Mathematics For Information Technology

Week 14: Matrices : Determinants of a matrix , Singular matrices, Inverse Matrices

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outline

- Matrices
- Determinants of a Matrix
- Singular Matrices
- Inverse Matrices

Learning outcome

- Understanding of matrices, including their representation, notation, and basic operations
- Able to calculate determinants of matrices
- Determine whether a matrix is singular or non-singular by evaluating its determinant.
- Able to find the inverse of square matrices

Introduction

- Consider the information given below in the university league tournament last season in Uganda.
- The result for three universities/schools, A, B, and C were as shown below

School	P	W	D	L
A	10	4	4	2
B	8	6	1	1
C	8	2	3	3

- The numbers above are arranged in a rectangular form.

- Such an arrangement of numbers is what is known as a matrix

Definition of a matrix

- A matrix is arrays of numbers in rectangular form with large brackets around them.

Or

- A matrix is a collection of information stored in rows and columns

Common terms used

- Below are some of the frequently used terms

Entry (an element)

- This is a number within the matrix. At times it is known as component. Consider the matrix below

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

- The numbers $2, 3, 4$ *and* 5 are the elements of the above matrix.

Rows of a matrix

- These are the lines of numbers that goes across the page. Considering the above matrix i.e.

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

(2 3) forms the first row and (4 5) forms another row. Therefore the matrix above has two rows.

Columns of a matrix

- These are the vertical lines of numbers. Considering

$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ i.e.

$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ forms the first column and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ forms another column.

- A matrix is represented with upper case letters.
- In identifying matrix, one has to use the position of row and column e.g.

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Then, a is the element in the first row and first column

  b is the element in first row and second column

  c is the element in the second row and first column

  d is the element in the second row and column

Order of a matrix

- This refers to the number of rows and columns in a given matrix and it is given by $order = Row \times column$
- Consider the matrix below:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- The matrix above has 2 rows and 2 columns. Therefore the order of the above matrix is 2×2

Example

- State the order of the following given matrices

$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, is a 2×2 matrix i.e. 2 rows and 2 columns

$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$, is a 2×3 matrix i.e. 2 rows and 3 columns

$\begin{pmatrix} 3 & 5 \\ 1 & -2 \\ 1 & 13 \end{pmatrix}$ is a 3×2 matrix i.e. 3 rows and 2 columns

$(0\ 1)$ is a 1×2 matrix i.e. 1 row and 2 columns

- The number of rows is denoted by m and columns by n. when stating the order of the matrix the number of rows is written first.
- This is followed by the number of columns, i.e. $order = m \times n$

Leading diagonal (major diagonal)

- This is a line of numbers that runs diagonally from the top left-hand corner to the bottom right-hand corner for the matrix with equal numbers of rows and columns (i.e. a square matrix).
- Consider the matrix below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & -1 \end{pmatrix}$$

- This matrix has the same number of rows and columns. It is a 3×3 matrix.
- Its leading diagonal is what has been enclosed in the loop i.e. $1, 5, and\ -1$

Minor diagonal

- This is a line of numbers that runs diagonally from the bottom left-hand corner to the top right-hand corner. For the matrix above its minor diagonal is what has been enclosed in the loop

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & -1 \end{pmatrix}$$

$1,5\ and\ 3$ are the entries of the minor diagonal

Types of matrices

Zero matrix

- This is a matrix whose all of its elements are zeros e.g.

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equal matrix:

- Two or more matrices are equal if and only if their corresponding elements are equal and are of same order

Example

Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 10 \\ 6 & -4 \end{pmatrix}$ therefore if matrix A is equal to matrix B, then

$$a = 4,$$

$$b = 10,$$

$$c = 6,$$

$$d = -4$$

Square matrix:

- This is a matrix with equal number of rows and columns e.g.

$$\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 2 & 5 & 4 \end{pmatrix}$$

Identity matrix

- This is a square matrix with 1 as an element in the leading diagonal and zeros elsewhere.

- At times it is known as unit matrix and it is denoted by the letter I

- Examples include

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

Three sales girls sold the following number of bottles of lotion on a certain day.

- Liz sold 9 bottles of Dear, 13 of scala and 6 of Venus.
- Suzie sold 8 bottles of movit lotion,7 of scala and 10 of Venus.
- Abe sold 15 bottles of movit lotion, 1 of Dear and 18 of scala.

Show this information in a 3×4 matrix.

solution

	Dear	scala	Venus	movit
Liz	9	13	6	0
Suzie	0	7	10	8
Abe	1	18	0	15

- The above matrix can shortly be written as

$$\begin{pmatrix} 9 & 13 & 6 & 0 \\ 0 & 7 & 10 & 8 \\ 1 & 18 & 0 & 15 \end{pmatrix}$$

- The table below shows the number of times that three couples attended various types of entertainment in one year.

Entertainment	Couple		
	Buwembo's	Frank's	Lukyamuzi's
Cinema	7	2	5
Dance	1	2	9
Play	5	8	1
Circus	0	3	2

- Write down the information in the table in the form of a matrix and state the order of the matrix

- Write the Frank's attendance as a column matrix. What is the order of this matrix?
- Write as a row matrix, the number of times the plays have been attended, and state the order of the matrix.

solution

- $\begin{pmatrix} 7 & 2 & 5 \\ 1 & 2 & 9 \\ 5 & 8 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, is the order 4×3
- $\begin{pmatrix} 2 \\ 2 \\ 8 \\ 3 \end{pmatrix}$, is the order 4×1
- $(5 \quad 8 \quad 1)$, is the order 1×3

Matrix operation (addition)

- If A and B are $m \times n$ matrix , the sum of A and B is defined to be the $m \times n$ matrix A+B obtained by adding corresponding entries i.e.

  $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$ for each i and j

Example

- Find A+B, given that $A = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$

Solution

$$A + B = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 0-6 \\ 3+6 & -6+7 \\ 5+3 & 1+0 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & -6 \\ 9 & 1 \\ 8 & 1 \end{pmatrix}$$

Properties of matrix addition

- For $m \times n$ matrices, A, B, and C, the following properties hold
- $A + B$ –Closure property
- $(A + B) + C = A + (B + C)$- associative property
- $A + B = B + A$ – commutative property
- The $m \times n$ matrix oconsisting of all zeros has the property that $A + 0 = A$, additive identity
- The $m \times n$ matrix (-A) has the property that A+(-A)=o

Scalar multiplication

- The product of a scalar α times a matrix A, denoted by $\alpha A$is defined to be the matrix obtained by multiplying each entry of A by α i.e

$$2\begin{pmatrix}1 & 2 & 3\\0 & 1 & 2\\1 & 4 & 2\end{pmatrix}=\begin{pmatrix}2 & 4 & 6\\0 & 2 & 4\\2 & 8 & 4\end{pmatrix}$$

Properties of scalar multiplication

- For $m \times n$ matrices A and B and for $\alpha\ and\ \beta$ the following properties hold
- Closure property: αA is again an $m \times n$ matrix
- Associative property: $(\alpha\beta)A = \alpha(\beta A)$
- Distributive property: $\alpha(A + B) = \alpha A + \alpha B$
- Identity property: $IA = A$

Example

Given the matrices

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

Find $2(A + B)$

Solution

$$2(A + B) = 2A + 2B$$

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 12 \\ 16 & 20 \end{pmatrix}$$

Determinant of a matrix

- A determinant is a number that is assigned to a square array of numbers in a certain way. i.e. The determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- is given by the formula:

$$\det(A) = ad - bc$$

- The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- is given by the formula:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

❖ Find the determinant of the matrix

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

solution

$$= \begin{vmatrix} 2 & 5 & 1 \\ 0 & 3 & 4 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} - 5\begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 2(3 \times (-2) - 4 \times 2) - 5(0 \times -2 - 4 \times 1) + 1(0 \times 2 - 3 \times 1)$$

$$= -28 + 20 - 3$$
$$= -11$$

- Therefore, the determinant of the matrix is -11.

Singular matrices

- Theses are matrices whose determinant is equal to zero i.e.

$$Det\ A = 0$$

Example

$$A = \begin{pmatrix} 39 & 91 \\ 51 & 119 \end{pmatrix}$$

$$\det A = (39 \times 119) - (51 \times 91)$$

$$= 4641 - 4641$$

$$0$$

- Hence it's a singular matrix since its determinant is zero

Adjoint of a matrix

- Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- To find the adjoint of matrix, we simply interchange the elements in the leading diagonal and change the signs in the minor diagonal. It is denoted as $Adj(A)$.

$$Adj(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

Given the matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 5 \end{pmatrix}$, find the adjoint of A

$$Adjoint\ of\ A = \begin{pmatrix} 5 & -3 \\ -2 & 2 \end{pmatrix}$$

Inverse of a matrix

- The inverse of a matrix A is given by $\frac{1}{detA} \times the\ adjoint\ matrix$. The inverse of a matrix A is denoted by A^{-1}. To get the adjoint, we interchange the entries of the major diagonal and multiply the entries of the minor diagonal by -1 i.e.
- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, Adjoint $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$Det\ A = ad - bc$$

$$A^{-1} = \frac{1}{\det A} \times Adjoint\ A$$

$$A^{-1} = \frac{1}{ad - bc} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- Note: The inverse of a singular matrix does not exist because we end up with a division by zero which is undefined.

Examples

- If $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, Find

1) A^{-1}

Solution

$$Det\ A = (3 \times 1) - (1 \times 0)$$

$$= 3$$

$$Adjoint\ A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$$

2) B^{-1}

Solution

$$Det\ B = (-1 \times 3) - (2 \times 1) = -5$$

$$Adjoint\ B = \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-5}\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

3) $(A+B)^{-1}$

Solution

$$Det(A+B) = (2 \times 4) - (3 \times 1) = 5$$

$$Adjoint(A+B) = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

$$(A+B)^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

- Note: $AA^{-1} = I$ where I is an identity matrix where an identity matrix which has the entries in the major diagonal equal to one and the entries in the minor diagonal all equal to zero e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is 2×2 identity matrix

References

- Strang, G. (2016). Linear algebra and its applications (5th ed.). Cengage Learning

- Davis C. L. (1988). Linear algebra and its applications. University of Maryland. Addison-wesley publishing company.

End of lecture 14

Next topic: Financial mathematics

Thank you