Discrete Mathematics

Lecture 9

Relations and Functions

Lecturer: Kahenya N.P

Introduction to lecture 9

Relations and functions are fundamental concepts in mathematics and computer science and in particular graph theory. This lecture forms a foundation for the lecture on trees and graph theory in general. Relations can be used to understand and solve problems in communications, networks, relationships between members of a social group among others.

References

These lecture notes have been derived from the following sources (Lipschutz & Lipson, 2007; Rosen, 2012).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Explain relations and functions.

(ii) Apply the concepts in solving problems involving relations and functions.

Introduction to Relations

Relations are structures that are used to represent the relationships between elements of a given set Relations can be used to solve problems such as;

(i) Determining which pairs of towns that connected by roads or trains.

(ii) Determining how a set of computers are networked.

(iii) Determine the best way to store information in a computer database.

Binary relations

Product sets are examples of binary relations where it is used to express relationship between elements of two sets. For example, let A and B be any two sets. The set of all ordered pairs of (a, b) where $a \in A$ and $b \in B$ is called the product or Cartesian product of A and B i.e.

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

For example, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the set of ordered pairs of real numbers.

Example 1: Let $A = [2,3\}$ and $B = \{a, b, c, d\}$ then;

$A \times B = \{(2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c), (3, d)\}$

$B \times A = \{(a, 2), (a, 3), (b, 2), (b, 3), (c, 2), (c, 3), (d, 2), (d, 3)\}$

$A \times A = \{(2,2), (2,3), (3,2), (3,3)\}$

Note that;

(i) $A \times B \neq B \times A$

(ii) The order in which the elements appear matters.

(iii) $n(A \times B) = n(A) \cdot n(B)$ where $n(A)$ is the cardinality of set A.

For instance, in the example above, $n(A \times B) = 8$ and $n(A) = 2$, $n(B) = 4$ clearly $n(A \times B) = n(A) \cdot n(B)$

In general, for any finite sets $A_1, A_2, \cdots, A_n$ the set of all ordered n- tuples $(a_1, a_2, a_3, \cdots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n$ is called the product of sets $A_1, A_2, \cdots, A_n$ denoted;

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n = A^n$$

Definition 1: Let A and B be sets. A binary relation or simply a relation from A to B is a subset of $A \times B$. That is, for each pair $a \in A$ and $b \in B$ exactly one of the following is true;

(i) $(a, b) \in R$ i.e 'a is R- related to b' written a R b.

(ii) $(a, b) \notin R$ i.e., 'a is not R-related to b' written a $\not R$ b.

Remark 1: If R is a relation from a set A to itself, that is if R is a subset of $A^2 = A \times A$ then we say that R is a relation on A.

Example 1: Let $A = \{1,2\}$ and $B = \{x, y\}$ and $R = \{(1, x), (1, y)\}$ then R is a relation from A to B since R is a subset of $A \times B$. Hence we have 1 R x, 1 R y but 2 $\not R$ x, 2 $\not R$ y.

Note that the domain of R is $\{1\}$ and the range of R *is* $\{x, y\}$.

Example 2: Let $A = \{\text{students in a class}\} = \{\text{Oti, Onyi, Kama, Chris, Njeri}\}$ and B be the set of courses i.e.,$B = \{\text{History, Maths, English}\}$. Let R be the relation that consists of the pairs (a,b) where a is a student enrolled in course b. for instance if Oti and Njeri are enrolled in maths then we have (Oti, Maths) and (Njeri, Maths) belong to R.

Examples of Relations

1) Set inclusion $\subseteq$ is a relation on any collections of sets e.g., $A \subseteq B$ or $A \nsubseteq B$.
2) In the set of integers, we have such relations as $m|n$ i.e., m divides n e.g., $4|12$, $5|25$ but $4 \nmid 17$.
3) Functions are relations. Where a function says, f from set X to a set Y assigns exactly one element of Y to each element of X. The graph of f is a set of ordered pairs (x, y) so that $y = f(x)$ which is a subset of $X \times Y$.

Inverse relation

Let R be any relation from set A to set B. The inverse of R denoted R^{-1} is the relation from B to A which consists of ordered pairs which when reversed belong to R i.e., $R^{-1} = \{(y, x)|(x, y) \in R\}$.

Example 1: Let set $A = \{1,2\}$ and $B = \{a, b\}$ then the inverse of $R = \{(1, a), (1, b), (2, a)\}$ is $R^{-1} = \{(a, 1), (b, 1), (a, 2)\}$.

Example 2: Consider a set A with n elements. A relation on the set A is a subset of the $A \times A$. The set $A \times A$ has n^2 elements. The set A with n elements has 2^n subsets, and therefore we can say that $A \times A$ has 2^{n^2}subsets. For instance, if the set $A = \{3, d, f, g\}$ then $A \times A$ has $2^{4^2} = 65536$ relations.

Representation of Relations

a) **Matrix of relation**

This can be done in a matrix form where we use 1 or 0 if there is a relation or no relation respectively.

For example, let sets $A = \{1,2,3\}$ and $B = \{a, b, c\}$ and $R = \{(1, b), (1, c), (2, c), (3, a)\}$.

Then the matrix of relation is

	a	**b**	**c**
1	0	1	1
2	0	0	1
3	1	0	0

In general, if $A = \{a_1, a_2, \cdots, a_m]$ and $B = \{b_1, b_2, \cdots, b_n\}$ are finite sets containing m and n elements, respectively. R is a relation from A to B then R can be represented by the $m \times n$ matrix $M_R = [m_{ij}]$ i.e.

$$M_R = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the matrix of R.

Hence for the above example its matrix of relation is; $M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Example 1: Let $M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and let sets $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$. Then by definition $(a_i, b_j) \in R$ iff $m_{ij} = 1$ thus; $R = \{(a, w), (a, z), (b, x), (b, y), (c, x), (c, z)\}$.

b) **Directed graphs/digraphs of R**

Relations can be represented using directed graphs, where the elements represent the vertices or nodes of the digraph and the relation between vertices is represented by a directed edge.

Example 1:

Let R be a relation on the set $A = \{1,2,3,4\}$ where $R = \{(1,2), (2,2), (2,4), (3,4), (4,1), (4,3)\}$ then its directed graph is;

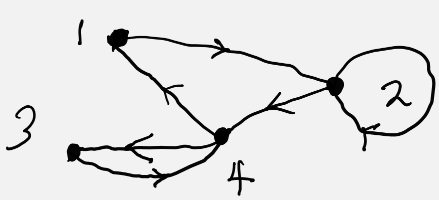


Figure 1: Digraph

Example 2: Let $A = \{1,2,3,4\}$ and $R = \{(1,1), (1,3), (1,4), (2,3), (3,1), (3,4), (4,4), (2,4)\}$. Then the diagraph of the relation is;

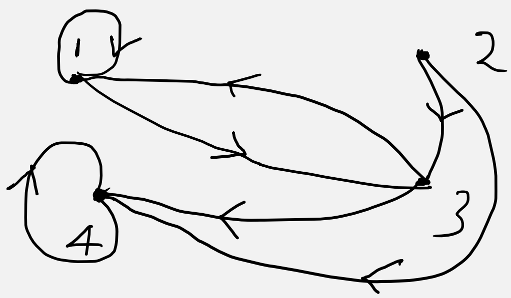


Figure 2: Digraph

Example 3: Find the relation determined by digraph below;

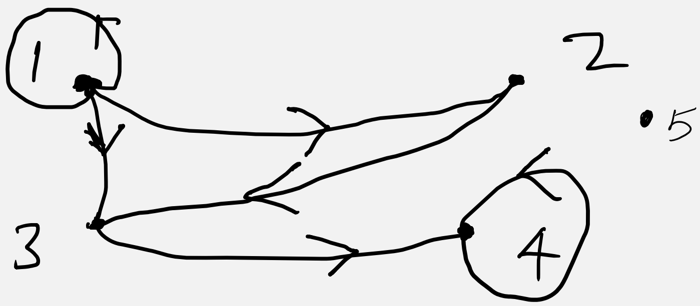


Figure 3: Digraph

Solution: Now let set $\mathrm{A} = \{1,2,3,4,5\}$ and $\mathrm{R} = \{(1,1), (1,2), (1,3), (2,3), (3,2), (3,4), (4,4)\}$.

Definition 1: The in-degree of a vertex is the number of edges terminating at the vertex or node or junction, while the out-degree of a vertex is the number of edges leaving a vertex.

Example 1: Let set $\mathrm{A} = \{1,2,3,4\}$ and let the relation on A represented by the matrix,

$$\mathrm{M_R} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Construct the digraph of R and list in-degree and out-degrees of all vertices.

Solution:

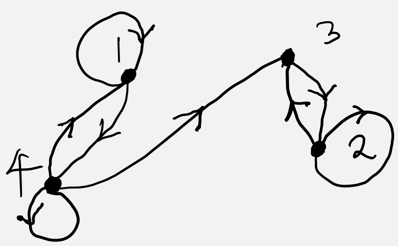


Figure 4: Digraph

Node	1	2	3	4
In-degrees	2	2	2	2
Out-degrees	2	2	1	3

c) **Graphs of functions**

Functions are relations and hence can be represented by graphing.

d) **Arrow diagram**

Relations can be represented in an arrow diagram. For example, let sets $A = \{1,2,3\}$ and $B = \{a, b, c\}$ and $R = \{(1, b), (1, c), (2, c), (3, a)\}$.

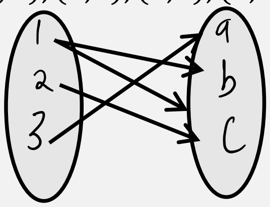


Figure 5: Arrow Diagram

Composition of relations

Definition 1: Let A, B, and C be any non-empty sets. Let R and S represent the relations from A to B, and B to C respectively. In other words, the relation R is a subset of $A \times B$ while the relation S is a subset of $B \times C$. Then the relation from A to C denoted $S \circ R$, the composition of S and R, is defined as;

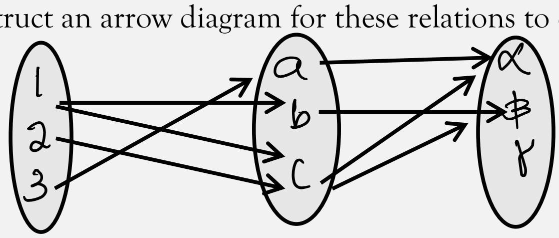
$$a(S \circ R)c$$

If for some $b \in B$ then aRb and bSc, that is,

$$S \circ R = \{(a, c) | \exists\, b \in B, (a, b) \in R, (b, c) \in S\}$$

Example 1: Consider the sets $A = \{1,2,3\}$, $B = \{a, b, c\}$ and $C = \{\alpha, \beta, \gamma\}$ and the relations $R = \{(1, b), (1, c), (3, a), (2, c)\}$ and $S = \{(a, \alpha), (b, \beta), (c, \alpha), (c, \beta)\}$. Determine $S \circ R$.

Solution: We can construct an arrow diagram for these relations to comprehend the composite relation.



From the arrow diagram we get;

$$S \circ R = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha)\}$$

Alternatively, consider the matrix of relations R and S.

$$M_R = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; M_S = \begin{matrix} a \\ b \\ c \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Then $M_R M_S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

The 2 implies that there are two ways from 1 to β. The non-zero entries indicate the relation $S \circ R$ exists.

Types of relations

(i) Reflexive

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$. That is R is reflexive if every element $a \in A$ is related to itself, otherwise it is irreflexive.

Example 1: Consider the set $A = \{a, b, c\}$ and the relations R on the set A namely;

$$R_1 = \{(a, a), (a, b), (a, c), (c, c)\}$$
$$R_2 = \{(a, b), (b, b), (c, c), (a, a), (a, c)\}$$
$$R_3 = \{(a, a), (b, b), (c, a), (c, b)\}$$

Only R_2 is reflexive since we have $(a, a) \in R, (b, b) \in R, (c, c) \in R$.

Note that the universal relation $A \times A$ is also reflexive.

(ii) Symmetric

A relation R on a set A is symmetric if whenever a R b then b R a.

A relation R on a set A is asymmetric if whenever a R b then b $\not R$ a i.e., $(a, b) \in R$ then $(b, a) \notin R$.

A relation R on a set A is antisymmetric if whenever a R b and b R a then a = b. Also, R is still antisymmetric if whenever $a \neq b$ then $(a, b) \notin R$ or $(b, a) \notin R$.

Remarks

- Symmetric is not the opposite of antisymmetric.
- A relation can have both of these properties or may lack.
- A relation can be both symmetric and antisymmetric if it contains some pair of the form (a, b), with $a \neq b$

Example 1: Let $A = \mathbb{Z} = \{\text{set of integers}\}$ and let $R = \{(a, b) \in A \times A | a < b\}$ where R is the relation less than. Is R symmetric, asymmetric, or antisymmetric?

Solution:

Symmetry: If $a < b$ then it is not true that $b < a$. Therefore, R is not symmetric.

Asymmetry: If $a < b$ then $b \nless a$ so R is asymmetric.

Antisymmetric: If $a \neq b$ then either $a \nless b$ or $b \nless a$. Therefore, R is a antisymmetric.

Example 2: Let $A = \{1,2,3,4\}$ and the relations R on A be as below. Determine which of the relations are symmetric and which are antisymmetric (Rosen, 2012) ;

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1)\}$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

Solutions: R_1, R_2 and R_3 are symmetric since (b, a) belongs to the relation whenever (a, b) does. For example (1, 2) and (2, 1) in R_1 and R_2 and (1, 4) and (4, 1) in R_3.

R_4, R_5 are both antisymmetric i.e., for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.

(iii) Transitive

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in A$ i.e. If a R b and b R c then a R c.

Example 1: Is the 'divides' relation on set of positive integers transitive?

Solution: Let a divides b and b divides c, does a divides c?

If a divides b then there exists an m such that $b = am \cdots$ (i)

Also, if b divides c then there exists an n such that $c = bn \cdots$ (ii)

From (i) and (ii) we have; $c = bn = (am)n = a(mn) \Rightarrow a|c$. Hence 'divides' relation is transitive.

Definition 1: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example 1: Let $A = \{1,2,3,4\}$ and $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$ then R is an equivalence relation.

Definition 2: Let A be a set. A partition of A is a subset of A such that every element of A is an element of exactly one of the subsets e.g. sets of odd and even numbers

Functions/Mappings/Transformations

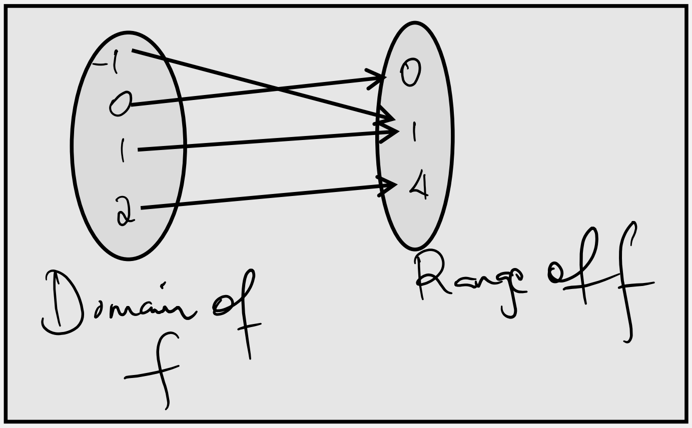
Definition 1: Let X and Y be sets. A function from X to Y is a rule that assigns to each element of X exactly one element of Y. A function is generally denoted with f, g, h etc.

If $f: X \rightarrow Y$ then X is called the Domain of f and Y is the codomain.

If $f: X \rightarrow Y$ the set of images $\{y \in Y: y = f(x) \text{ for some } x \in X\}$ is called the range of f. Note that range is a subset of codomain.

Example 1: Consider the function $f: x \rightarrow x + 1$ where $0 \leq x \leq 3,\ x \in \mathbb{R}$ then the domain of f is the set $X = \{0,1,2,3\}$ and the range (codomain) is the set $Y = f(x) = \{1,2,3,4\}$.

Definition 2: A function f is onto if its range is equal to its domain e.g., consider the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ over $-1 \leq x \leq 2$

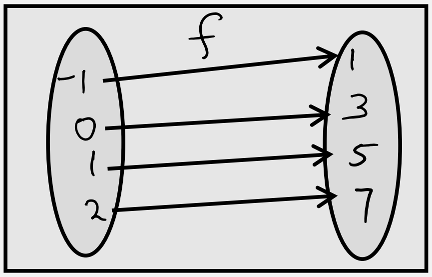


Remark: f is onto if every element of the codomain is the image of at least one element of the domain.

Definition 3: A function f from set A to B is one-to-one if no two distinct elements of the domain have the same image i.e. If $f(a) = f(b)$ then $a = b$.

Definition 4: A function f from set A to set B is onto if $\forall b \in B, \exists a \in A$ such that $f(a) = b$

Example 1: Consider the function $f: x \rightarrow 2x + 3$ over $-1 \leq x \leq 2$



f is 1-1 and onto mapping.

Definition 5: A function $f: X \to Y$ is said to be invertible if its inverse relation f^{-1} is also a function. Note that a function is not necessarily invertible.

Theorem 1: Let $f: X \to Y$ be a function

(i) Then f^{-1} is a function from B to A iff f is 1-1.

(ii) If f^{-1} is a function, then the function $f^{-1}$is also 1-1.

(iii) f^{-1} is everywhere defined iff f is onto.

(iv) f^{-1} is onto iff f is everywhere defined.

Remark 1: Let f be a function $f: X \to Y$. Then we say that f is everywhere defined if $Dom(f) = X$

Theorem 2: A function f has an inverse iff f is onto and 1-1.

Definition 6: (Composite function) Consider the functions $f: X \to Y$ and $g: Y \to Z$ i.e., where the codomain of f is domain of g. We define the function from X to Z as the composition of f and g written $f \circ g$ as follows

$$(g \circ f)(x) = g(f(x))$$

Definition 7: (Inverse of a function) Let f be $1 - 1$ function, then g is the inverse of f denoted f^{-1} if;

(i) $(f \circ g)(x) = x, \forall\, x \in D(g)$

(ii) $(g \circ f)(x) = x, \forall x \in D(f)$

Determining the inverse of a function $y = f(x)$

<u>Steps</u>

(i) Check if f is $1 - 1$

(ii) If yes, then solve for x i.e. let $x = f^{-1}(y)$.

(iii) Exchange x and y to get $y = f^{-1}(x)$

(iv) Check that $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)$

Example 1: Determine whether $f(x) = 5x$ is a $1 - 1$ function

Solution: Suppose $f(x) = \alpha \Rightarrow \alpha = 5x \therefore x = \frac{\alpha}{5} \Rightarrow f(x)$ is $1 - 1$.

Example 2: Determine if $f(x) = x^2$ is a one-to-one function.

Solution: Let $f(x) = \alpha \therefore \alpha = x^2 \Rightarrow x = \pm\sqrt{\alpha}$. Therefore $f(x)$ is not one-to-one (since we are getting two values of x).

Example 3: If f be defined by $f(x) = x^3 - 2$ and $g(x) = \sqrt[3]{(x+2)}$. Determine if g is the inverse of f.

Solution:

Step 1: Determine if f is $1-1$. Let $f(x) = \alpha \Rightarrow x^3 - 2 = \alpha \therefore x = \sqrt[3]{(x+2)}$ i.e.f is $1-1$.

Step 2: By definition if g is the inverse of f then $(f \circ g)(x) = (g \circ f)(x) = x$.

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{(x+2)}\right) = \left(\sqrt[3]{(x+2)}\right)^3 - 2 = x$$

$$(g \circ f)(x) = g(f(x)) = g(x^3 - 2) = \sqrt[3]{(x^3 - 2) + 2} = x$$

$$\Rightarrow g = f^{-1}$$

Applications of Relations

It is used in relational data model, where a database consists of records that a n-tuples and are made up of fields. Tables are used to show relations in a database. The database query language SQL (Structured Query Language) is used to carry out the operations of relations in a database.

Exercise

1) Determine if f is one-to-one given
   a) $f(x) = 2x - 1$
   b) $f(x) = \sqrt{81 - x^2}$
2) Find the inverse of the following
   a) $f(x) = \frac{2x+7}{3}$
   b) $f(x) = 8 - x^2$
   c) $g(x) = \frac{11}{x}$
   d) $f(x) = \sqrt{13 + 2x}$
3) Attempt the exercise on (Rosen, 2012, p. 581)

References

Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.

Rosen, K. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.