

Mathematics For Information Technology

Week 10: Integration: Basic integration, Change of variables, Integration by inspection, Integration by parts

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outline

- ❖ Intended learning outcome
- ❖ Basic Integration
- ❖ Integration by inspection
- ❖ Change of variables
- ❖ Integration by parts

Intended learning outcome

- ❖ Understand the basic concepts of integration, including Antiderivatives, and the relationship between differentiation and integration.
- ❖ Apply change of variables techniques to simplify integrals and make them easier to evaluate.
- ❖ Use integration by inspection to quickly evaluate integrals by recognizing basic integration formulas
- ❖ Apply integration by parts to evaluate integrals, and choose appropriate u and dv terms.

Basic integration

- Integration is the process or act or an instance of making up a whole or is a process of uniting parts together.
- Involves finding the Antiderivatives of a function, which is the inverse operation of differentiation

The reverse of differentiation

- ❖ Given the equation $y = f(x)$, then by differentiation ,
we can find the gradient function $\frac{dy}{dx}$
- ❖ We can obtain the equation of the curve using the reverse process called integration.
- ❖ Suppose $\frac{dy}{dx} = 2x$. To find y we require a function that differentiates to give $2x$.

- ❖ Clearly $y = x^2$ is such a function
- ❖ We say that $y = x^2 + c$, where c is a constant, gives $2x$ when differentiated.
- ❖ Thus if we integrate $2x$ we obtain $x^2 + c$. Given the gradient function of a curve, integration gives the family of curves to which that gradient function applies.

❖ Generally, given $\frac{dy}{dx} = ax^n$ then $y = \frac{ax^{n+1}}{n+1} + c$, for $n \neq -1$

❖ If $y = \frac{ax^{n+1}}{n+1} + c$

$$\frac{dy}{dx} = \frac{a(n+1)x^n}{n+1}$$

$$\frac{dy}{dx} = ax^n$$

❖ Using the integral sign $\int dx$, we can write this as

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c, n \neq -1$$

❖ With the dx meaning that we are integrating with respect to the variable x

Types of integrals

❖ **Indefinite integrals.** These are integrals without limits.

$$\int x^n dx = \frac{x^{(n+1)}}{n+1} + c$$

❖ **Definite integrals.** These are integrals with limits

$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

Example: solve the following

1) $\int 4x^5 dx$

$$\int 4x^5 dx = \frac{4x^{5+1}}{5+1} + c$$

$$= \frac{4x^6}{6} + c$$

$$= \frac{2x^6}{3} + c$$

$$2) \int \left(5 - x^2 + \frac{18}{x^4} \right) dx$$

$$= \int (5 - x^2 + 18x^{-4}) dx$$

$$= 5x - \frac{x^3}{3} + \frac{18x^{-3}}{-3} + c$$

$$= 5x - \frac{x^3}{3} + \frac{6x^{-3}}{-1} + c$$

$$= 5x - \frac{x^3}{3} - \frac{6}{x^3} + c$$

$$3) \int_1^3 4x^5 dx$$

$$= \frac{4x^{5+1}}{5+1} \Big|_1^3$$

$$= \frac{4x^6}{6} \Big|_1^3$$

$$= \left(\frac{2(3)^6}{3} - \frac{2(1)^6}{3} \right)$$

$$= \left(2 \times 243 - \frac{2}{3} \right)$$

$$= (486 - 0.67)$$

$$= 485.33$$

Integration by inspection

❖ Consider the integral function

$$\int f'(x)[f(x)]^n dx$$

❖ Integration by inspection is obtained as

$$\frac{d}{dx} [f(x)]^{n+1} = (n + 1)[f(x)]^{n+1-1} f'(x)$$

$$\frac{1}{n+1} \frac{d}{dx} [f(x)]^{n+1} = f'(x) [f(x)]^n$$

❖ Introducing integrals on both sides with respect to x ,
we obtain

$$\int f'(x) [f(x)]^n dx = \frac{1}{n+1} \int [f(x)]^{n+1} dx + c$$

Example: Find

1) $\int 20(7 + 5x)^3 dx$

❖ We suspect that this may have come from differentiating $(7 + 5x)^4$ as a function of a function i.e.

$$\frac{d}{dx} (7 + 5x)^4 = 4(7 + 5x)^3 \times 5$$

$$\frac{d}{dx} (7 + 5x)^4 = 20(7 + 5x)^3$$

❖ Introducing integrals on both sides with respect to x

$$\int \frac{d}{dx} (7 + 5x)^4 dx = \int 20(7 + 5x)^3 dx$$

$$\int 20(7 + 5x)^3 dx = (7 + 5x)^4 + c$$

$$2) \int (4x - 2)(x^2 - x + 4)^5 dx$$

❖ From

$$\frac{d}{dx} (x^2 - x + 4)^6 = 6(x^2 - x + 4)^5 (2x - 1)$$

$$\frac{1}{3} \frac{d}{dx} (x^2 - x + 4)^6 = (4x - 2)(x^2 - x + 4)^5$$

❖ Introducing integrals on both sides with respect to x

$$\int \frac{1}{3} \frac{d}{dx} (x^2 - x + 4)^6 dx = \int (4x - 2)(x^2 - x + 4)^5 dx$$

$$= \frac{1}{3} (x^2 - x + 4)^6 + c$$

Change of variables

- ❖ Suppose we want to integrate $x^2(2x^3 + 3)^5$ with respect to x .
- ❖ We could expand the bracket then we integrate each term
- ❖ It is however much better to notice that x^2 is a scalar multiple of $\frac{d}{dx}(2x^3 + 3)$ and then it follows that

$$\int x^2(2x^3 + 3)^5 dx = \frac{1}{36} (2x^3 + 3)^6 + c$$

❖ If we fail to notice this function of function, an alternative method is to change variable from x to some other suitable chosen variables depending on the following theory

❖ Suppose $y = \int f(x) dx$ where $x = g(u)$, i.e. $\frac{dy}{dx} = f(x)$ and

$$f(x) = f[g(u)]$$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = f(x) \times \frac{dx}{du}$$

❖ Integrating with respect to u , gives

$$y = \int f(x) \frac{dx}{du} du,$$

$$y = \int f(g(u)) \frac{dx}{du} du, \text{ but } y = \int f(x) dx$$

$$\int f(x) dx = \int f(g(u)) \frac{dx}{du} du$$

❖ Thus, to integrate $f(x)$ with respect to x , dx is replaced by $\frac{dx}{du} du$ and the integrand is then expressed in terms of the new variable, u .

Example

1) Find $\int x^2(2x^3 + 3)^5 dx$

Let $u = 2x^3 + 3$ then $\frac{du}{dx} = 6x^2$

Thus $\int x^2(2x^3 + 3)^5 dx = \int x^2(2x^3 + 3)^5 \frac{dx}{du} du$

$$= \int x^2(2x^3 + 3)^5 \frac{1}{6x^2} du$$

$$= \int \frac{u^5}{6} du$$

$$= \frac{u^6}{36} + c$$

$$= \frac{(2x^3 + 3)^6}{36} + c$$

$$2) \quad \int 2x(x + 2)^5 dx$$

- Let $u = x + 2$, then $\frac{du}{dx} = 1$

$$\int 2x(x + 2)^5 dx = \int 2x(x + 2)^5 \frac{dx}{du} du$$

$$= \int 2(u - 2)u^5 du$$

$$= \frac{2}{7}u^7 - \frac{2}{3}u^6 + c$$

$$= \frac{2}{21}u^6(3u - 7) + c$$

$$= \frac{2}{21}(x + 2)^6(3x - 1) + c$$

$$3) \int_0^5 \frac{x}{\sqrt{x+4}} dx$$

- Let $u = x + 4$ then $\frac{du}{dx} = 1$

x	0	5
u	4	9

$$\int_0^5 \frac{x}{\sqrt{x+4}} dx = \int_4^9 \frac{x}{\sqrt{x+4}} \frac{dx}{du} du$$

$$\int_4^9 \frac{u-4}{\sqrt{u}} du$$

$$\left[\frac{2}{3} u^{\frac{3}{2}} - 8u^{\frac{1}{2}} \right]_4^9$$

$$= \left(\frac{2}{3} \times 27 - 8 \times 3 \right) - \left(\frac{2}{3} \times 8 - 8 \times 2 \right)$$

$$= 4 \frac{2}{3}$$

$$4) \int \frac{1}{25+16x^2} dx$$

Let $u = 4x$, then $\frac{du}{dx} = 4$

$$= \int \frac{1}{25 + u^2} \times \frac{1}{4} du$$

$$= \frac{1}{4} \int \frac{1}{25 + u^2} du$$

$$= \tan^{-1} \left(\frac{u}{5} \right) + c$$

$$= \frac{1}{20} \tan^{-1} \left(\frac{4x}{5} \right) + c$$

Integration by parts

❖ Recall $\frac{d(uv)}{dx} = \frac{vdu}{dx} + \frac{udv}{dx}$

$$\frac{d(uv)}{dx} dx = \frac{vdu}{dx} dx + \frac{udv}{dx} dx$$

❖ Introduction integrals throughout

$$\int d(uv) = \int \frac{vdu}{dx} dx + \int \frac{udv}{dx} dx$$

$$uv = \int \frac{vdu}{dx} dx + \int \frac{udv}{dx} dx$$

$$\int \frac{udv}{dx} dx = uv - \int \frac{vdu}{dx} dx$$

Example: Evaluate

$$1) \int_0^{\frac{\pi}{2}} x \cos x dx$$

$$\text{let } u = x, \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 1 \text{ and } v = \sin x$$

$$\int_0^{\frac{\pi}{2}} x \cos x dx = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \cos x dx &= [x \sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

$$2) \int x \ln x \, dx$$

- Let $u = \ln x$, $v = \frac{x^2}{2}$

$$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$= \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} + c \right]$$

$$3) \int x(3x - 2)^4 dx$$

- Let $u = x$, $\frac{dv}{dx} = (3x - 2)^4$

$$\frac{du}{dx} = 1 \text{ and } v = \frac{1}{15}(3x - 2)^5$$

$$= \frac{x}{15}(3x - 2)^5 - \int \frac{1}{15}(3x - 2)^5 dx$$

$$= \frac{x}{15}(3x - 2)^5 - \frac{1}{15 \times 18}(3x - 2)^6 + c$$

$$= \frac{1}{270}(3x - 2)^5(15x + 2) + c$$

References

- ❖ Sadler, A.J.& Thorning, D.W.S. (2004). Understanding pure mathematics. Oxford university press.
- ❖ Backhouse, J.K.& Houldsworth, S.P.T.(1985). Pure mathematics 1. PEARSON EDUCATION LIMITED.



Thank you for listening

Next lecture we shall look at Graph theory