Mathematics For Information Technology

Week 7: Differentiation: Derivatives of trigonometric functions

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outline

- Intended learning outcome
- Sum and product formulae
- Inverse trigonometric functions
- Differentiation from first principles
- Derivative of trigonometric functions
- Derivatives of inverse trigonometric functions

Intended learning outcomes

- Differentiate the trigonometric functions with respect to their arguments.
- Apply the derivatives of the other trigonometric functions, such as tangent, cotangent, secant, and cosecant, to solve calculus problems.
- Understand the derivatives of inverse trigonometric functions, such as arcsin, arccos, and arctan.

Sum and product formulae

- Consider the following

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B$$

$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

Here, the right-hand sides of the identities are in factors,

$$P = A + B \; and \; Q = A — B$$

Adding,

$$P + Q = 2A$$

$$;.A = \frac{P + Q}{2}$$

- Subtracting,

$$P - Q = 2B$$

$$:.B = \frac{P - Q}{2}$$

- Substituting into the four identities above,

$$\cos P + \cos Q = 2\cos\left(\frac{(P+Q)}{2}\right)\cos(\frac{P-Q}{2})$$

$$cosP - cosQ = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

$$sinP + sinQ = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$sinP - sinQ = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Example: Solve the equation $sinx + sin5x = sin3x$ for $0^0\ to\ 180^0$ inclusive

solution

$$sinx + sin5x = sin3x$$

$$2sin3xcos2x = sin3x$$

$$2sin3xcos2x - sin3x = 0$$

$$sin3x(2cos2x - 1) = 0$$

$$sin3x = 0 \text{ or } 2cos2x - 1 = 0$$

$$3x = 0^0, 180^0, 360^0, 540^0$$

$$x = 0^0, 60^0, 120^0, 180^0$$

$$cos2x = \frac{1}{2}$$

$$2x = 60^0, 300^0, 420^0$$

$$x = 30^0, 150^0, 210^0$$

$$x = 0^0, 30^0, 60^0, 120^0, 150^0, 180^0$$

Inverse trigonometric functions

- The functions $f(x) = sinx, g(x) = cosx$ and $h(x) = tanx$ are many to many and therefore have no inverse functions.
- However if we restrict the sine function to the domain $-90 \leq x \leq 90$,
- The cosine function to the domain $0^0 \leq x \leq 180$ and
- The tangent function to the domain $-90^0 < x < 90^0$, the functions are one to one

- We can then consider the inverse functions $f^{-1}(x) = \sin^{-1} x, g^{-1}(x) = \cos^{-1} x$ and $h^{-1}(x) = \tan^{-1} x$
- Thus
- $\sin^{-1} x$ is defined as the angle θsuch that$-90 \leq x \leq 90$
- $\cos^{-1} x$ is defined as the angle θsuch that $0^0 \leq x \leq 180$
- $\tan^{-1} x$ is defined as the angle θ such that $-90 \leq x \leq 90$

Example 1: Without the use of mathematical tables or a calculator, find $tan\theta$ if $\theta = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{7}{24}$

solution

- Let $\alpha = \tan^{-1}\frac{5}{12}, tan\alpha = \frac{5}{12}$

$$\beta = \tan^{-1}\frac{7}{24}, tan\beta = \frac{7}{24}$$

- Thus $\theta = \alpha + \beta$

$$tan\theta = \tan(\alpha + \beta)$$

$$= \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$$

$$tan\theta = \frac{\frac{5}{12} + \frac{7}{24}}{1 - \frac{5}{12}\frac{7}{24}}$$

$$tan\theta = \frac{204}{253}$$

Example 2: Solve the equation $\cos^{-1} x + \cos^{-1} x\sqrt{3} = \frac{\pi}{2}$

Solution

Let $\alpha = \cos^{-1} x\,, cos\alpha = x$

$$\beta = \cos^{-1} x\sqrt{3}\,, cos\beta = x\sqrt{3}$$

Thus $\alpha + \beta = \frac{\pi}{2}$

Introducing cos on both sides

$$\cos(\alpha + \beta) = \cos\left(\frac{\pi}{2}\right)$$

$$cos\alpha cos\beta - sin\alpha sin\beta = 0$$

$$x.x\sqrt{3} - \sqrt{(1-x^2)(1-3x^2)} = 0$$

$$x^2\sqrt{3} = \sqrt{(1-x^2)(1-3x^2)}$$

Squaring both sides

$$3x^4 = 1 - 4x^2 + 3x^4$$

$$4x^2 = 1$$

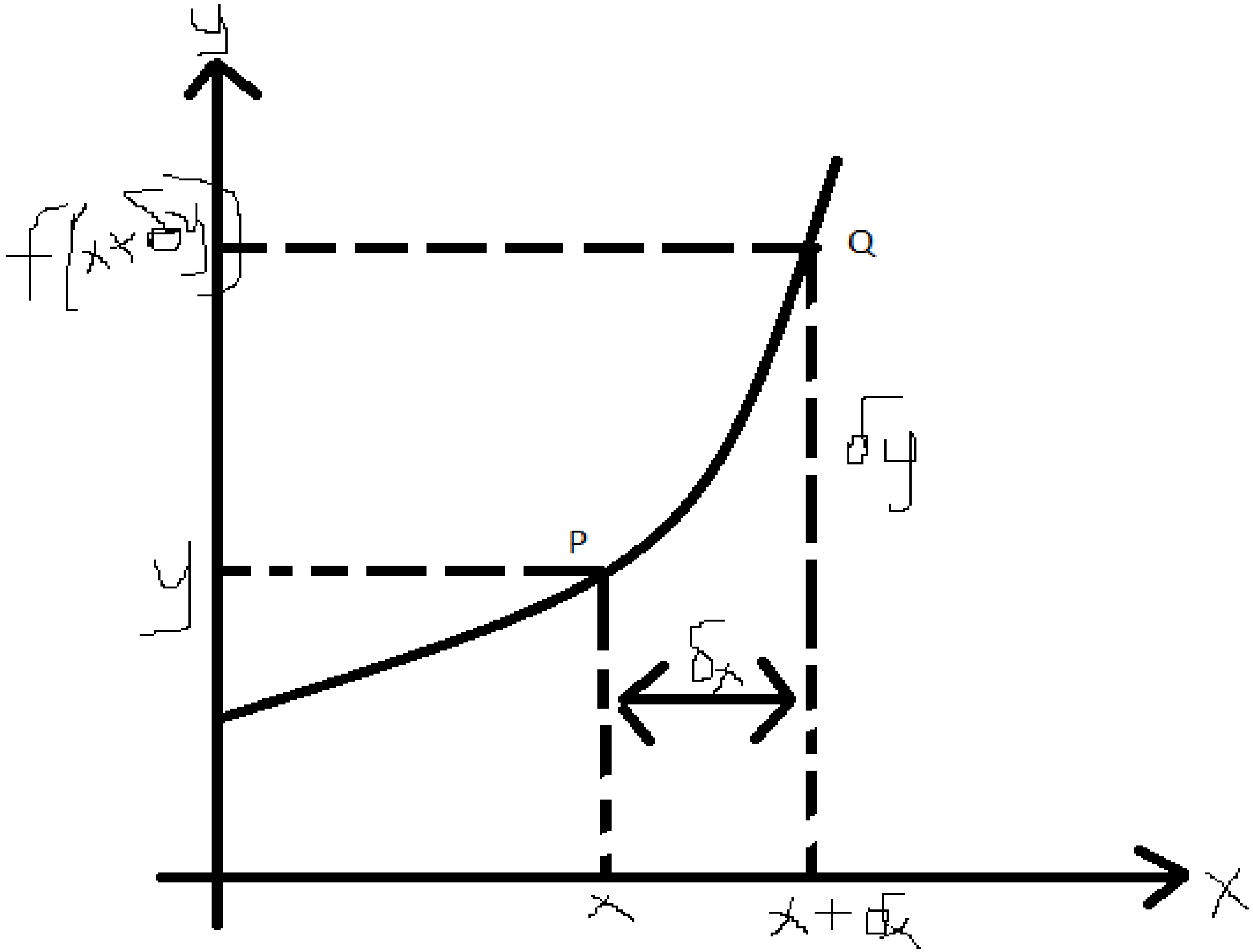
$$x = \pm\frac{1}{2}$$

Differentiation from first principles

- This is the process of calculating the ratio of the incremention change in a function y of x to the incremental change in x

- As away of determining an expression for $\frac{dy}{dx}$ and then finding the limit value of this ratio as δx approaches zero, the unit found is generally denoted by $\frac{dy}{dx}$ and it is called the differential coefficient of y with respect to x

- All notation for differential coefficients are $\frac{dy}{dx}$, $f'(x)$, $\frac{d(f(x))}{dx}$
- Differentiation from first principles can be done as below
- Consider the curve $y = f(x)$ as below and let $p(x, y)$ be a point on a curve $y = f(x)$


Q
P

$$\frac{\delta y}{\delta x} = \frac{[f(x+\delta x) - f(x)]}{x + \delta x - x}$$

$$\frac{\delta y}{\delta x} = \frac{[f(x+\delta x) - f(x)]}{\delta x}$$

Introducing limits

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} \left(\frac{[f(x+\delta x) - f(x)]}{\delta x}\right),$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)]}{\delta x}$$

Example 1: differentiate $y = x^2$ using first principles

solution

Using the definition $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x)-f(x)]}{\delta x}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[(x + \delta x)^2 - x^2]}{\delta x}$$

$$= \lim_{\delta_x \to 0} \frac{[x^2 + 2x\delta_x + \delta_x^2 - x^2]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{[2x\delta_x + \delta_x^2]}{\delta_x}$$

$$= 2x$$

Example 2: differentiate $f(x) = 4x + 2x^2$

Using the definition of first principles

$$f'(x) = \lim_{\delta_x \to 0} \frac{[4(x + \delta_x) + 2(x + \delta_x)^2 - 4x - 2x^2]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{[4x + 4\delta_x + 2x^2 + 4x\delta_x + 2\delta_x^2 - 4x - 2x^2]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{[4\delta_x + 4x\delta_x + 2\delta_x^2]}{\delta_x}$$

$$= 4 + 4x$$

Derivative of trigonometric functions

Differentiate the following using first principles

1) $y = sinx$

solution

From $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)]}{\delta x}$

$$\frac{dy}{dx} = \lim_{\delta_x \to 0} \frac{[\sin(x + \delta_x) - sinx]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{\left[2\cos\left(\frac{(x + \delta_x + x)}{2}\right)\sin\left(\frac{x + \delta_x - x}{2}\right)\right]}{\delta_x}$$

But $\sin\left(\frac{\delta_x}{2}\right) \to \frac{\delta_x}{2}$

$$= cosx$$

2) $y = cosx$

solution

From $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)]}{\delta x}$

$$\frac{dy}{dx} = \lim_{\delta_x \to 0} \frac{[\cos(x + \delta_x) - cosx]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{\left[-2\, sin\left(\frac{(x + \delta_x + x)}{2}\right) \sin\left(\frac{x + \delta_x - x}{2}\right)\right]}{\delta_x}$$

But $\sin\left(\frac{\delta_x}{2}\right) \rightarrow \frac{\delta_x}{2}$

$$= -sinx$$

3) $y = cosecx = \frac{1}{sinx}$

solution

From $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x)-f(x)]}{\delta x}$

$$\frac{dy}{dx} = \lim_{\delta_x \to 0} \frac{[1/\sin(x+\delta_x) - 1/sinx]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{[(sinx - \sin(x+\delta_x)/\sin(x+\delta_x)\, sinx]}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{\left[2\cos\left(\frac{x+x+\delta_x}{2}\right)\sin\left(\frac{x-x-\delta_x}{2}\right)\right]}{\sin(x+\delta_x) sinx\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{\left[-\cos\left(x + \frac{\delta_x}{2}\right)\right]}{\sin(x + \delta_x)\, sinx}$$

$$= -\frac{cosx}{\sin^2 x}$$

$$= -cosecxcotx$$

4) $y = \sin^2 x$

solution

From $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[f(x+\delta x)-f(x)]}{\delta x}$

$$\frac{dy}{dx} = \lim_{\delta_x \to 0} \frac{[\sin^2(x+\delta_x) - \sin^2 x]}{\delta_x}$$

$$\lim_{\delta_x \to 0} \frac{(\sin(x+\delta_x) + sinx)(\sin(x+\delta_x) - sinx)}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{2\cos\left(\frac{x + x + \delta_x}{2}\right)\sin\left(\frac{x + \delta_x - x}{2}\right).2\sin\left(\frac{x + \delta_x + x}{2}\right)\cos(\frac{x + \delta_x - x}{2})}{\delta_x}$$

$$= \lim_{\delta_x \to 0} \frac{2\cos\left(x + \frac{\delta_x}{2}\right)\sin\left(\frac{\delta_x}{2}\right).2\sin\left(x + \frac{\delta_x}{2}\right)\cos\left(\frac{\delta_x}{2}\right)}{\delta_x}$$

$$= 2cosxsinx$$

Differentiation of Inverse Trigonometric functions

Example 1: differentiate $y = \sin^{-1} x$

solution

- $x = siny$ and $\frac{dx}{dy} = cosy \ldots \ldots \ldots \ldots (1)$

- Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, and $\sin^2 y + \cos^2 y = 1$

- We can rearrange (1) to give $\frac{dy}{dx} = \frac{1}{\sqrt{(1-\sin^2 y)}}$

$$= \frac{1}{\sqrt{(1 - x^2)}}$$

- Thus, if $y = \sin^{-1} x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$

Example 2: differentiate $y = \cos^{-1} x$

solution

$$y = \cos^{-1} x$$

- Then $x = cosy$ and $\frac{dx}{dy} = -siny \dots\dots\dots\dots (1)$

- Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, and $\sin^2 y + \cos^2 y = 1$

- We can rearrange (1) to give $\frac{dy}{dx} = \frac{-1}{\sqrt{(1-\cos^2 y)}}$

$$= \frac{-1}{\sqrt{(1 - x^2)}}$$

- Thus, if $y = \cos^{-1} x$, then $\frac{dy}{dx} = \frac{-1}{\sqrt{(1-x^2)}}$

Example 3: differentiate $y = \tan^{-1} x$

solution

$$y = tan^{-1} x$$

- Then $x = tany$ and $\frac{dx}{dy} = \sec^2 y \ldots\ldots\ldots\ldots (1)$

- Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, and $1 + \tan^2 y = \sec^2 x$

- We can rearrange (1) to give $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$$= \frac{1}{1 + x^2}$$

- Thus, if $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

References

- Sadler, A.J.& Thorning, D.W.S. (2004).Understanding pure mathematics. Oxford university press.
- Backhouse, J.K.& Houldsworth, S.P.T.(1985).Pure mathematics 1. PEARSON EDUCATION LIMITED.

Thank you for listening

Next lecture we shall look at Gradient of a curve, Chain rule, Products and quotients