

Mathematics For Information Technology

Week 9: Differentiation: Implicit differentiation, Rates of change

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outline

- ❖ Intended learning outcome
- ❖ Implicit functions
- ❖ Rates of change

Intended learning outcomes

- ❖ Able to differentiate implicitly defined functions with respect to a given variable.
- ❖ Apply implicit differentiation to find the derivatives of functions that cannot be easily solved for a given variable.
- ❖ Students will be able to solve problems involving rates of change.

Implicit functions

❖ These are the functions in which y cannot be given as an expression in x or these are functions in which y cannot be expressed in terms of x

❖ They are in the form

$$ax^n + by^n + cx^ny^n = d$$

❖ Where a , b , c and d are constants

- ❖ Given two variables x and y is given without having y as an explicit or clearly defined function of x .
- ❖ Thus in $y = 3x^2 - 7x + 1$, y is given as an explicit function of x , whereas in $y^2 - 3yx = x^2$, y is not given explicitly as a function of x
- ❖ An implicit function involving y and x can be differentiated with respect to x as it stands

❖ Suppose $y^2 + 4x = 6x^2$

❖ Differentiating each term with respect to x

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(4x) = \frac{d}{dx}(6x^2)$$

$$\frac{d}{dy}(y^2) \frac{dy}{dx} + \frac{d}{dx}(4x) = \frac{d}{dx}(6x^2)$$

$$\frac{2ydy}{dx} + 4 = 12x$$

❖ It should be noted that we had to rearrange

$\frac{d}{dx}(y^2)$ using the chain rule,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \frac{dy}{dx}$$

to enable y^2 to be differentiated with respect to y

Example: Differentiate the following with respect to x

$$1) y^3 + 6x = x^2$$

Solution

Differentiating each term with respect to x

$$3y^2 \frac{dy}{dx} + 6 = 2x$$

$$\frac{dy}{dx} = \frac{2x - 6}{3y^2}$$

$$2) \quad 3y^2 + 2y + xy = x^3$$

Solution

Differentiating each term with respect to x

$$6y \frac{dy}{dx} + 2 \frac{dy}{dx} + \left(x \frac{dy}{dx} + y \right) = 3x^2$$

$$\frac{dy}{dx} (6y + 2 + x) = 3x^2 - y$$

$$\frac{dy}{dx} = \frac{3x^2 - y}{x + 6y + 2}$$

$$3) \quad x^2 y^2$$

Solution

$$= 2xy^2 + 2y \frac{dy}{dx} x^2$$

$$= 2y \frac{dy}{dx} x^2 + 2xy^2$$

Example: Find the gradient of the curve $x^2 + 2xy - 2y^2 + x = 2$. at the point $(-4,1)$

Solution

$$x^2 + 2xy - 2y^2 + x = 2$$

$$2x + 2 \left[y + x \frac{dy}{dx} \right] - 4y \frac{dy}{dx} + 1 = 0$$

$$2x + 2y + 2x \frac{dy}{dx} - 4y \frac{dy}{dx} = -1$$

$$(2x - 4y) \frac{dy}{dx} = -1 - 2x - 2y$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 4y}$$

- At (-4,1)

$$\frac{dy}{dx} = \frac{-1 - 2(-4) - 2(1)}{2(-4) - 4(1)}$$

$$\frac{dy}{dx} = \frac{5}{-12}$$

Example: Find the equation of the tangent and the normal to the curve $3x^2 - xy - 2y^2 + 12 = 0$ at the point (2,3)

Solution

we want to check the point (2,3) if it lies on the curve

$$\begin{aligned} &= 3x^2 - xy - 2y^2 + 12 \\ &= 3(2)^2 - 2(3) - 2(3)^2 + 12 \\ &= 12 - 6 - 18 + 12 \\ &= 0 \end{aligned}$$

Hence the point lies on the line

Differentiating the equation of the curve with respect to x

$$6x - x \frac{dy}{dx} - y - 4y \frac{dy}{dx} + 0 = 0$$

$$(x + 4y) \frac{dy}{dx} = 6x - y$$

$$\frac{dy}{dx} = \frac{6x - y}{x + 4y}$$

At the point (2,3)

$$\frac{dy}{dx} = \frac{6(2) - 3}{2 + 4(3)}$$

$$\frac{dy}{dx} = \frac{12 - 3}{2 + 12}$$

$$\frac{dy}{dx} = \frac{9}{14}$$

Equation of tangent is

$$\frac{y - 3}{x - 2} = \frac{9}{14}$$

$$14y - 42 = 9x - 18$$

$$14y = 9x + 24$$

❖ Gradient of normal is $-\frac{1}{\text{gradient of tangent}}$

$$= -\frac{1}{14}$$

❖ Gradient of normal = $-\frac{14}{9}$

Equation of the normal

$$\frac{y - 3}{x - 2} = -\frac{14}{9}$$

$$9y - 27 = -14x + 28$$

$$9y = -14x + 55$$

Example: find the equation of tangent and the normal to the curve at the given point on the curve $x^3 + 3x^2y = 2y^2$ at the point $(-1,1)$

Solution

We want to check the point if it lies on the line

$$(-1)^3 + 3(-1)^2(1) = 2(1)^2$$

$$-1 + 3 = 2$$

Hence the LHS equals RHS

$$x^3 + 3x^2y = 2y^2$$

Differentiating with respect to x

$$3x^2 + 6xy + 3x^2 \frac{dy}{dx} = 4y \frac{dy}{dx}$$

$$4y \frac{dy}{dx} - 3x^2 \frac{dy}{dx} = 3x^2 + 6xy$$

$$(4y - 3x^2) \frac{dy}{dx} = 3x^2 + 6xy$$

$$\frac{dy}{dx} = \frac{3x^2 + 6xy}{4y - 3x^2}$$

At point (-1,1)

$$\frac{dy}{dx} = \frac{3(-1)^2 + 6(-1)(1)}{4(1) - 3(-1)^2}$$

$$\frac{dy}{dx} = \frac{3 - 6}{4 - 3}$$

$$\frac{dy}{dx} = -\frac{3}{1}$$

$$= -3$$

The equation of tangent is given

$$\frac{y - 1}{x + 1} = -3$$

$$y = -3x - 3 + 1$$

$$y = -3x - 2$$

Gradient of the normal

$$-\frac{1}{\text{gradient of tangent}}$$

$$= -\frac{1}{-3} = \frac{1}{3}$$

Equation of normal

$$\frac{y - 1}{x + 1} = \frac{1}{3}$$

$$3y - 3 = x + 1$$

$$3y = x + 4$$

Rates of change

- ❖ The rate of change of the variable due to the change in another is determined by chain rule e.g.

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$$

- ❖ As by definition

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right)$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\frac{\delta x}{\delta y}} \right)$$

❖ But as $\delta_x \rightarrow 0$, then also $\delta_y \rightarrow 0$, $\frac{\delta_x}{\delta_y} \rightarrow \frac{dx}{dy}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Example: The side of a cube is increasing at the rate of $6\text{cm}/\text{s}$. find the rate of increase in the volume when the length of the side is 5cm

Solution

Let the side be s

$$\frac{ds}{dt} = 6\text{cm}/\text{s}$$

$$v = s^3$$

$$\frac{dv}{ds} = 3s^2$$

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{ds} \cdot \frac{ds}{dt} \\ &= 3s^2 \times 6 \\ &= 18s^2 \text{ cm}^3/\text{s}\end{aligned}$$

When $s = 5$

$$\begin{aligned}\frac{dv}{dt} &= 18 \times 5^2 \\ &= 450 \text{ cm}^3/\text{s}\end{aligned}$$

Example : The area of a circle is increasing at the rate of $3\text{cm}^2/\text{s}$.
Find the rate of change of the circumference when radius is 2cm

Solution

$$\frac{dA}{dt} = 3\text{cm}^2/\text{s}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r, c = 2\pi r, \frac{dc}{dr} = 2\pi$$

$$\frac{dc}{dt} = \frac{dc}{dr} \cdot \frac{dr}{dt}$$

But $\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt}$

$$= \frac{1}{2\pi r} \cdot 3$$

$$\frac{dr}{dt} = \frac{3}{2\pi r}$$

$$\frac{dc}{dt} = 2\pi \cdot \frac{3}{2\pi r}$$

$$\frac{dc}{dt} = \frac{3}{r} \text{ cm/s}$$

When $r = 2$

$$\frac{dc}{dt} = \frac{3}{2} \text{ cm/s}$$

Example: A container in the shape of a right circular cone of height 10cm and radius 1cm is catching drops from a tap linking at the rate of $0.1\text{ cm}^3/\text{s}$. find the rate at which the surface area of the water is increasing when the water is halfway up the cone ($h = 5$)

Solution

$$\frac{dv}{dt} = 0.1\text{cm}^3/\text{s}$$

$$\frac{r}{1} = \frac{h}{10}$$

But $v = \frac{1}{3} \pi r^2 h$

$$v = \frac{1}{3} \pi \left(\frac{h}{10} \right)^2 h$$

$$= \frac{1}{300} \pi h^3$$

$$\frac{dv}{dh} = \frac{1}{100} \pi h^2$$

$$A = \pi r^2$$

$$A = \frac{\pi h^2}{100}$$

$$\frac{dA}{dh} = \frac{1}{50} \pi h$$

But $\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$= \frac{100}{\pi h^2} \times 0.1$$

$$\frac{dh}{dt} = \frac{10}{\pi h^2}$$

$$\frac{dA}{dt} = \frac{dA}{dh} \frac{dh}{dt}$$

$$= \frac{1}{50} \pi h \times \frac{10}{\pi h^2}$$

$$\frac{dA}{dt} = \frac{1}{5h}$$

At $h = 5$

$$\frac{dA}{dt} = \frac{1}{25} \text{ cm}^2/\text{s}$$

References

- ❖ Sadler, A.J.& Thorning, D.W.S. (2004). Understanding pure mathematics. Oxford university press.
- ❖ Backhouse, J.K.& Houldsworth, S.P.T.(1985). Pure mathematics 1. PEARSON EDUCATION LIMITED.



Thank you for listening

Next lecture we shall look at Integration