Discrete Mathematics

Lecture 2

Logic and Validity of Arguments

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Introduction to lecture 2

This lecture is a continuation of lecture 1 and introduces the application of propositional logic in testing the validity of arguments.

References

These lecture notes have been derived from the following sources, Rosen (2012), Koman et al. (2001), Mathew (2022), Seymour (2020) and Susanna (2003).

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Define terms used in logic.

(ii) Determine the validity of arguments.

Definition of terms

Definition 1: Logic is the science of correct reasoning. It is fundamental to critical thinking and problem solving.

Definition 2: Logic is the study of whether an argument is robust and leads to valid conclusions

Definition 3: Reasoning is defined as drawing of inferences or conclusions from known or assumed facts.

Definition 4: Deductive reasoning is the type of logic that involves application of a general statement to a specific instance e.g., the use of a formula to solve a particular problem.

Definition 5: Inductive reasoning is drawing conclusions from a specific example.

Definition 6: An argument is a discussion in which there is a disagreement. It is made up of premises and conclusions.

Definition 7: A valid argument is where the conclusion of argument is guaranteed, otherwise it is invalid. Argument is said to be valid if the conclusion follows logically from the assumptions or premises.

Remark 1: An argument that is valid does not imply that the conclusion is true.

Example 1: Consider the following argument;

Major premise *p*: All lecturers are men.

Minor premise *q*: my mother is a lecturer

∴ My mother is a man

Note that the conclusion is valid though not true.

Remark 2: An argument being invalid does not mean the conclusion is false.

Example 1: Consider the argument;

Major premise *p:* All professional soldiers are snipers

Minor premise *q*: Kimani is a sniper

∴ Kimani is a professional soldier.

The argument is invalid, although the conclusion may not be false.

Remark 3: We use lower case letters such as *p, q, r, or s* to denote a premise/assumption, conclusion, and proposition.

Testing the validity of an argument

The lecture will delve into the following methods of testing the validity of an argument;

(i) Venn diagram

(ii) Truth table

(iii) Rules of inferences

Venn Diagrams and Arguments

Venn diagrams can be used to determine if an argument is valid or invalid. Three intersecting circles are needed to diagram a categorical syllogism. A categorical syllogism is a deductive argument that consists of three categorical statements namely the two premises or assumptions, and the conclusion. This works well when dealing with predicate logic.

Rules to follow:

(i) Draw the set representing the universal premise first (this will be explained in the next lecture on predicate logic).

(ii) Letter x is indicated on the line intersecting a section of the whole area is so designated in the premise.

(iii) You should only draw circles for premises/assumptions.

(iv) The argument or syllogism is only valid if the conclusion is self-evident in the Venn diagram drawn.

Example 1: Consider the following argument and use a Venn diagram to determine if it is valid

p: All drivers are men

q: My mother is a driver

∴My mother is a man

Solution: Note that the set of drivers is contained in the set of men. Since *my mother* X is contained in the set of *drivers,* it implies then that *my mother* is also in the set of *men*. Hence the argument is valid.

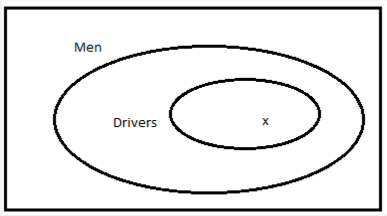

Figure 1

Alternatively: We first plot the universe: *All drivers are men* i.e., region 1 and 2. Then for the premise *my mother is a driver* is region 1. The conclusion *my mother is a man* is self-evident from region 1.

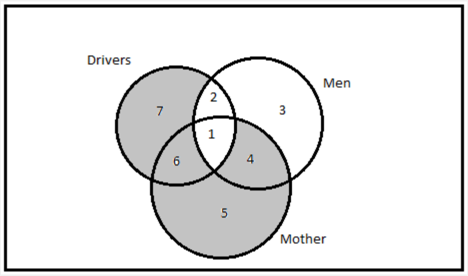

Figure 2

Example 2: Consider the following argument and use a Venn diagram to determine if it is valid.

p: All professional models are singers

q: Otieno is a singer

∴Otieno is a professional model.

Solution

If Y represents *Otieno as a singer* (Assumption *q*), X represents *Otieno is a professional model,* then the different positions means that the argument is invalid (though the conclusion is not necessarily false).

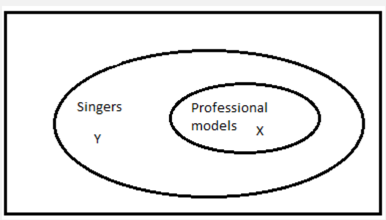

Figure 3

Alternatively: We first plot the universe i.e., All professional models are singer i.e. region 1 and 2.

Then *Otieno is a singer* is region 1 and 4. The conclusion *Otieno is a professional model is not self-evident.*

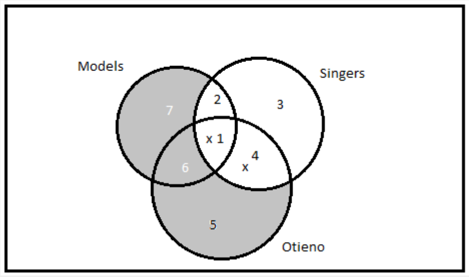

Figure 4

Example 3

Consider the argument;

p: No gecko is warm-blooded

q: All mammals are warm-blooded

∴*Geckos are not mammals*

Let x be gecko, then as shown below the position is unique hence the argument is valid.

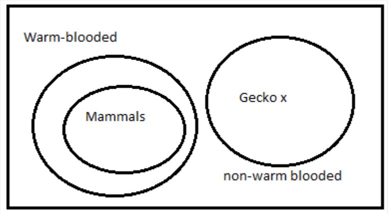

Figure 5

Alternatively: We first plot the universe

$q: All\ mammals\ are\ warm-blooded$ – this is region 1 and 2 where all mammals are warm-blooded.

$p: No\ gecko\ is\ warm-blooded$ – we shade region 1 and 4 since no gecko is warm-blooded

It is self-evident that geckos are not mammals (region 5)

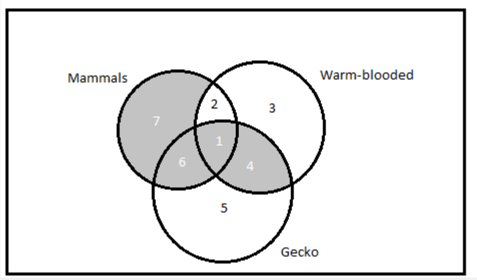

Figure 6

Truth Tables and Validity of Arguments

Truth table Procedure

Option 1: Finding the critical rows

Apply the following procedures;

(i) Construct the truth table for the argument.

(ii) Determine the critical rows i.e., rows in which all the assumptions are true.

(iii) Check the conclusion of all critical rows i.e., rows where the premises are all TRUE.

(iv) Then;

a. If in each critical row the conclusion is TRUE then the argument is valid.

b. If there is a row in which the conclusion is FALSE then the argument is invalid.

Example 1: Show that the argument; $p \rightarrow q,\ q \rightarrow p\ \therefore\ p \lor q$ is invalid

Solution (Option 1):

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \lor q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

From the table above, the last row is critical row i.e. it has all the assumptions true, but the conclusion is false. Hence, the argument is invalid.

Optional 2

In this option we determine if the argument is a tautology. If the argument is a tautology, then it is valid otherwise it is invalid.

Solution : We check to see if $((p \rightarrow q) \land (q \rightarrow p)) \rightarrow (p \lor q)$ is a tautology.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \lor q$	$\big((p \rightarrow q) \land (q \rightarrow p)\big) \rightarrow (p \lor q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	F

From the last column it is clear that $((p \rightarrow q) \land (q \rightarrow p)) \rightarrow (p \lor q)$ is NOT a tautology. Hence the argument is invalid.

Example 2: Determine if the validity of the argument is valid: p ⟶ q; r ⟶ q; ¬p ∧ r ∴ q

Solution (by Option 1): Construct the truth table and determine the critical rows.

p	q	r	$\neg p$	$p \to q$	$r \to q$	$\neg p \wedge r$	q
T	T	T	F	T	T	F	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	F	F
T	F	F	F	F	T	F	F
F	T	T	T	**T**	**T**	**T**	**T**
F	T	F	T	T	T	F	T
F	F	T	T	T	F	T	F
F	F	F	T	T	T	F	F

Row 5 is the only critical row, since all the assumptions are TRUE. Note that the conclusion is TRUE and hence the argument is valid.

Option 2: We check if the argument is a tautology i.e. $[(p \to q) \wedge (r \to q) \wedge (\neg p \wedge r)] \to q$ is a tautology.

p	q	r	$\neg p$	$p \to q$	$r \to q$	$\neg p \wedge r$	$(p \to q) \wedge (r \to q) \wedge (\neg p \wedge r)$	$[(p \to q) \wedge (r \to q) \wedge (\neg p \wedge r)] \to q$
T	T	T	F	T	T	F	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	F	F	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	T	T	F	T	F	T
F	F	F	T	T	T	F	F	T

The argument is a tautology (the last row is all TRUE) and hence it valid.

Example 3: Rewrite the following argument in symbolic form, and hence determine its validity.

$$\textit{If the carpenter is tall, the carpenter will not buy the flowers.}$$

$$\textit{The carpenter bought the flowers}$$

$$\therefore \textit{The carpenter is tall}$$

Solution: Let p: *the carpenter is tall*; q: *the carpenter bought the flowers*. The argument in logic connectives becomes;

$$\begin{array}{l} p \to \neg q \\ q \\ \hline \therefore p \end{array}$$

Alternatively, you can write the argument as follows; $p \to \neg q, q \therefore p$

			Assumptions		Conclusion
p	q	$\neg q$	$p \to \neg q$	q	p
1	1	0	0	1	1
1	0	1	1	0	1
0	1	0	**1**	**1**	0
0	0	1	1	0	0

Row three is the critical row, however the conclusion is false. Hence the argument is invalid.

Rules of inference

Sometimes using truth tables to test for the validity of argument can be tedious. For example to test the validity or argument with 5 different propositional variables it will require $2^5 = 32$ rows. Hence we use the logic laws and rules of inference to establish the validity of such arguments. A rule of inference is a compound proposition that is a tautology and involves an implication.

Note that when using the critical rows, the assumptions must be all true and so must be the conclusion for the argument to be valid. In proving validity without using tables , it sufficient to assume the assumptions are true and then proceed to show if the conclusion is true.

Examples of standard rules of inferences

Rule of inference	Tautology	Name
P, p → q ∴ q	[p ∧ (p → q)] → q	Modus ponens
¬q, p → q ∴ ¬p	[¬q ∧ (p → q)] → ¬p	Modus tollens
p ∴ p ∨ q	p → (p ∨ q)	Addition
p ∧ q ∴ p	(p ∧ q) → p	Simplification
p → q, q → r ∴ p → r	[(p → q) ∧ (q → r)] → (p → r)	Hypothetical syllogism
p ∨ q, ¬p ∴ q	((p ∨ q) ∧ ¬p) → q	Disjunctive Syllogism
p, q ∴p ∧ q	p ∧ q → p ∧ q	Conjunction

Example 1: Without use of tables, determine the validity of the argument;

$$p \rightarrow \neg q, q \therefore p$$

Solution: We assume the premises are TRUE and proceed to determine the truth value of the conclusion.

(1) q is a premise, assume it is T.
(2) $\neg q$ is F from (1) above.
(3) $p \rightarrow \neg q$ is a premise we assume it is T.
(4) P is false from (3) above (since (3) can only be true when both p and $\neg q$ are false.

Since the premises are all true but conclusion is false then the argument is invalid.

Example 2: Consider the following argument:

It is not noisy, and it is hot.
I will go to class only if it is noisy.
If I don't go to class then I will take a nap.
If I take a nap, then I will be at the party by 7pm.
Therefore, I will be at the party by 7pm.

Determine if the argument is valid.

Solution: Let *p: it is noisy; q: It is hot; r: I will go to class; s: I will take a nap; and t: I will be at the party by 7pm.* Then;

(1) $\neg p \wedge q$ *Premise*
(2) ¬p Simplification rule using (1)
(3) $r \rightarrow p$ *Premise*
(4) $\neg r$ *Modus Tollens using (2) (3)*
(5) $\neg r \rightarrow s$ *Premise*
(6) s Modus Ponens using (4) (5)
(7) $s \rightarrow t$ *Premise*
(8) t Modus Ponens using (6) (7). The argument is valid.

Exercise

1) Determine the validity of the following arguments;

(a) p: My watches are the only items I have that are made of silver.

q: I find all my accessories valuable.

r: None of my watches is of any value.

∴ My accessories are not made of silver.

(b) p: All my friends are dancers.

q: Chris is my friend.

r: None of my classmates is a dancer.

∴ Chris is not my classmate.

2) Consider the following assumptions;

p: *Musicians are happy people*

q: Every teacher is wealthy

r: No happy person is wealthy

Use a Venn diagram to determine the validity of each of the following conclusions;

(i) No musician is wealthy.

(ii) Teacher are happy people.

(iii) No person can be both a musician and a teacher.

3) Use a Venn diagram to check the validity of the following argument. (4 mks)

All beggars are homeless.
Kim is not a beggar.
Therefore, Kim is not homeless.

4) Attempt exercises in (Kenneth, 2012, p. 78).

References

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Venn diagram and validity of arguments