Discrete Mathematics

Lecture 1

Propositional logic

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Introduction to Lecture 1

This lecture introduces the concept of propositional logic, its application and importance to computing. Major importance of logic is to automate reasoning. It is used to design digital circuits, construction, and verification of validity of computer programs.

References

These lecture notes have been derived from the following resources (Kenneth, 2012; Koman et al., 2001; Lipschutz & Lipson, 2007; Susanna, 2003).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Define terms used in propositional logic.

(ii) Carry out operations involving propositional logic.

(iii) Apply logic in circuit design.

Definition of Terms

Definition 1: A proposition is a statement that can be assigned a truth value. That is, a statement that can either True (T) or False (F). For example, *12+7 = 19*; *10-2=7*; *all cows have tail* etc.

Some statements are not proposition since it is impossible to assign a truth value to them.

For example; *come tomorrow*; *I am telling the truth* etc.

We use lower case letters such as p, r, q, s, t etc. to denote a proposition.

For example, proposition *p*: *Today is Sunday.*

Definition 2: Some propositions used in mathematical proofs are generally assumed to be true. An axiom is a proposition that is assumed to be true such as the commutative axiom.

For example, $a + b = b + a$ – commutative axiom.

Definition 3: We say that the proposition ¬p or Not p is true if the proposition p is false and is false if the proposition p is true. This is the negation of a proposition.

For example, state the negation of, *p: the book cover is red.*

The negation of the proposition is; $\neg p$: *the book cover is not red.*

Definition 4: A conjunction consists of two or more statements connected by the word '*AND*'. For example, suppose p and q are two simple statements then p ∧ q is called the conjunction of p with q.

Definition 5: A disjunction consists of two or more statements connected with the word *'OR'*. Suppose p and q are two simple statements, then p ∨ q is called the disjunction of p with q.

Definition 6: A conditional proposition consists of two or more statements connected by the words, *′ if · · ·then · · · ′*

Suppose *p*: *I am happy* and *q*: *Today is Friday* are two simple statements, then; p ⟶ q is called the conditional of p with q. Read as, *'If I am happy then today is Friday'.*

Remark 1: p ⟶ q can also be read as *'p only if q'* or *'p implies q'*

Definition 7: The **converse** of the conditional *'if p then q'* is *'if q then p'* that is the converse of p ⟶ q is q ⟶ p.

For example, the converse of *'If I am happy, then today is Friday'* is *'if today is Friday, then I am happy'*

The **inverse** of the conditional *'If p the q'* is *'If not p then not q'* i.e., the inverse of p ⟶ q is ¬p ⟶ ¬q. The inverse of the statement, *'If I am happy then today is Friday'* is *'If I am not happy, then today is not Friday'.*

The **contrapositive** of the conditional *'If p then q'* is *'If not q then not p'* i.e., $\neg q \rightarrow \neg p$. The contrapositive of the conditional statement, *'If I am happy then today is Friday'* is *'If today is not Friday, then I am not happy'.*

Remark 2: $p \rightarrow q \equiv \neg q \rightarrow \neg p$ i.e., the conditional is equivalent to contrapositive.
Remark 3: $q \rightarrow p \equiv \neg p \rightarrow \neg q$ i.e., converse is logically equivalent to inverse. These equivalents are very useful in mathematical proofs.

Definition 8: A biconditional is a statement of the form; $(p \rightarrow q) \wedge (q \rightarrow p)$ and is symbolized as; $p \leftrightarrow q$. For example, *She dances if and only if it is a full moon*. This statement is logically equivalent to, *if she dances then it is a full moon and if it is a full moon then she dances.*
Remark 2: $p \leftrightarrow q$ is read as *'p if and only if q'*. Also symbolized as *p iff q* .

Definition 9: A **compound proposition** is a proposition that contains one or more simple propositions. A compound statement is formed by inserting the word 'not' into a simpler statement or by joining two or more simpler statements with connective words such as *'and'*, *'or'*, *'if…then…'*, *'only if'*, *'if and only if'* etc. The compound statement could be a negation, a conjunction, a disjunction, a conditional or combination thereof.

Example 1: Let *p: the gate is open. q: the water is flowing*. State symbolically;
(a) If the gate is open then the water is flowing. **Solution:** $p \rightarrow q$
(b) If the gate is not open then the water is flowing. **Solution:** $\neg p \rightarrow q$
(c) If the water is flowing then the gate is open. **Solution:** $q \rightarrow p$

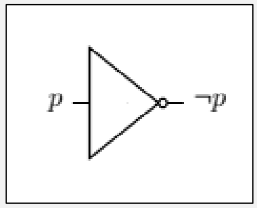
Truth Tables and Logic Gates

A truth table displays the relationships between the truth values of propositions.

Example 1: Let p and q be any propositions, then the truth tables and corresponding logic gates is as below;

(a) ¬p

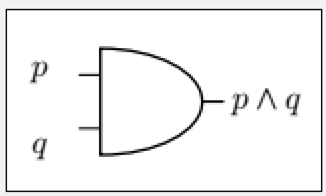
p	¬p
T	F
F	T



The NOT gate corresponds to the negation of p. It is also referred to as an inverter.

(b) p ∧ q

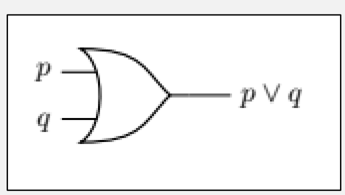
p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0



The AND gate corresponds to the conjunction proposition.

(c) p ∨ q

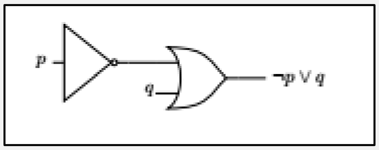
p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0



The OR corresponds to the disjunction proposition.

(d) p → q Implication statement is false exactly when p is true, and q is false (or you can also say that; propositional p → q is true if the proposition p is false or if the proposition q is true or both and is false otherwise.)

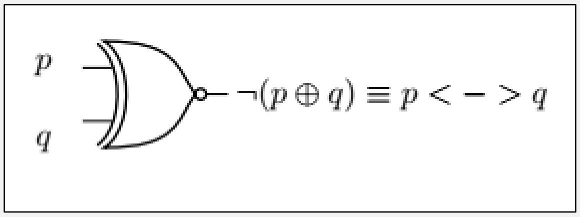
p	q	p → q
1	1	1
1	0	0
0	1	1
0	0	1



The conditional proposition corresponds to the logic gate above.

(e) p ↔ q Double conditional statement is true exactly when p and q have the same truth value (or you can say that the propositional p ↔ q is true if the two are both true or false and is false otherwise).

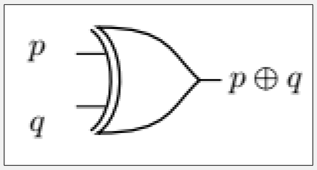
p	q	p ⇔ q
1	1	1
1	0	0
0	1	0
0	0	1



The XNOR gate corresponds to the biconditional statement.

(f) Exclusive statement p⊕q (i.e., p 0-plus q) is the compound proposition that is true exactly when atomic proposition is true and the other is false.

p	q	p ⊕ q
1	1	0
1	0	1
0	1	1
0	0	0



The XOR gate corresponds to the exclusive proposition.

Logic and bit operations

A bit stands for a binary digit. Bit can be 0 or 1. A bit string is a string of bits. There is a direct connection between bit and bits operations used in programming and propositional logic.

T =1, F = 0, ¬ = NOT, ∧ = AND, ∨ = OR, and ⊕ = XOR.

Logic is used to help in the structuring of computer programs. Logic is used to automate reasoning.

The operations of intersection, union and complement correspond to the logical connectives AND, OR and NOT respectively.

Example 1: Find the bitwise OR, bitwise AND and bitwise XOR of the bit strings 1101 0011 and 1001 1101

Solution:

1100 0011
100 1101
1100 1111 OR
1000 0001 AND
0101 1110 XOR

Laws of Logic & Logical Equivalence

Two propositions p and q are equivalent if they have the same truth values under all circumstances, denoted $p \equiv q$.

The following are some laws of logic and their logical equivalence;

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Biconditional

$p \rightarrow q \equiv \neg p \vee q$ implication law

$\neg\neg p \equiv p$ double negation

$p \wedge p \equiv p$, $p \vee p \equiv p$ Idempotent laws

$p \wedge q \equiv q \wedge p$, $q \vee p \equiv p \vee q$ Commutative laws

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, $(p \vee q) \vee r \equiv p \vee (q \vee r)$ Associative laws

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ Distributive laws

$\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$ De Morgan's laws for logic i.e. conjunction and disjunction interchange under negation.

$p \wedge T \equiv p$, $p \vee F \equiv p$ Identity laws

$p \wedge (p \vee q) \equiv p$, $p \vee (p \wedge q) \equiv p$ Absorption laws

Example 1: Determine the negation of the following proposition; *If he cries, he will go home*.

Solution: Suppose we let; *p: he cries*; *q: he will go home*. Then our statement can be written as; $p \rightarrow q$. Then its negation is; $\neg(p \rightarrow q)$ q. $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

In words then we have, *He cries, and he will not go home.*

Note that the negation: $\neg(p \rightarrow q)$ is read as; *It is not the case that if he cries, he will go home.*

Example 2: Determine the negation of the following statement, *He sings if and only if the piano is white*. Hence design a logic gate for the negation.

Solution: Suppose we let; *p: he sings*; *q: the piano is white*.

Using logic connectives, our statement can be written as; $p \leftrightarrow q$.

Its negation is then; $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$.

In words this is; *He sings if and only if the piano is not white* i.e., $p \leftrightarrow \neg q$.

One can also write it as; *He does not sing if and only if the piano is white i.e.,* $\neg p \leftrightarrow q$.

$$\neg(p \leftrightarrow q) \equiv p \oplus q$$

This corresponds to XOR gate i.e.

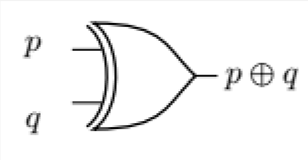


Figure 1

Example 3: Consider the statement; ***If I do not have a high fever then the diode will not show red.***

Let p: *if I do have a high fever* and q: *the diode shows red*. Design a logic gate for;

(i) The proposition i.e. $\neg p \rightarrow \neg q$

(ii) The inverse i.e., $p \rightarrow q$

(iii) The negation of the contrapositive i.e., $\neg(q \rightarrow p)$

Solution: It can be shown that. $p \rightarrow q \equiv \neg p \vee q$. Hence we have,

(i) $\neg p \rightarrow \neg q \equiv \neg\neg p \vee \neg q \equiv p \vee \neg q$ i.e., *I do have a high fever, or the diode will not show red.*

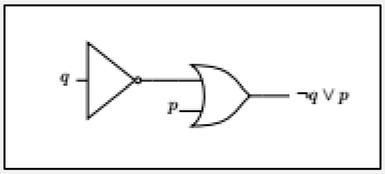


Figure 2

(ii) $p \rightarrow q \equiv \neg p \vee q$ i.e., *I don't have a high fever, or the diode will show red.*

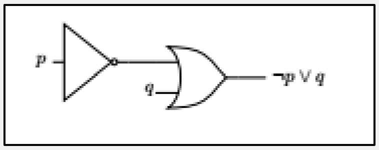


Figure 3

(iii) $\neg(q \rightarrow p) \equiv \neg(\neg q \vee p) \equiv q \wedge \neg p$ i.e., *The diode shows red, and I don't have a high fever.*

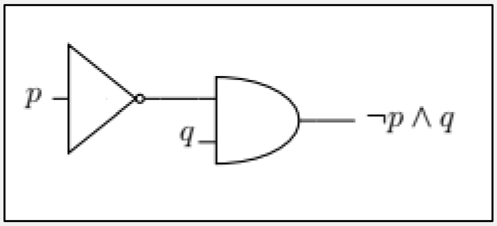


Figure 4

Example 5: Consider the following statement; '*It is not the case that both the virus and the parasites are not in the blood sample'*. Compare the statement with the following statement; '*Either the virus or the parasites are in the blood sample'*. Use a truth table to determine if the two statements are logically equivalent. Design logic gates for the two statements.

Proof: Suppose we let *p: the virus is in the blood sample q: the parasites are in the blood sample.* Using logical connectives, we can have the statement;

'*It is not the case that both the virus and the parasites are not in the blood sample'* as $\neg(\neg p \wedge \neg q)$.

While the statement; '*Either the virus or the parasites are in the blood sample'* can be written as; $p \vee q$

p	q	¬p	¬q	¬p ∧ ¬q	¬(¬p ∧ ¬q)	p ∨ q
1	1	0	0	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	0	0

The two statements are logically equivalent going by the 6th and 7th columns.

Exercise: Design their logic gate. Note the most complicated circuit.

Example 6: Write down propositions for the converse and contrapositive of, '*If 60 is divisible by 4 then 60 is an even number'*. Determine if the two statements are logically equivalent i.e.

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Solution: Suppose we let; *p: 60 is divisible by 4; q: 60 is an even number*

Converse: The statement is $p \rightarrow q$, hence the converse is $q \rightarrow p$ i.e., *if 60 is an even number, then 60 is divisible by 4.*

Contrapositive: The contrapositive is $\neg q \rightarrow \neg p$ i.e., *if 60 is not an even number then 60 is not divisible by 4.* Next we use the truth table to show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (contrapositive)

p	q	p → q	¬q	¬p	¬q → ¬p
1	1	1	0	0	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

Note that since the columns for both $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are the same then the two are logically equivalent.

Tautology and Contradiction

Tautology is a compound proposition that is always true, regardless of the truth values of the basic propositions which comprise it. In other words, a tautology is a proposition that is true on logical ground only.

A **contradiction** is a proposition that is always false regardless of the truth value of the basic propositions that comprise it.

A proposition is **satisfiable** if its truth table contains the value T at least once. For example, p ∧ q is satisfiable. A proposition is a **contingency** if it is satisfiable but not a tautology.

Example 1: Show that the proposition, '*It is raining and sunny and it is not raining*' is a contradiction.

Solution: Suppose we let; *p: It is raining q: It is sunny.*

Then our statement using logical connectives become; (p ∧ q) ∧ ¬p. We next use a truth table;

p	Q	¬p	p ∧ q	(p ∧ q) ∧ ¬p
1	1	0	1	0
1	0	0	0	0
0	1	1	0	0
0	0	1	0	0

The last column is all 0s. This implies that the proposition is false and therefore a contradiction.

Example 2: Use a truth table to determine if the following is a tautology; p ∨ ¬p

Solution:

P	¬p	p ∨ ¬p
1	0	1
0	1	1

The last column is all 1s, hence it is a tautology.

Exercises

1) Show if the following is a tautology $[(p \rightarrow q) \land \neg q] \rightarrow \neg p$.

2) Show using the truth table that $(p \rightarrow q) \land (p \rightarrow r) \equiv (q \land r)$.

3) Show using the truth table that $(p \land T) \equiv P$.

4) Verify the absorption law.

5) Show without using truth tables that $p \lor q \equiv \neg p \rightarrow q$.

6) Draw the logic gate for *it is not the case that if today is Sunday, then it will rain.*

7) Find the negation of p : $-3 < x \leq 11$

8) Show using a truth table if the following statements are logically equivalent: '*The car is a convertible and a SUV*' and '*It is not the case that if the car is a SUV then it is not convertible*'.

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