Mathematics For Information Technology

Week 8: Differentiation: Gradient of a curve, Chain rule, Products and quotients

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outline

- Intended learning outcome
- Gradient of a curve
- Function of a function (chain rule)
- Product rule
- Quotient rule

Intended learning outcomes

- Define gradient of the curve
- Define the product rule as a method for differentiating the product of two functions
- Define the chain rule as a method for differentiating composite functions
- Understand the quotient rule and how it can be used to differentiate the quotient of two functions

Gradient of a curve

- Gradient of a curve refers to the gradient of the tangent to the curve at a given point.
- The gradient of a curve is obtained by differentiating the curve given the curve $y = ax^n + b$
- Its gradient is given as $\frac{dy}{dx} = nax^{n-1}$

Example : Find the $y - coordinates$ and the gradient at the point on the following curves corresponding to the given value of x

1) $y = x^2 - 2x + 1\,;\;\; x = 2$

Solution

$$y = x^2 - 2x + 1$$

$$y = (2)^2 - 2(2) + 1$$

$$y = 4 - 4 + 1$$

$$y = 1$$

To find the gradient, we are going to differentiate y wrt x

$$\frac{dy}{dx} = 2x - 2$$

At $x = 2$

$$\frac{dy}{dx} = 2(2) - 2$$

$$\frac{dy}{dx} = 2$$

2) $y = (x + 2)(x + 4);\; x = -1$

solution

$$y = (-1 + 2)(-1 + 4)$$
$$y = 1 \times 3$$
$$y = 3$$

To find the gradient, we differentiate y with respect to x

$$y = x^2 + 6x + 8$$
$$\frac{dy}{dx} = 2x + 6$$

but $x = -1$

$$\frac{dy}{dx} = 2(-1) + 6$$
$$\frac{dy}{dx} = 4$$

3) $y = 3x^2 - 2x^3$; $x = -2$

solution

$$y = 3(-2)^2 - 2(-2)^3$$

$$y = 12 + 16$$

$$y = 28$$

To obtain the gradient we differentiate y with respect to x

$$\frac{dy}{dx} = 6x - 6x^2$$

But at $x = -2$

$$\frac{dy}{dx} = 6(-2) - 6(-2)^2$$

$$\frac{dy}{dx} = -12 - 24$$

$$\frac{dy}{dx} = -36$$

Example: Find the coordinates of the points on the following curve at which the gradient has the given values

1) $y = x^2, \ \ gradient = 8$

Solution

$$\frac{dy}{dx} = 2x$$

But $\frac{dy}{dx} = 8$

$$2x = 8$$

$$x = 4$$

When $x = 4$ then

$$y = (4)^2$$

$$y = 16$$

The coordinates are $(4, 16)$

2) $y = x^3 - 6x^2 + 4; \quad gradient = -12$

Solution

$$\frac{dy}{dx} = 3x^2 - 12x$$

But $\frac{dy}{dx} = -12$

$$3x^2 - 12x = -12$$

$$3x^2 - 12x + 12 = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2)=0$$

$$x=2\ or\ x=2$$

When $x=2$

$$y=(2)^3-6(2)^2+4$$

$$y=8-24+4$$

$$y=-12$$

The coordinates are $(2,-12)$

3) $y = x(3 - x); gradient = 0$

solution

$$y = 3x - x^2$$

$$\frac{dy}{dx} = 3 - 2x$$

But $\frac{dy}{dx} = 0$

$$3 - 2x = 0$$

$$x = \frac{3}{2}$$

To obtain the coordinates

$$y = 3\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2$$

$$y = \frac{9}{2} - \frac{9}{4}$$

$$y = \frac{9}{4}$$

Therefore the coordinates are $\left(\frac{3}{2}, \frac{9}{4}\right)$

Function of a function (chain rule)

- If y is a function of U and U is a function of x, y is a function of a function

- A function is differentiated by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

Example: Differentiate the following with respect to x

1) $y = (x-5)^3$

solution

Let $u = x - 5$

$$\frac{du}{dx} = 1$$

$$y = u^3$$

$$\frac{dy}{du} = 3u^2$$

From $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\frac{dy}{dx} = 3u^2 \times 1$$

$$\frac{dy}{dx} = 3(x-5)^2$$

2) $y = (x^2 + 2)^5$

Solution

Let $u = x^2 + 2$

$$\frac{du}{dx} = 2x$$

$$y = u^5$$

$$\frac{dy}{du} = 5u^4$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = 5u^4 \times 2x$$

$$\frac{dy}{dx} = 5(x^2 + 2)^4 . 2x$$

$$\frac{dy}{dx} = 10x(x^2 + 2)^4$$

3) $y = \sqrt{(2x-3)^4}$

solution

- Let $u = 2x - 3$

$$\frac{du}{dx} = 2$$

$$y = (u^4)^{\frac{1}{2}}$$

$$y = u^2$$

$$\frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= 2u.2$$

$$= 4(2x - 3)$$

4) $y = \sin^2 x$

Solution

Let $u = sinx$

$$\frac{du}{dx} = cosx$$

But $y = u^2$

$$\frac{dy}{du} = 2u$$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = 2ucosx$$

$$\frac{dy}{dx} = 2sinxcosx$$

Product rule

- If $y = uv$, where u and v are functions of say x

$$y + \delta y = (u + \delta u)(v + \delta v)$$

$$\delta_y = (u + \delta u)(v + \delta v) - uv$$

$$= uv + u\delta v + v\delta u + \delta u\delta v - uv$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}$$

- As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta v}{\delta x} \to \frac{dv}{dx}, \frac{\delta u}{\delta x} \to \frac{du}{dx}$ $and\ \delta u \delta v \to 0$

$$\frac{dy}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

Examples: Differentiate the following

1) $y = (x + 1)(x + 2)^2$

Solution

From $\frac{dy}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

Let $u = x + 1$

$$\frac{du}{dx} = 1$$

Let $v = (x + 2)^2$

$$\frac{dv}{dx} = 2(x + 2)$$

$$\frac{dy}{dx} = (x+1)2(x+2) + (x+2)^2.1$$

$$\frac{dy}{dx} = (x+2)[2x+2+x+2]$$

$$\frac{dy}{dx} = (x+2)(3x+4)$$

2) $y = (x-2)^2(x^2-2)$

solution

- Let $u = (x-2)^2$

$$\frac{du}{dx} = 2(x-2)$$

Let $v = (x^2-2)$

$$\frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = (x-2)^2 . 2x + (x^2-2) . 2(x-2)$$

$$\frac{dy}{dx} = 2x(x-2)^2 + 2(x^2-2)(x-2)$$

$$\frac{dy}{dx} = 2(x-2)[x(x-2) + (x^2-2)]$$

$$\frac{dy}{dx} = 2(x-2)(x^2 - 2x + x^2 - 2)$$

$$\frac{dy}{dx} = 2(x-2)(2x^2 - 2x - 2)$$

3) $y = x^3 cos5x$

Solution

Let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

Let $v = cos5x$

By using chain rule

$$\frac{dv}{dx} = -5sin5x$$

Therefore applying the product rule we obtain

$$\frac{dy}{dx} = x^3.-5sin5x + cos5x\ 3x^2$$

$$\frac{dy}{dx} = -5x^3sin5x + 3x^2cos5x$$

$$\frac{dy}{dx} = x^2(-5xsin5x + 3cos5x)$$

Quotient rule

- If $y = \frac{u}{v}$, where u and v are functions of x

$$y = UV^{-1}$$

$$\frac{dy}{dx} = U.-V^{-2}\frac{dv}{dx} + V^{-1}\frac{du}{dx}$$

$$= -\frac{U}{V^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$$

$$= \frac{-\frac{Udv}{dx} + \frac{Vdu}{dx}}{V^2}$$

$$\frac{dy}{dx} = \frac{\frac{Vdu}{dx} - \frac{Udv}{dx}}{V^2}$$

Example: Differentiate the following

1) $y = \frac{x}{x+1}$

Solution

Let $u = x$ and $v = x + 1$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{(x+1).1 - x.1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x+1-x}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

2) $y = \frac{x^2+1}{(x+1)^2}$

Solution

Let $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

Let $v = (x + 1)^2$

$$\frac{dv}{dx} = 2(x + 1)$$

$$\frac{dy}{dx} = \frac{(x + 1)^2 . 2x - (x^2 + 1). 2(x + 1)}{(x + 1)^4}$$

$$\frac{dy}{dx} = \frac{2(x+1)[x(x+1) - (x^2+1)]}{(x+1)^4}$$

$$\frac{dy}{dx} = \frac{2(x+1)(x-1)}{(x+1)^4}$$

$$\frac{dy}{dx} = \frac{2(x-1)}{(x+1)^3}$$

References

- Sadler, A.J.& Thorning, D.W.S. (2004).Understanding pure mathematics. Oxford university press.

- Backhouse, J.K.& Houldsworth, S.P.T.(1985).Pure mathematics 1. PEARSON EDUCATION LIMITED.

- Thank you for listening to me

- In our next lecture we are going to extend our idea of differentiation to implicit function and rates of change