtained by replacing each column of the coefficient matrix with the constant terms.

- The coefficient matrix is:

$$= \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix}$$

- The determinant of the coefficient matrix is:

$$|A| = 1 \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 3 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= -2 + 14 - 33$$

$$= -21$$

- To compute the determinant of the matrix obtained by replacing the first column with the constant terms, we replace the first column of the coefficient matrix with the column of constant terms:

$$= \begin{vmatrix} 1 & 2 & -3 \\ -1 & -1 & 1 \\ 3 & 4 & -2 \end{vmatrix}$$

- The determinant of this matrix is:

$$|A_1| = 1\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= -2 + 2 + 3$$

$$= 3$$

$$A_2 = \begin{vmatrix} 1 & 1 & -3 \\ 2 & -1 & 1 \\ 3 & 3 & -2 \end{vmatrix}$$

- The determinant of this matrix is:

$$|A_2| = 1\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= -1 + 7 - 27$$

$$= -21$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & 4 & 3 \end{vmatrix}$$

- The determinant of this matrix is:

$$|A_3| = 1\begin{vmatrix} -1 & -1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= 1 - 18 + 11$$

$$= -6$$

- Using Cramer's Rule, we can solve for the variables $x, y, and\ z$ as follows:

$$x = \frac{|A_1|}{|A|} = \frac{3}{-21}$$

$$y = \frac{|A_2|}{|A|} = \frac{-21}{-21} = 1$$

$$z = \frac{|A_3|}{|A|} = \frac{-6}{-21} = \frac{2}{7}$$

- Use Cramer's Rule to find the value of k that makes the following system of linear equations inconsistent:

$$2x - y + z = 3$$
$$x + 3y - z = 2$$
$$3x + 2y + kz = 1$$

Solution

Determining the coefficient matrix

$$= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 3 & 2 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

- To use Cramer's Rule, we need to first find the determinant of the coefficient matrix and the determinant of each of the matrices.
- The determinant of the coefficient matrix is:

$$= \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 3 & 2 & k \end{vmatrix}$$

$$= 2(3k + 2) - (-1)(k + 3) + 1(2 - 9)$$

$$= 6k + 4 + k + 3 + 2 - 9$$

$$= 7k$$

- To find the determinant of the matrix formed by replacing the first column with the constants, we get:

$$A_1 = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & k \end{vmatrix}$$

$$= 3(3k + 2) + 1(2k + 1) + 1(4 - 3)$$

$$= 9k + 6 + 2k + 1 + 1$$

$$= 11k + 8$$

$$A_2 = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & k \end{vmatrix}$$

$$= 2(2k + 1) - 3(k + 3) + 1(1 - 6)$$

$$= 4k + 2 - 3k - 9 - 5$$

$$= k - 12$$

$$A_3 = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 2(3-4) + 1(1-6) + 3(2-9)$$

$$= -2 - 5 - 21$$

$$= -28$$

$$x = \frac{11k + 8}{7k}$$

$$y = \frac{k - 12}{7k}$$

$$z = -\frac{28}{7k}$$

- For the system of equations to be inconsistent, the determinant of the coefficient matrix must be 0. Therefore:

$$7k = 0$$

$$k = 0$$

- Therefore the system of linear equation has no solution

Application of crammer's rule

- Suppose a company has a computer network consisting of three servers. Each server has a different processing capacity, measured in transactions per second (tps). The company wants to load-balance the network by assigning a portion of the workload to each server. The load assigned to each server depends on its processing capacity and the total workload. Suppose the total workload is 900 tps, and the load assigned to the servers is as follows:

$$Server\ 1:\ 2x + y + z = 450\ tps$$

$$Server\ 2:\ x + 3y + z = 300\ tps$$

$$Server\ 3:\ x + y + 4z = 150\ tps$$

Use Cramer's rule to find the portion of the workload assigned to each server.

- We can use Cramer's rule to solve this system of equations. The first step is to calculate the determinant of the coefficient matrix:

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 2(11) - 1(3) + 1(-2)$$

$$= 22 - 3 - 2$$

$$= 17$$

- Next, we need to calculate the determinants of the matrices obtained by replacing each column of the coefficient matrix with the column vector on the right-hand side of the equation.

$$= \begin{vmatrix} 450 & 1 & 1 \\ 300 & 3 & 1 \\ 150 & 1 & 4 \end{vmatrix}$$

$$= 450 \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 300 & 1 \\ 150 & 4 \end{vmatrix} + 1 \begin{vmatrix} 300 & 3 \\ 150 & 1 \end{vmatrix}$$

$$= 450(11) - 1(1050) + 1(-150)$$

$$= 4950 - 1050 - 150$$

$$= 3750$$

$$= \begin{vmatrix} 2 & 450 & 1 \\ 1 & 300 & 1 \\ 1 & 150 & 4 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 300 & 1 \\ 150 & 4 \end{vmatrix} - 450 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 300 \\ 1 & 150 \end{vmatrix}$$

$$= 2(1050) - 450(3) + 1(-150)$$

$$= 2100 - 1350 - 150$$

$$= 600$$

$$= \begin{vmatrix} 2 & 1 & 450 \\ 1 & 3 & 300 \\ 1 & 1 & 150 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & 300 \\ 1 & 150 \end{vmatrix} - 1 \begin{vmatrix} 1 & 300 \\ 1 & 150 \end{vmatrix} + 450 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 2(150) - 1(-150) + 450(-2)$$

$$= 300 + 150 - 900$$

$$= -450$$

$$x = \frac{3750}{17} = 220.6$$

$$y = \frac{600}{17} = 35.3$$

$$z = -\frac{450}{17} = -26.5$$

References

- Strang, G. (2016). Linear algebra and its applications (5th ed.). Cengage Learning
- Davis C. L. (1988). Linear algebra and its applications. University of Maryland. Addison-wesley publishing company.

End of lecture 4

Next topic: trigonometry

Thank you