

Discrete Mathematics

Lecture 3

Predicate logic

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Introduction to Lecture 3

This lecture introduces the concept of predicate logic (predicate calculus), its importance to computing, and its application. The universal and existential quantifiers are the most important quantifiers in mathematics and computer science. PROLOG is a programming language using Logic that was developed in the 1970s by computer scientists working in AI. It makes use of predicates.

References

These lecture notes have been derived from the following sources , Rosen (2012), Koman et al. (2001), Seymour (2020), and Susanna (2003).

Intended learning outcomes

At the end of this lecture, you will be able to;

- (i) Define terms used in predicate logic.
- (ii) Solve problems involving predicate logic.

Introduction to predicate logic

Propositional logic cannot represent all types of assertions that are used in computer science and mathematics. It cannot express certain types of relationship between propositions such as equivalence.

Example 1: The assertion $x > 3$ where x is a variable is not a proposition. Proposition logic cannot deal with this.

Example 2: 'Not all birds fly' is equivalent to 'some birds don't fly'. In propositional logic we treat the assumption independently.

Say, p : *Not all birds fly* q : *Some birds don't fly*. Unfortunately, in propositional logic, there is no mechanism to find out whether p is equivalent to q .

Definition of Terms

Definition 1: A predicate is a verb that describes a property of objects or a relationship among objects represented by variables.

Example 1: Consider the following statements; *The car Otieno is driving is black*; *The shoes are black*. *My trouser is black*. The phrase '*is black*' is a predicate and describes the property of being black.

If we let p be the name for the predicate '*is black*', then the sentences that assert an object is black can be represented as $p(x)$, where x represent an arbitrary object i.e. $p(x)$ means '*x is black*'.

Definition 2: A predicate with variables is not a proposition e.g., $x > 4$. However, a predicate with variables can be made a proposition by following ways;

- (a) Assigning a value to the variable.
- (b) Quantifying the variable using a quantifier.

Example 1: We can assign x is 7 and hence the statement $x > 4$ becomes $7 > 4$ which is a proposition. It is a proposition with truth value, TRUE.

Example 2: The second way is to quantify the statement. There are two types of quantifiers;

- (a) Universal quantifier \forall
- (b) Existential quantifier \exists

Consider the statement $x > 4$. We can use the universal quantifier denoted \forall to change it to; '*For every x in the set of natural numbers, $x > 4$* ', which is better expressed as, $\forall x \in \mathbb{N}, x > 4$. It is now a proposition. The statement is false in the set of natural numbers. One need also to specify the set. In this case the set of natural numbers \mathbb{N} is referred to as the universe of discourse or the domain.

Example 2: The statement $x > 4$ can be written as, 'For some x in the set of natural numbers, $x > 4$ ', which can also be expressed as, $\exists x \in \mathbb{N}, x > 4$. This statement is true in the set of $\mathbb{N} = \{1, 2, 3, \dots\}$. Since we only need one number to make the statement true e.g., if we assign x the value 10 then the statement is true.

Reading Quantified Formula

The expression $\forall x \in \mathbb{D}, p(x)$ denotes the universal quantification of the atomic formula $p(x)$ in the domain \mathbb{D} . One can read it as, 'For all x in \mathbb{D} , $p(x)$ holds', or 'For each x in \mathbb{D} , $p(x)$ holds' or 'For every x in \mathbb{D} , $p(x)$ '.

The expression $\exists x, p(x)$ denotes the existential quantification of $p(x)$. One can read it as, 'There exists an x such that $p(x)$ holds' or 'There is at least one x such that $p(x)$ holds' or 'for some x such that $p(x)$ holds'

$\forall x \in \mathbb{D}$ means 'Every x in the universe/domain' while $\exists x \in \mathbb{D}$ means 'at least one x in the universe/domain'. Hence, $\exists x, p(x)$ means that $p(x)$ is true for at least one x of the universe.

Example 1: Let the universe of discourse be the set of all phones in the market and let $f(x, y)$ denote, 'x has a higher memory than y'. Then in English we have,

a. $\forall x, \forall y, f(x, y)$

For every phone x then the following holds; for every phone y , phone x has a higher memory than phone y .

b. $\exists x, \exists y, \neg f(x, y)$

For some phone x in the market there exists a phone y such that phone x has no higher memory than phone y .

Example 2: Let the universe of discourse be the set of all phones in the market and let $f(x, y)$ denote, 'x has a higher memory than y'. Then we can write the following using logic connectives,

a. There exists a phone x which satisfies the following; for every phone y , x has a higher memory than y .

$$\exists x, \forall y, f(x, y)$$

b. For every phone x the following holds; for some phone y , x does not have a higher memory than phone y .

$$\forall x, \exists y, \neg f(x, y)$$

Remark 1: Universal quantifier \forall and connective AND: Suppose it is possible to list all the elements in the universe of discourse then the universal quantification, $\forall x, p(x)$ is equivalent to the conjunction;

$$p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_n)$$

This conjunction is true if and only if $p(x_1), p(x_2), \dots, p(x_n)$ are all true.

Example 1: Determine the truth value of the statement $\forall x, p(x), x \in \mathbb{N}, 1 \leq x \leq 3$ where $p(x)$ is the statement $x^3 \geq 8$

Solution: Our statement is the same as the conjunction

$$p(1) \wedge p(2) \wedge p(3)$$

Note that $p(1)$ is false since $1^3 < 8$. Hence the truth value of our statement is false.

Remark 2: Existential quantifier \exists and connective OR: Suppose all the elements of the universe of discourse can be listed, then the existential quantification $\exists x, p(x)$ is equivalent to the disjunction;

$$p(x_1) \vee p(x_2) \vee \cdots \vee p(x_n)$$

The above disjunction is true if and only if at least one of $p(x_1), p(x_2), \dots, p(x_n)$ is true. We need to show that only one case is true for the whole statement to be true.

Example 1: Determine the truth value of the statement $\exists x, p(x), x \in \mathbb{N}, 1 \leq x \leq 5$ where $p(x)$ is the statement $x^3 < 4$

Solution: Our statement is the same as the conjunction

$$p(1) \vee p(2) \vee p(3) \vee p(4) \vee p(5)$$

Note that $p(1)$ is true since $1^3 < 4$. Hence the truth value of our statement is true.

Remark 3: Two statements P and Q involving predicates and quantifier are logically equivalent denoted $P \equiv Q$ if and only if the two have the same truth value (no matter the predicates P and Q takes or even the domain).

Example 1: $\forall x(p(x) \wedge q(x)) \equiv \forall x p(x) \wedge \forall x q(x)$ - Universal quantifier is distributive over conjunction (but not over disjunction).

$\exists x(p(x) \vee q(x)) \equiv \exists x p(x) \vee \exists x q(x)$ - existential quantifier is distributed over disjunction (but not over conjunction).

Interpreting the truth value

Example 1: State if the following propositions are true or false;

(a) $(\forall n \in \mathbb{N})(n + 4 > 3)$

Solution: Let $N = \{0, 1, 2, 3, \dots\}$. If we consider $n = 0 \in N$ then our statement will be $0 + 4 > 3$. This is true, and since 0 is the smallest in our set then we can conclude that the statement is true of all values of n .

(b) $\forall n \in \mathbb{N}, n + 2 > 8$

Solution: Let $N = \{0, 1, 2, 3, \dots\}$. If we consider $n = 0 \in N$ then our statement will be $0 + 2 > 8$. This is false, we can conclude that the statement is false.

Remark 1: If $\{x | x \in A, p(x)\} = A$, then $\forall x, p(x)$ is true, otherwise $\forall x, p(x)$ is false.

Example 3: State if the following propositions are true or false;

(a) $\exists n \in \mathbb{N}, n + 3 < 7$

Solution: Let $N = \{0, 1, 2, 3, \dots\}$. If we consider $n = 0 \in N$ then our statement will be $0 + 3 < 7$. This is True, we can conclude that the statement is True. We just have to show it is true for one element.

(b) $\exists n \in \mathbb{N}, n + 6 < 2$

Solution: Let $N = \{0, 1, 2, 3, \dots\}$. If we consider $n = 0 \in N$ then our statement will be $0 + 6 < 2$. This is False, we can conclude that the statement is False, since zero is the smallest.

Remark 2: If $\{x | p(x)\} \neq \emptyset$ then $\exists x, p(x)$ is true, otherwise $\exists x, p(x)$ is false.

Negating quantified statements

Quantified statements with more than one variable may be negated by successively applying the De Morgan's laws for quantifiers;

(a) $\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$ That is, *It is not true that for all $x \in A$, $p(x)$ is true*, is equivalent to, *There exists an $x \in A$ such that $p(x)$ is false*.

(b) $\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$ That is, *It is not true that for some $x \in A$, $p(x)$ is true*, is equivalent to, *For all $x \in A$, $p(x)$ is false.*

Each \forall is changed to \exists and each \exists is changed to \forall as the negation symbol \neg passes through the statement from left to right.

Example 1: Consider the statement; *All computer science lecturers are male.* Its negation reads as; *It is not the case that all computer science lecturers are male.*

Equivalently we have; *There exists at least one computer science lecturer who is not a male.*

Now suppose p denote the set, *computer science lecturers.* Then the negation can be written as;

$$\neg(\forall x \in p)(x \text{ is a male}) \equiv (\exists x \in p)(x \text{ is not a male})$$

Alternatively, when $t(x)$ denote; x is male then the above statement becomes;

$$\neg(\forall x \in p)t(x) \equiv (\exists x \in p)\neg t(x)$$

Alternatively; $\neg \forall x t(x) \equiv \exists x \neg t(x)$

Remark 1: Alternatively suppose we have it as *For every lecturer x , if x is the computer science lecturer, then x is male.* Then let $P(x)$ represents the statement that the lecturer in computer science and $Q(x)$ represent x is a male. Then our statement can be expressed as;

$$\forall x (P(x) \rightarrow Q(x))$$

Again recall $p \rightarrow q \equiv \neg p \vee q$. Hence we have;

$$\forall x (P(x) \rightarrow Q(x)) \equiv \forall x [\neg P(x) \vee Q(x)]$$

Remark 2: Consider the statement *Some cars in the parking lot are white in color.* We can let $P(x)$ be the predicate x are white in color. Where x is the domain of cars. Then our statement *some cars in the parking lot are white in color* can be written as $\exists x, P(x)$.

However, if our interest is more than this category of cars then we can write our statement as;

There is a car x in the parking lot and is white in color.

In this case we introduce $S(x)$ to represent car x in the parking lot. Hence our statement becomes;

$$\exists x (S(x) \wedge P(x))$$

In other words, *There is a car in the parking lot and the car is white in color.*

Example 2: Negate the following statements and simplify as possible;

Every student has at least one course where the lecturer is a man.

Solution: Suppose we let x be student, y is a course, $p(x, y)$ be a course by student x where the lecturer is a man. Then our statement can be written as; $\forall x, \exists y, p(x, y)$

Hence the negation of the statement is; $\neg(\forall x, \exists y, p(x, y))$ i.e., *it is not the case that every student has at least a course where the lecturer is a man.*

Alternatively, $\neg(\forall x, \exists y, p(x, y)) \equiv \exists x, \forall y, \neg p(x, y)$ i.e., *some students have all courses where the lecturer is not a man.* Alternatively, *There is a student such that in every course the lecturer is not a man.*

Example 3: Negate each of the following statements;

(a) $\exists x \forall y, p(x, y)$ **Solution:** $\neg(\exists x \forall y, p(x, y)) \equiv \forall x \exists y, \neg p(x, y)$

(b) $\forall x \forall y, p(x, y)$ **Solution:** $\neg(\forall x \forall y, p(x, y)) \equiv \exists x \exists y, \neg p(x, y)$

(c) $\exists y \exists x \forall z, p(x, y, z)$ **Solution:** $\neg(\exists y \exists x \forall z, p(x, y, z)) \equiv \forall x \forall y \exists z, \neg p(x, y)$

Example 4: Negate each of the following statements;

(a) All lions live in the savannah. **Solution:** At least one of lions does not live in the savannah.

(b) All mathematicians are women. **Solution:** At least one mathematician is a man.

(c) Some students are Kenyans. **Solution:** None of the students is Kenyans or All students are not Kenyans.

Exercise

1) Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements;

(i) $\exists x \in A, x + 3 = 9$

(ii) $\forall x \in A, x + 2 < 7$

(iii) $\exists A, x + 3 > 6$

(iv) $\forall x \in A, x + 1 \leq 8$

2) Determine the truth value of each of the following statements where the universal set $U = \{1, 2, 3\}$

(i) $\exists x, \forall y, x^2 < y + 1$

(ii) $\forall x \exists y, x^2 + y^2 < 11$

(iii) $\forall x \forall y, x^2 y > 10$

3) Let $A = \{1, 2, 3, \dots, 7\}$ and let $p(x, y)$ denote ' $x + y = 9$ '. Then $p(x, y)$ is a proposition. Determine the truth value of;

(i) $\forall x \exists y, p(x, y)$.

(ii) $\exists y \forall x, p(x, y)$.

4) Use logic connectives to work the negation of each of the following statements;

(a) If the teacher is absent, then some students do not complete the assignment.

(b) All students completed their assignment, and the teacher is present.

(c) Some of the students did not complete their assignment or the teacher is absent.

5) Write using logic connectives

(a) For every car x , if x is a car in the parking lot, then x is white in color or has tinted window.

(b) There is a student x , if x is a student in class, then x is doing discrete mathematics and theology.

6) Attempt exercises in (Rosen, 2012, p. 53).

References

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