Mathematics For Information Technology

Week 6: Trigonometry: Compound and half angle formulae

Nagulama Moses

Lecturer

Department of Information Technology

Kumi University

Email: mnagulama@gmail.com

outline

- Intended learning outcome
- Pythagoras theorem
- Compound angles
- Double angles
- Half angles

Intended learning outcome

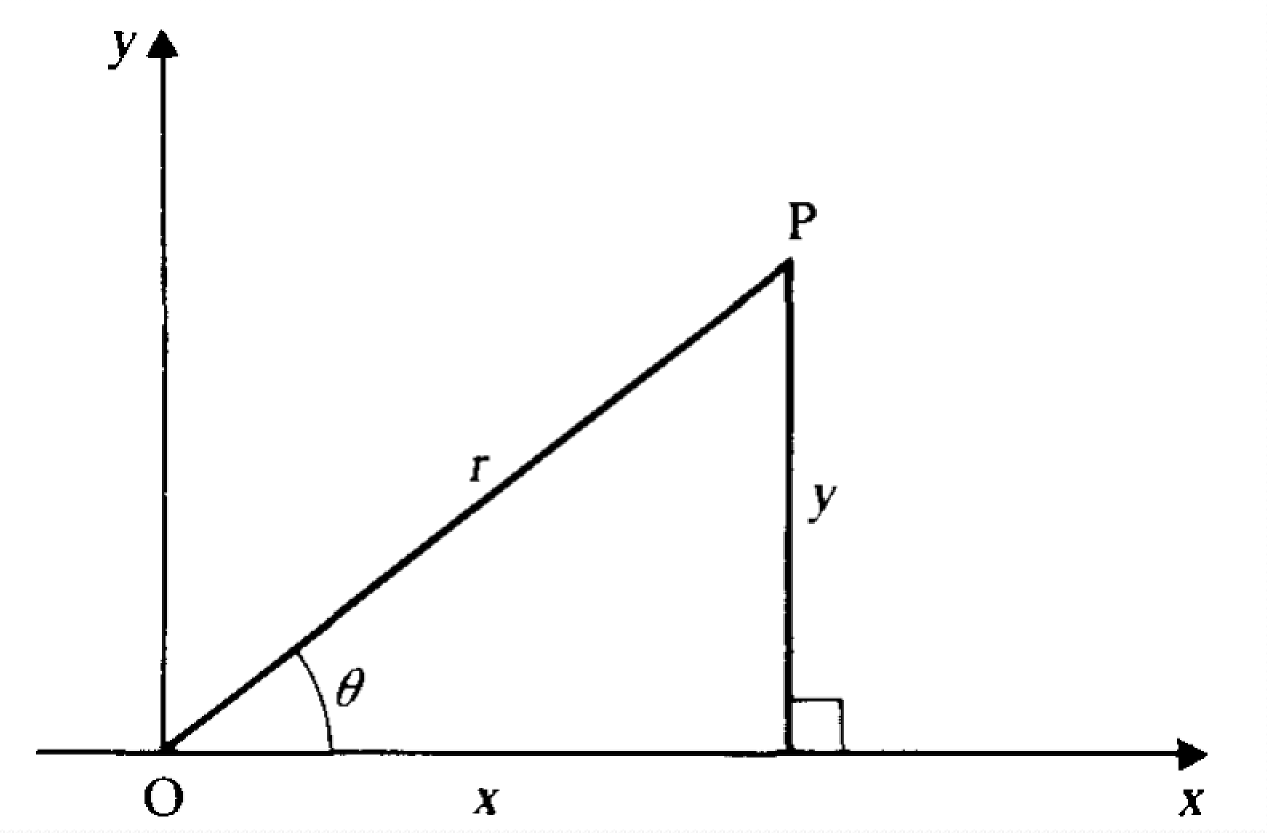
Students should be able to understand and apply the formulas that relate the trigonometric functions of compound angles, such as $\sin(A+B), \cos(A+B), and \tan(A+B)$.

Students should be able to use the formulas for trigonometric functions of compound angles to solve problems involving angles in various contexts.

Students should be able to understand and apply the formulas that relate the trigonometric functions of half angles, such as $\sin(\theta/2), \cos(\theta/2), and \tan(\theta/2)$.

Pythagoras theorem

- Consider the figure below



Using Pythagoras theorem

$$x^2 + y^2 = r^2 \dots\dots\dots\dots. (***)$$

But $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$, so we divide by r^2 obtaining

$$x = rcos\theta, y = rsin\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$$\cos^2\theta + \sin^2\theta = 1 \dots\dots\dots\dots\dots\dots\dots (1)$$

Two similar identities can be deduced from this.

Dividing equation (1) through by $\cos^2\theta$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta \ldots\ldots\ldots\ldots..(2)$$

Dividing the original identity by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = cosec^2\theta \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Example

Solve the equation $1 + cos\theta = 2\sin^2\theta$, for values of θ between $0° \; and \; 360°$.

Solution

$$1 + cos\theta = 2(1 - \cos^2\theta)$$

$$1 + cos\theta = 2 - 2\cos^2\theta$$

$$2\cos^2\theta + cos\theta - 1 = 0$$

$$2\cos^2\theta + 2cos\theta - cos\theta - 1 = 0$$

$$2cos\theta(cos\theta + 1) - (cos\theta + 1) = 0$$

$$(cos\theta + 1)(2cos\theta - 1) = 0$$

$$cos\theta = -1$$

$$\theta = 180^0$$

$$cos\theta = \frac{1}{2}$$

$$\theta = 60^0, 300^0$$

$$\therefore \quad \theta = 60^0, 180^0, 300^0$$

Simplify $\frac{1}{\sqrt{(x^2-a^2)}}$ when $x = a\, cosec\theta$.

Solution

$$= \frac{1}{\sqrt{a^2 cosec^2\theta - a^2}}$$

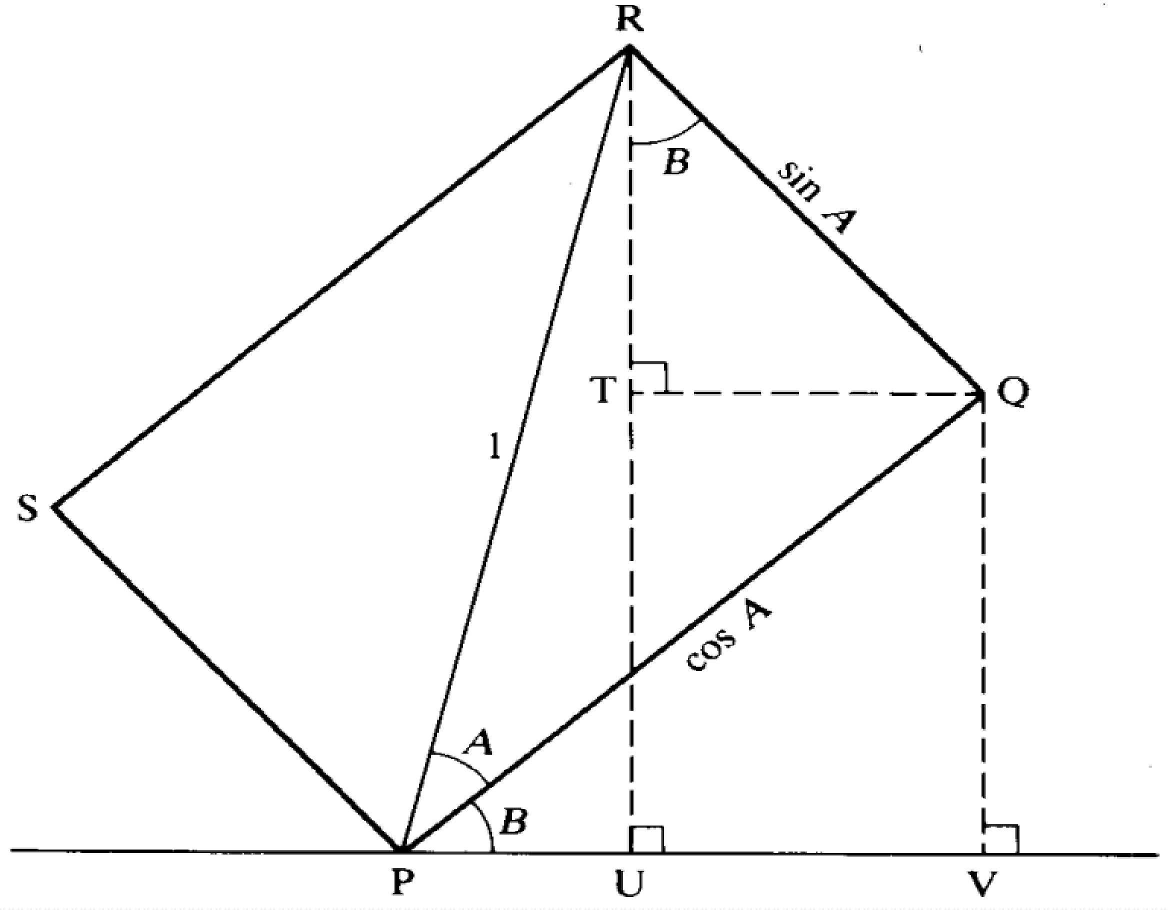
$$= \frac{1}{\sqrt{a^2(cosec^2\theta - 1)}}$$

$$= \frac{1}{a\sqrt{\cot^2\theta}}$$

$$= \frac{1}{acot\theta}$$

Compound angles

- Consider the figure below



- What is the height of R above P?
- One way to find this out is to drop a perpendicular RU from R to the horizontal through P, then from the triangle RPU,

$$RU = \sin(A + B) \ldots\ldots *$$

- Alternatively, since $RQ = \sin A, PQ = cosA$ and angle $QRU = B$, the height of R above P can be found in two parts.

First, the height of R above Q, $RT = \sin A\ cosB$.

Secondly, the height of Q above P, $QV = cosA \sin B$.

$$RT + QV = sinAcosB + cosAsinB \; \ldots .**$$

Thus, equating the height of R above P obtained in the two ways

Equating * and **

$$\sin(A + B) = \sin A\, cosB + cosA \sin B \; \ldots \ldots (1)$$

How far to the right of P is R?

- In triangle RPU, $PU = cos(A + B)$.
- Alternatively, the distance of Q to the right of P, $PV = cosA\ cosB$,

The distance of R to the left of Q, $QT = \sin A \sin B$.

- So, equating the distance of R to the right of P obtained in these

two ways

$$\cos(A + B) = cosA\ cosB - \sin A \sin B \ldots\ldots\ldots.(2)$$

- The height of R above P is now $\sin(A - B)$.
- R is a distance $\sin A \cos B$ above Q, but Q is a distance $\cos A \sin B$ below P,
- Therefore

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \ldots\ldots\ldots (3)$$

- Further, R is a distance $\cos(A - B)$ to the right of P.
- Q is a distance $\cos A \cos B$ to the right of P, but R is now a distance $\sin A \sin B$ to the right of Q,
- Therefore

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \ldots\ldots. (4)$$

- We can deduce more two identities from the above four
- From (1) and (2)

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\, cosB\ +cosA\sin B}{cosA\ cosB\ —\sin A\sin B}$$

$$\tan(A+B)=\frac{tanA+tanB}{1-tanAtanB}\ldots\ldots\ldots\ldots..(5)$$

- From (3) and (4)

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$

$$=\frac{\sin A\, cosB\ \ -cosA\sin B}{cosA\ cosB+\sin A\sin B}$$

$$\tan(A-B)=\frac{tanA-tanB}{1+tanAtanB}\ldots\ldots\ldots\ldots\ldots..(6)$$

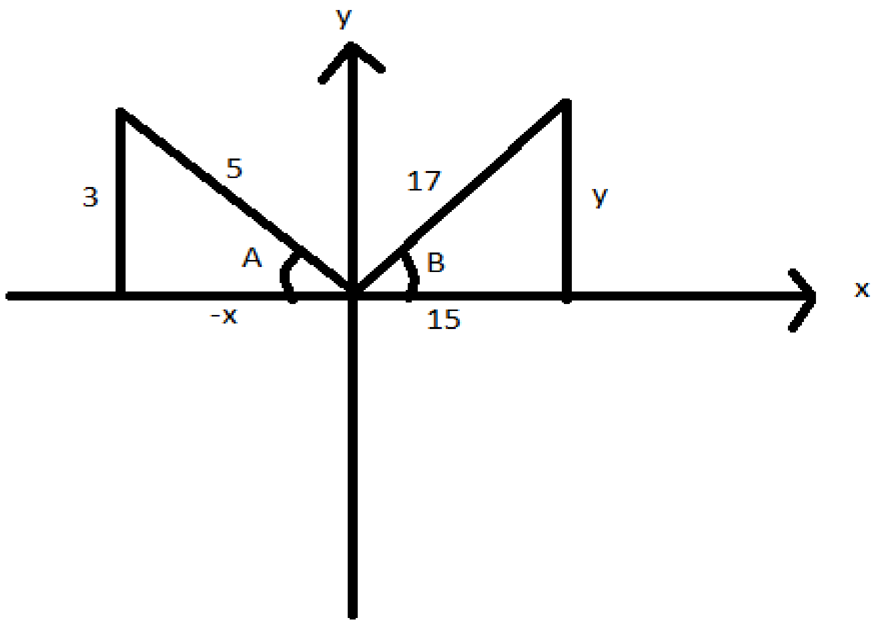
Example

If $\sin A = \frac{3}{5}$ and $\cos B = \frac{15}{17}$, where A is obtuse and B is acute, find the exact value of sin(A + B), cos(A+B) and tan(A+B).

solution

$$\sin(A + B) = sinAcosB + cosAsinB$$

- Using Pythagoras theorem to find $cosA$, and $sinB$



$$x^2 + 9 = 25$$

$$x^2 = 16$$

$$x = 4$$

$$15^2 + y^2 = 17^2$$

$$y^2 = 64$$

$$y = 8$$

$$sinA = \frac{3}{5}, \qquad cosA = -\frac{4}{5}$$

$$sinB = \frac{8}{17}, \qquad cosB = \frac{15}{17}$$

$$\sin(A+B) = sinAcosB + cosAsinB$$

$$= \frac{3}{5} \times \frac{15}{17} + \left(-\frac{4}{5} \times \frac{8}{17}\right)$$

$$= \frac{45}{85} - \frac{32}{85}$$

$$\sin(A+B) = \frac{13}{85}$$

$$\cos(A + B) = cosAcosB - sinAsinB$$

$$= -\frac{4}{5} \times \frac{15}{17} - \left(\frac{3}{5} \times \frac{8}{17}\right)$$

$$= -\frac{60}{85} - \frac{24}{85}$$

$$\cos(A + B) = -\frac{84}{85}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{13}{85}\div\frac{-84}{85}$$

$$=\frac{13}{85}\times\frac{-85}{84}$$

$$\tan(A+B)=-\frac{13}{84}$$

Solve the equation $cos\theta cos20^0 + sin\theta sin20^0 = 0.75$ for values of θ from 0^0 to 360^0

Solution

$$cos\theta cos20^0 + sin\theta sin20^0 = 0.75$$

$$\cos(\theta - 20) = 0.75$$

$$\theta - 20 = \cos^{-1}(0.75)$$

$$\theta - 20 = 41.41^0, 318.59^0$$

$$\theta = 61.41^0, 338.59^0$$

Double angles

- Double angles are obtained from compound angles by setting A=B

$$\sin 2A = sinAcosA + cosAsinA$$

$$sin2A = 2sinAcosA \ldots \ldots .. (1)$$

$$cos2A = cosAcosA - sinAsinA$$

$$= \cos^2 A - \sin^2 A \ldots \ldots \ldots .. (2)$$

Identity (2) can result into

$$cos2A = 1 - 2\sin^2 A$$

$$cos2A = 2\cos^2 A - 1$$

Example

Solve the equation $3\cos 2\theta + sin\theta = 1$, for values of θ from $0° \ to \ 360°$ inclusive.

Solution

$$3(1 - 2\sin^2\theta) + sin\theta = 1$$

$$3 - 6\sin^2\theta + sin\theta = 1$$

$$6\sin^2\theta - sin\theta - 2 = 0$$

$$6\sin^2\theta + 3sin\theta - 4sin\theta - 2 = 0$$

$$3sin\theta(2sin\theta + 1) - 2(2sin\theta + 1) = 0$$

$$(2sin\theta + 1)(3sin\theta - 2) = 0$$

$$sin\theta = -\frac{1}{2}$$

$$\theta = -30^0, 210^0, 330^0$$

$$sin\theta = \frac{2}{3}$$

$$\theta = 41.8^0, 138.2^0$$

$$\therefore \quad \theta = 41.8^0, 138.2^0, 210^0, 330^0$$

- Find the values of $\sin 2\theta$ and $\cos 2\theta$. when $sin\theta = \frac{3}{5}$ where θ is an acute angle

Solution

Using Pythagoras theorem

$$x^2 = 25 - 9$$

$$x^2 = 16$$

$$x = 4$$

Therefore $sin\theta = \frac{3}{5}, cos\theta = \frac{4}{5}$

$$sin2\theta = 2sin\theta cos\theta$$

$$= 2 \times \frac{3}{5} \times \frac{4}{5}$$

$$sin2\theta = \frac{12}{5}$$

$$cos2\theta = \cos^2\theta - \sin^2\theta$$

$$cos2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$$

$$cos2\theta = \frac{16}{25} - \frac{9}{25}$$

$$cos2\theta = \frac{7}{25}$$

Half angles

From

$$sinA = \sin\left(\frac{A}{2} + \frac{A}{2}\right) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$cosA = \cos\left(\frac{A}{2} + \frac{A}{2}\right) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$

Which results to

$$cosA = 2\cos^2\frac{A}{2} - 1$$

$$cosA = 1 - 2\sin^2\frac{A}{2}$$

Example

If $\sin x = \frac{3}{5}$ and x is in the second quadrant, find the exact value of $\sin(\frac{x}{2})$ using the half-angle formula for sine.

Solution

- From

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$cos x = 1 - 2\sin^2\frac{x}{2}$$

$$\frac{4}{5} = 1 - 2\sin^2\frac{x}{2}$$

$$\sin^2\frac{x}{2} = \frac{1}{10}$$

$$\sin\left(\frac{x}{2}\right) = \pm\frac{1}{\sqrt{10}}$$

- Since sine is positive in the second quadrant then

$$\sin\left(\frac{x}{2}\right) = \frac{1}{\sqrt{10}}$$

Reference

- Sadler, A.J.& Thorning, D.W.S. (2004).Understanding pure mathematics. Oxford university press.
- Backhouse, J.K.& Houldsworth, S.P.T.(1985).Pure mathematics 1. PEARSON EDUCATION LIMITED.
- Stewart, J. (2015). Calculus: Early transcendentals (8th ed.). Cengage Learning.

Thank you for listening to me

Next lecture: Derivatives of trigonometric functions