

Discrete Mathematics

Lecture 10

Introduction to Graph Theory

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Introduction to lecture 10

This lecture is a continuation of the lecture on Relations and functions. Graphs are associated with the diagraphs of a symmetric relations. Graph theory is applicable to data structures, understanding networks, and relationships.

References

These lecture notes have been derived from the following sources (Lipschutz & Lipson, 2007; Rosen, 2012; Wilson, 1998).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

- (i) Define terms used in graph theory.
- (ii) Apply the concepts in solving problems involving graph theory.

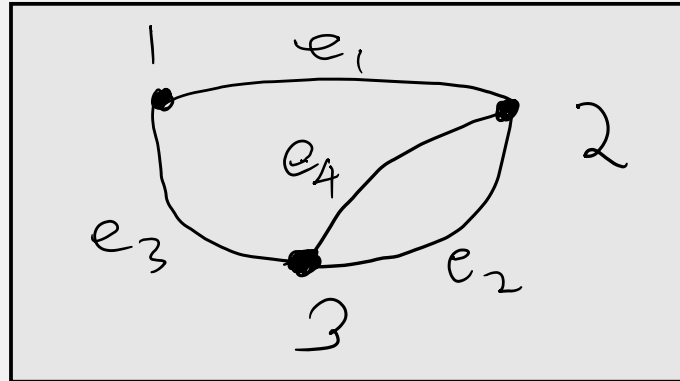
Definition of Basic Terms

Definition 1: A graph G consists of a finite set V of objects called the vertices, a finite set E of objects called edges, and a function f that assigns to each edge a subset $\{v, w\}$ where v and w are vertices.

The graph is denoted $G = (V, E, f)$.

Example 1: If e is an edge and $f(e) = \{v_i, v_j\}$, we say that e is an edge between v_i and v_j and that e is determined by v_i and v_j . The vertices v_i and v_j are called the endpoints of e .

Example 2: Let the vertices $V = \{1,2,3\}$ and edges $E = \{e_1, e_2, e_3, e_4\}$. Let f be defined by $f(e_1) = (1,2), f(e_2) = f(e_4) = \{2,3\}, f(e_3) = \{1,3\}$. Then G is a graph represented by the picture.



Definition 2: (Degree of a vertex) The degree of a vertex is the number of edges having that vertex as an endpoint. The degree of a vertex is denoted $\deg(v)$.

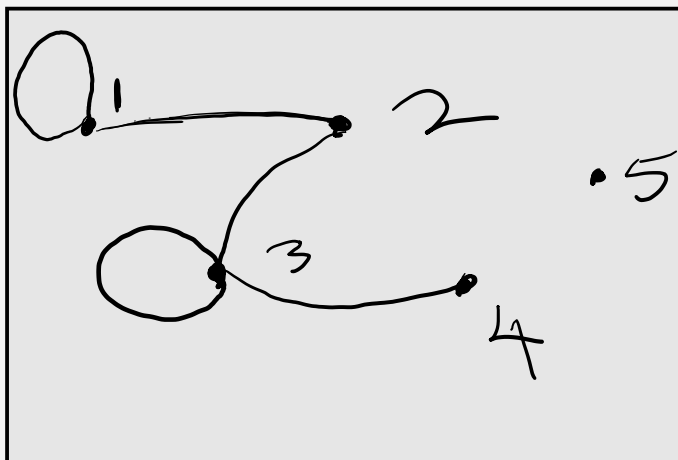
Definition 3: (loop) A graph may contain an edge from a vertex to itself referred to as a loop. A loop contributes two to the degree of the vertex.

Definition 4: (Isolated vertex) A vertex with degree 0 is referred to as an isolated vertex.

Theorem 1 (Handshaking Theorem): Let graph G be undirected graph with n edges. Then

$$2n = \sum_{v \in V} \deg(v)$$

Example 3: Consider the graph below.



Vertex	Degree
1	3
2	2
3	4
4	1
5	0

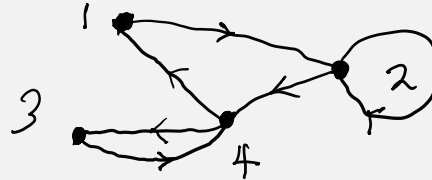
The graph above has 5 edges and sum of degrees of vertices is 10.

Theorem 2: Let the graph $G(V, E)$ be a directed graph (with directed edges) then

$$\sum_{v \in V} \text{deg}^-(v) = \sum_{v \in V} \text{deg}^+(v) = |E|$$

Where E is the edges, and $\text{deg}^-(v)$ and $\text{deg}^+(v)$ are the in-degree and out-degree of a vertex v respectively.

Example 1: Consider the directed graph below 6 edges;



Vertex	1	2	3	4	Sum
In-degrees	1	2	1	2	6
Out-degrees	1	2	1	2	6

Clearly;

$$\sum_{v \in V} \text{deg}^-(v) = 6 = \sum_{v \in V} \text{deg}^+(v) = 6 = |E|$$

Definition 5: (A path) A path in a graph is a sequence $\{x\} = v_1, v_2, \dots, v_k$ of vertices, each adjacent to the next, having an edge between each v_i and v_{i+1} such that no edge is chosen more than once.

Definition 6: (Circuit) A circuit is a path that begins and ends with the same vertex.

Definition 7: (Simple path) A path v_1, v_2, \dots, v_k is called simple if no vertex appears more than once.

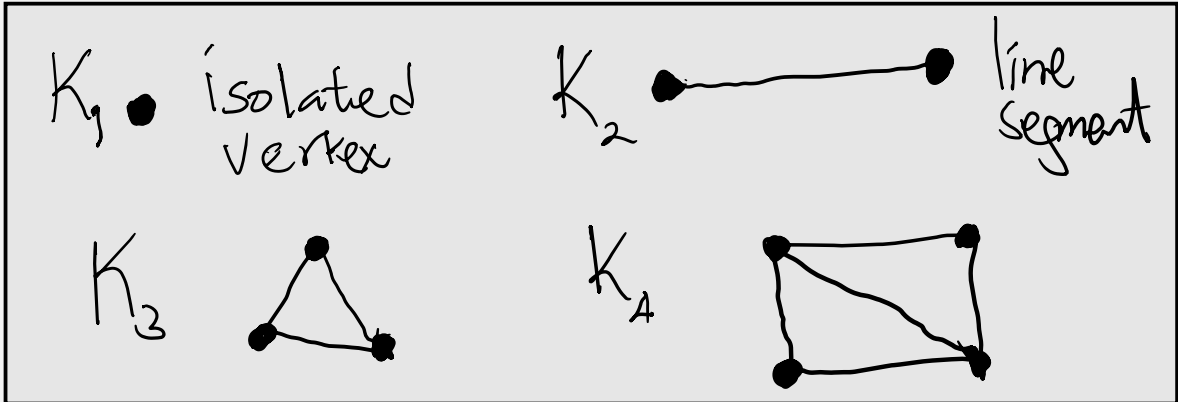
Definition 8: (Simple circuit) A circuit v_1, v_2, \dots, v_{k-1} is called simple if the vertices v_1, v_2, \dots, v_{k-1} are all distinct or unique.

Definition 9: (Euler path) A path in a graph G is called a Euler path if it includes every edge exactly once. Similarly, a Euler circuit is a Euler path that is a circuit.

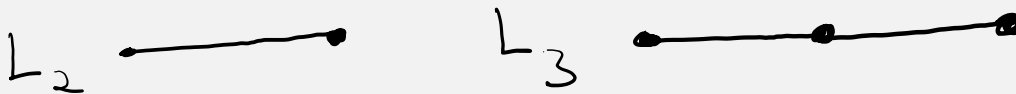
Definition 10: (Hamiltonian path) A path that contains each vertex exactly once. Similarly, a circuit that contains each vertex exactly once except for the first vertex, which is also the last.

Families of graphs

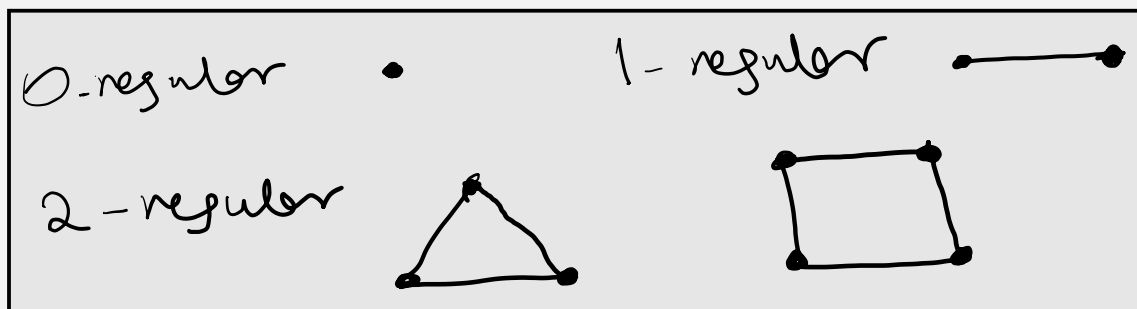
- a) **Discrete graphs:** Let D_n denote the graph with n vertices and no edges, for each integer $n \geq 1$, then D_n is a discrete graph with n vertices e.g. $D_2 \dots D_5 \dots$
- b) **Complete graph:** Let K_n denote the graph with vertices $\{v_1, \dots, v_n\}$ and with edge $\{v_i, v_j\}$ for every i and $j, \forall n \geq 1$ i.e., every vertex in K_n is connected to every other vertex. Then K_n is called a complete graph. The degree of each vertex of K_n is $(n - 1)$.



- c) **Linear graph:** Let L_n denote the graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and with edges $\{v_i, v_{i+1}\}, \forall n \geq 1$ and $1 \leq i \leq n$, then L_n is called a linear graph of n vertices.

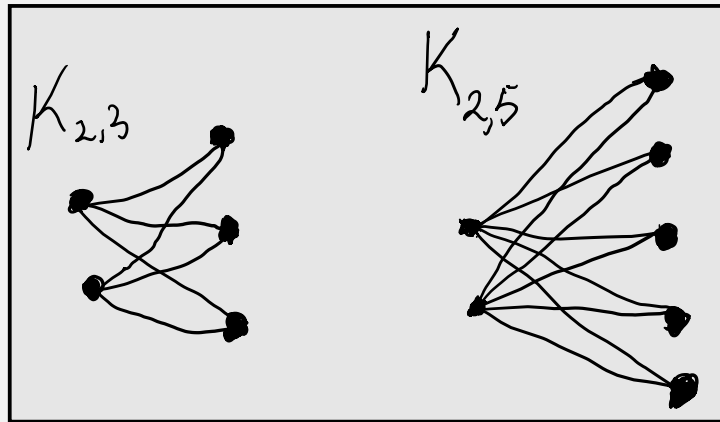


- d) **Regular graph:** A graph G is regular of degree k or k -regular if every vertex has degree k i.e. a graph is regular if every vertex has the same degree.



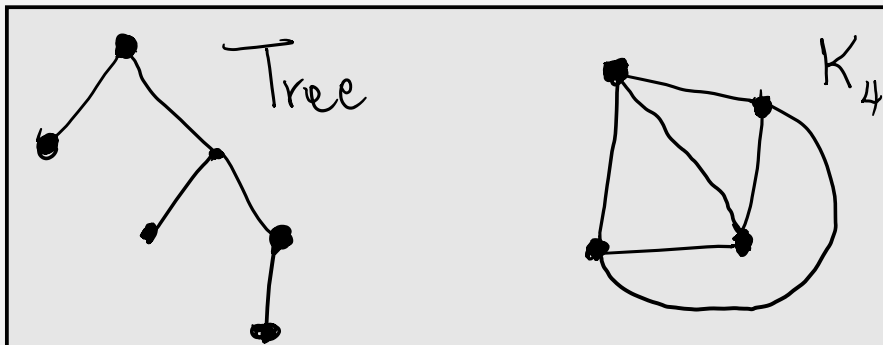
Remark: A complete graph K_n with n vertices, is a regular graph of degree $(n - 1)$.

- e) **Bipartite graphs:** A graph G is said to be bipartite if its vertices can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N denoted K_{mn} , where $m \leq n$.



Bipartite graphs are used to model problems that require application of matching. For example, in assigning different tasks to a set of workers or users.

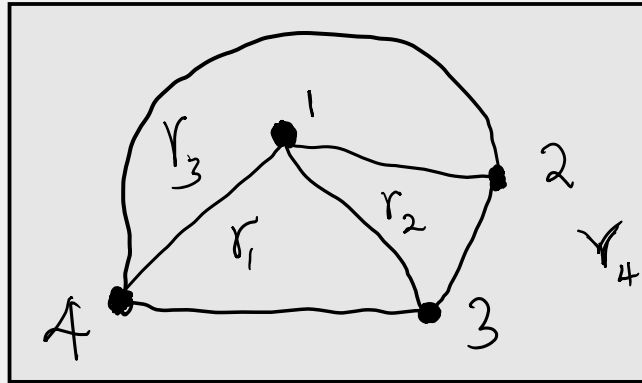
- f) **Planar graphs:** A graph that can be drawn on a plane that its edges do not cross is said to be planar e.g., tree graphs, K_4 , maps.



- g) **Maps:** A map is a finite planar multigraph. A map divides the plane into various regions. A degree of a region denoted $\deg(r)$ is the length of the cycle which borders r .

Theorem 1: The sum of the degrees of the regions of a map equal to twice the number of edges.

Example 1: Consider the K_4 below;



The map above has 4 regions r_1, r_2, r_3, r_4

$$\deg(r_1) = 3; \deg(r_2) = 3; \deg(r_3) = 3; \deg(r_4) = 3$$

Sum of degrees is 12. The graph has 6 edges therefore (sum of degrees) = 2(No. of edges)

Euler's Formula: $R + V - E = 2$ where V is the number of vertices, E is the number of edges and R is the number of any connected regions or faces. For objects, we use faces, however, in some instances it does not sum to 2. It may sum to 1 or zero e.g., the mobius strip and torus. Hence the more general formula is the Euler's characteristic i.e., $F + V - E = \chi$

Representing Graphs in Computer Memory

We have two ways of maintaining a graph G in the memory of a computer.

- (i) Sequential representation of G i.e., using the graph adjacency matrix and incidence matrices.
- (ii) Linked representation i.e., using the linked list of neighbors.

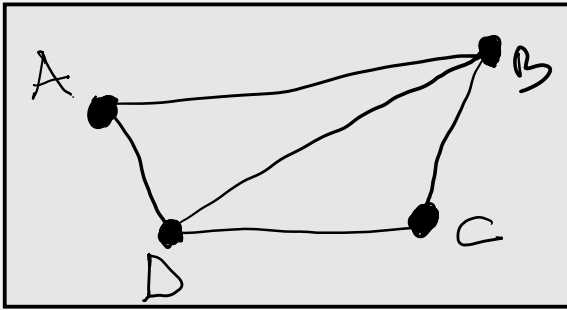
Adjacency matrix

Suppose G is a graph with n vertices and suppose the vertices have been ordered say v_1, v_2, \dots, v_n .

Then the adjacency matrix $A = [a_{ij}]$ of the graph G is the matrix defined by;

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Example 1: Consider the graph below;



Adjacency Matrix

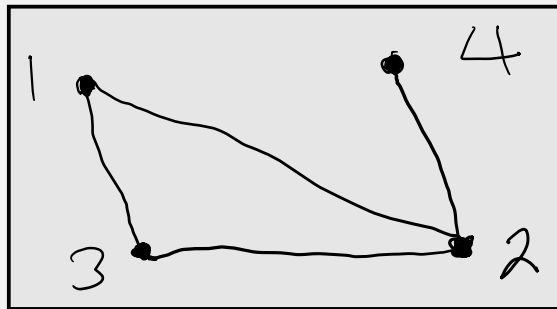
$$\begin{matrix} \mathbf{A} & \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\ \mathbf{B} & \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} \\ \mathbf{C} & \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\ \mathbf{D} & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Remarks: Graphs can be represented by matrices that are based on incidence of vertices and edges. The above adjacency matrix is for a simple graph (that has no loops). Such matrices are symmetric and have only zero and ones as entries.

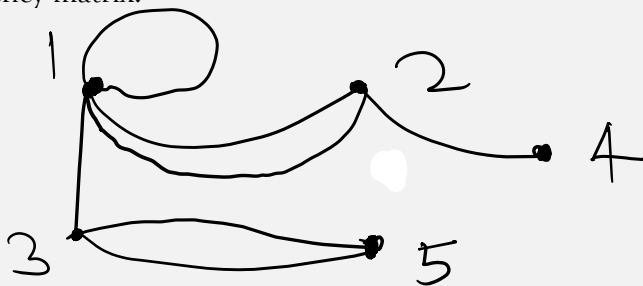
Example 2: Consider the adjacency matrix below of a graph with four vertices. Plot the graph.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Solution: Assume the vertices are labelled 1,2,3, and 4 in that order.



Example 3: Consider the following pseudograph with multiple edges and loops. Determine its adjacency matrix.



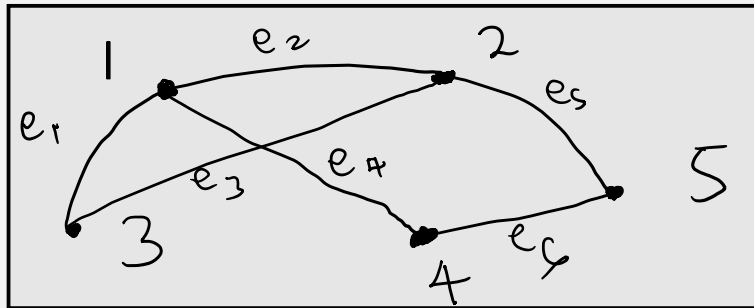
$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

Incidence matrices

Let G be undirected graph with vertices v_1, v_2, \dots, v_n and edges e_1, e_2, \dots, e_n , then the incidence matrix for G with respect to the vertices and edges is the matrix $M = [a_{ij}]$ where;

$$a_{ij} = \begin{cases} 1 & \text{where edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example 1: Consider the undirected graph below.

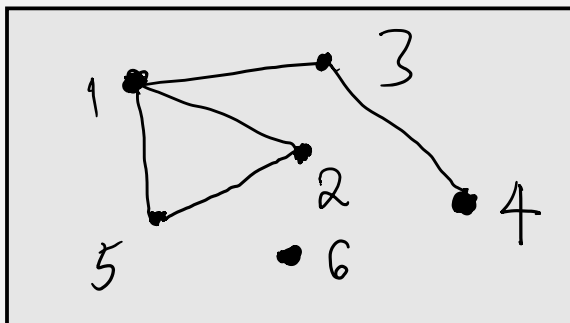


$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix}$$

Linked representation/ Adjacency structure

Adjacency matrix system consumes more memory. In this method, each vertex of G is followed by its adjacency list i.e., its list of adjacent vertices or neighbors. If a vertex has no neighbor we denote it with \emptyset in the list.

Example 1: Consider the graph below

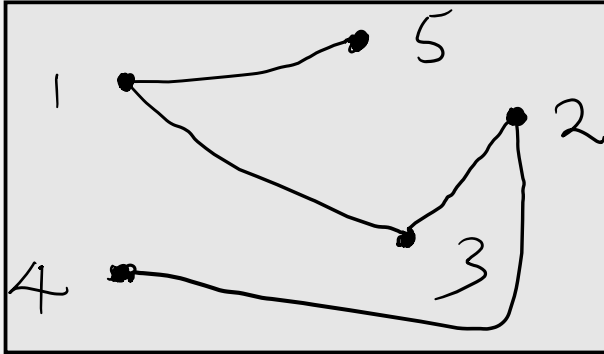


<u>Vertex</u>	<u>Adjacency list</u>
1	2,3,5
2	1,5
3	1,4
4	3
5	1,2
6	\emptyset

Example 2: Plot the graph represented by the following adjacency list

Vertex	Adjacency list
1	3,5
2	3,4
3	1,2
4	2
5	1

Solution:



Exercise

- 1) Define with relevant example the following graphs;
 - a. Wheels
 - b. n-cubes
 - c. complete bipartite graph
- 2) Define the graphs below and illustrate with an example for each case;
 - a. Complete graph
 - b. Regular graph
- 3) Let set $A = \{1,2,3,4\}$ with relation

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,3), (3,4), (2,4), (4,1), (4,4)\}.$$
 Hence write its M_R and Construct the diagraph for the relation
- 4) Attempt exercise (Rosen, 2012, p. 650,665,675)

References

- Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.
- Rosen, K. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.
- Wilson, R. J. (1998). *Graph Theory* (4th ed.). Addison Wesley Longman Ltd.