

Discrete Mathematics

Lecture 11

Trees

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Introduction to lecture 11

This lecture builds up on the lecture on introduction to graph theory. Trees are connected graphs and have many applications in computer science, communication, sociology, and many other fields. The idea of trees in a computer can be seen in the file system that consists of directories, subdirectories, and files. The administrative structure of an institution, family lineage etc. adopts the structure of a tree. Trees are used to model the process of solving mathematical expressions.

References

These lecture notes have been derived from the following sources (Lipschutz & Lipson, 2007; Rosen, 2012; Wilson, 1998).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

- (i) Define terms used in trees.
- (ii) Apply the concepts in solving problems involving trees.

Definition of Terms

Definition 1: A graph $G(V,E)$ is connected if there is a path from any vertex to any other vertex in the graph, otherwise it is disconnected.

Definition 2: A cycle is a path in a graph that has the following properties:

- (i) It includes at least one edge.
- (ii) There are no repeated edges.
- (iii) The first and last vertices coincide i.e., are the same but there are no other repeated vertices.

Remarks

- (i) A cycle with n -edges is called an n -cycle.
- (ii) A cycle is a circuit with some additional properties.
- (iii) A 1-cycle is a loop.

Definition 3: A tree is a connected graph with no cycles.

Properties of a tree

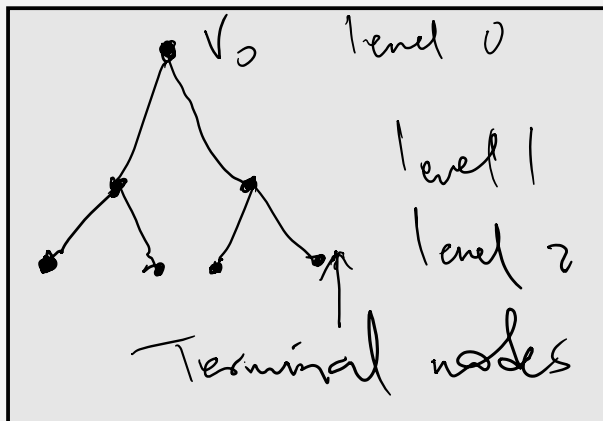
- (i) For each pair of vertices u and v in a tree, there is exactly one path from u to v that does not repeat any vertices or edges.
- (ii) Inserting an edge between any two vertices of a tree produces a graph containing a cycle.
- (iii) Removing any edge from a tree produces a disconnected graph

Definition 4: A rooted tree (T, V_0) is a tree in which one of the vertices is specified as the root V_0 .

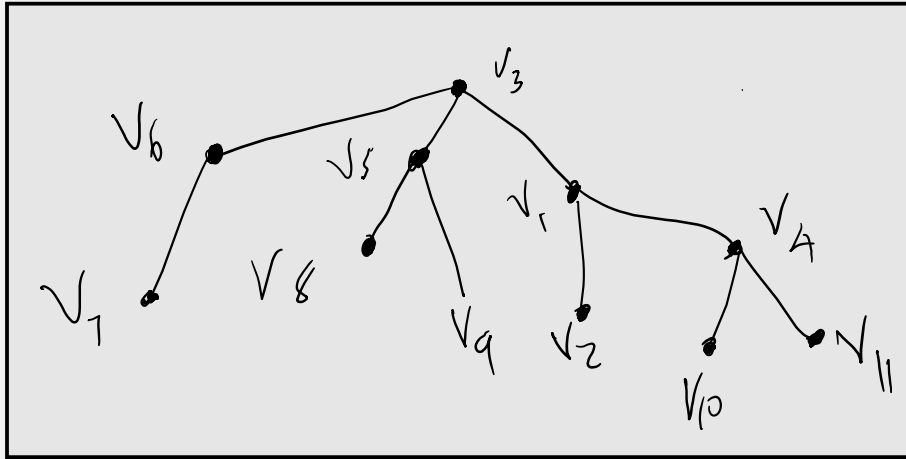
Rooted trees are usually drawn with the root at the top.

Theorem 1: Let (T, V_0) be a rooted tree then;

- (i) There are no cycles in T .
- (ii) V_0 is the only root of T .
- (iii) Each vertex in T , other than V_0 has in-degree one, and V_0 has in-degree 0.



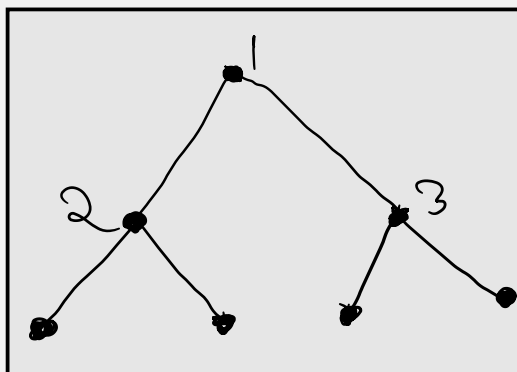
Example 1: Let $A = \{v_1, v_2, \dots, v_{11}\}$ and let the tree $T = \{(v_1, v_2), (v_1, v_4), (v_3, v_6), (v_4, v_{10}), (v_5, v_8), (v_6, v_7), (v_3, v_5), (v_4, v_{11}), (v_5, v_9), (v_3, v_1)\}$. Show that T is a rooted tree and determine the root.



T is a rooted tree with v_3 as the root.

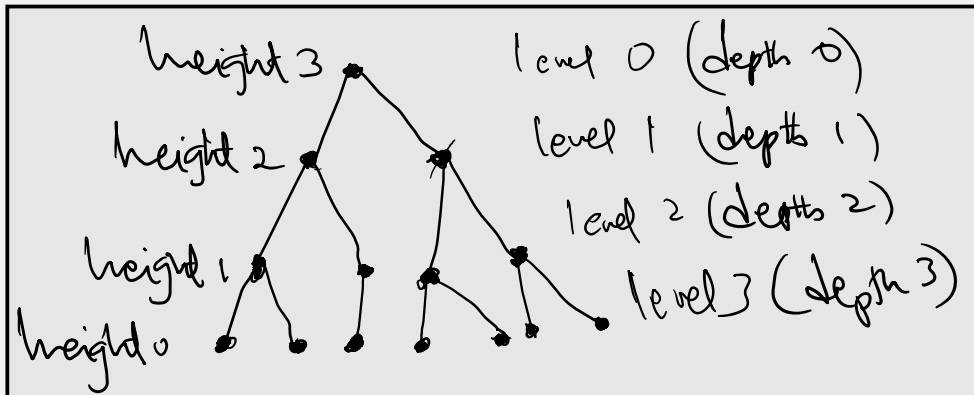
Definition 5: If n is a positive integer, then a tree T is said to be a n -tree if every vertex has at most n offspring.

Definition 6: A tree T is said to be complete if all its levels except possibly the last, have the maximum number of possible nodes, and if all the nodes at the last level appear as far left as possible. A node that has no child is called a leaf node. Any two nodes v_i and v_j which are connected by an edge are referred to as adjacent nodes. For example;



Definition 7: The depth of a tree d_n is number of edges from the root node to the particular node. The root node has a depth of zero. While the height of a tree is the number of edges from the leaf node to the particular node in the longest path. The leaf nodes has height zero. The root node is at

level 0. The root node's children are at level 1, and so on as seen in the diagram below.

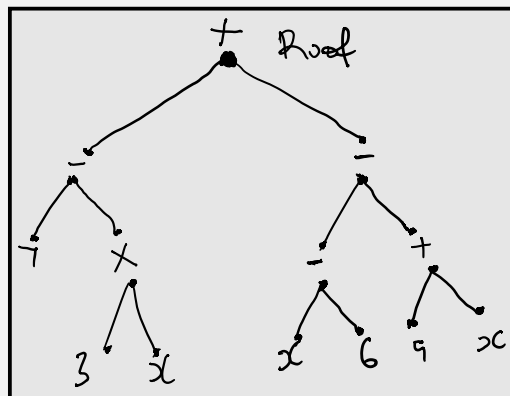


Definition 8: (Binary tree) A 2-tree is called a binary tree, that is, if each node has either 0 or 2 children. The nodes with 2 children are called internal and nodes with 0 children are called external nodes. We use circles for internal nodes and squares for external nodes.

Definition 9: (Labelled trees) In computer science, the set of vertices is not important compared to the utility of the tree which is emphasized by the label of the vertices.

Example 1: Consider the algebraic expression $(7 - (3 \times x)) + ((x - 6) - (9 + x))$

Note that '+' is the central operator. The expression can be graphically represented by a labelled binary tree, where the root is the central operator i.e.



Definition 10: (Spanning tree) A subgraph T of a connected graph G is called a spanning tree of G if T is a tree and T include all the vertices of G .

Definition 11: (Minimum spanning tree) A minimal spanning tree of G is a spanning tree whose total weight is as small as possible.

Remark 1: The weight of a minimum spanning tree is unique but the minimum spanning tree itself is not. A graph G may have more than 1 spanning trees.

Remark 2: Given a connected graph G with n vertices then the minimum spanning tree T must have $n - 1$ edges.

Algorithms for finding the minimum spanning tree

Greedy algorithms such as Prim-Jarnik's and Kruskal's algorithms are used to determine the minimum spanning tree.

Prim-Jarnik's algorithm

Steps

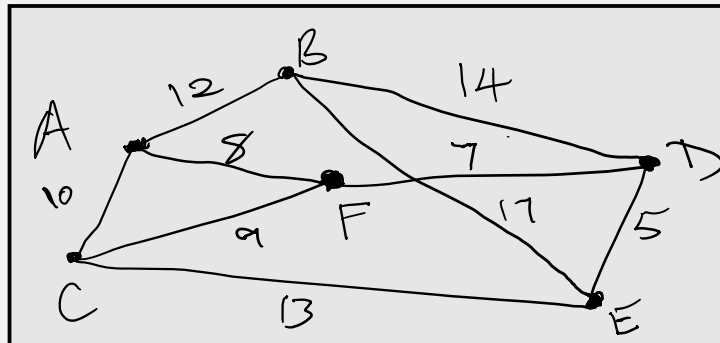
1. Initialize a tree with a single vertex, chosen arbitrarily.
2. Choose any edge with the smallest weight.
3. Proceed sequentially by adding edges of minimum weight incident a vertex in the tree, that does not result to a circuit
4. Stop when $n - 1$ edges have been added.

Kruskal's algorithm

Steps

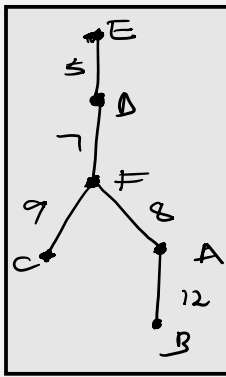
1. Choose an edge with minimum weight.
2. Successively add edges with minimum weight, that do not form a circuit, with any of the edge already taken.
3. Stop when $n - 1$ edges have been added.

Example 1: Find a minimum spanning tree of the weighted graph G below.

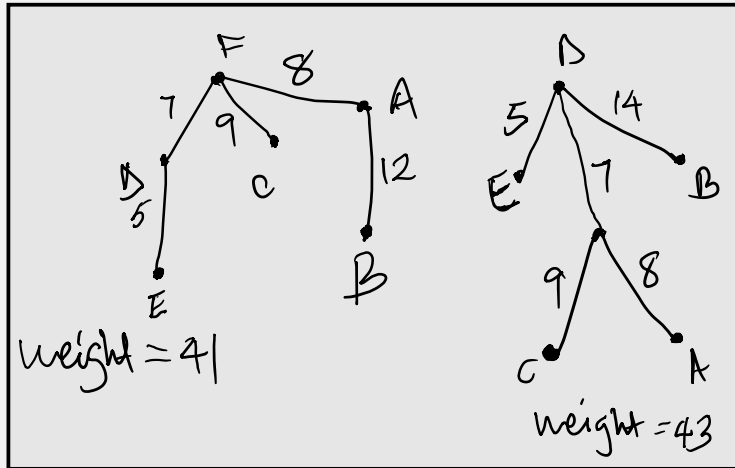


Consider the following trees: Using the Kruskal's algorithm

Edges	DE	DF	FA	FC	AC	AB	CE	BD	Total
Weight	5	7	8	9	10	12	13	14	41
Take	Yes	Yes	Yes	Yes	No	Yes	No	No	
Why					Circuit		Circuit	Circuit	

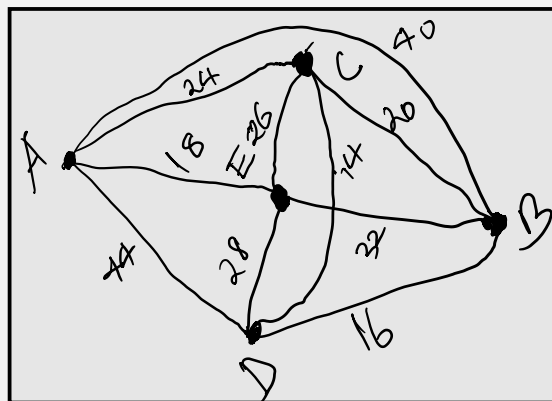


Using the Prim-Jarnik's algorithm, we can start at vertex F and F and have the following subtrees.



The first tree has weight of 41 and the second tree has a weight. The minimum spanning tree has a weight of 41. It is not an easy task to determine a tree that has a minimum value especially when the graph is complex.

Example 2: Five towns are connected by roads. The road network can be represented with the connected graph below (4-regular graph) with weights that represent the distance between the nodes or towns. Determine the minimum tree connecting all the towns.



Solution: Using the Kruskal's algorithm we have;

Edges	CD	DB	AE	BC	AC	CE	DE	BE	AB	TOTAL
Weight	14	16	18	20	24	26	28	32	40	72 km

Using the Prim's algorithm

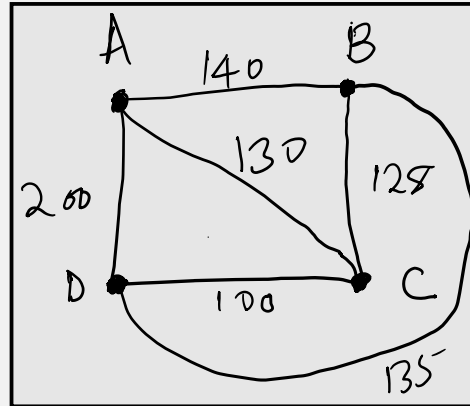
We start at vertex C

Edge	CD	DB	CA	AE	Total
Weight	14	16	24	18	72 km

Application

Local area networks and minimum spanning trees

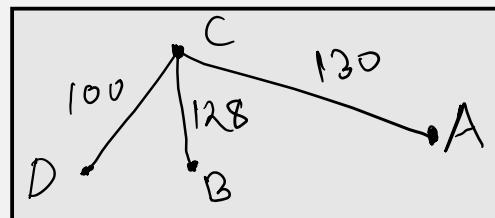
Example 1: Consider four buildings A, B, C, and D. The buildings are to be linked with data cables. The length of the cables is as shown in the graph(3-regular graph) below. Determine the minimum length of cable that connects all the four buildings.



We use the Kruskal's algorithm to determine the minimum tree.

Edge	CD	BC	AC	BD	AB	Total
Weight	100	128	130	135	140	358
				Reject	Reject	

It is cheaper and sufficient to install the 3 shortest cables; CD, BC, and AC. Any two buildings can easily connect with one another. The three cables form a tree (a connected graph) and forms a minimal spanning tree, with C as the root, that provide the shortest distance.



Traveling-salesperson problem

Definition 1: (Weight matrix) Let G be a weighted graph with vertices v_1, v_2, \dots, v_n in which the weights represent a penalty of some kind. The weight matrix of G is the $n \times n$ matrix for which the entry M_{ij} in the i^{th} row and j^{th} column is given by;

$$M_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{the weight of the edge from } i \text{ to } j \text{ if } V_i = V_j \\ \infty & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ are not adjacent} \end{cases}$$

A weight matrix can be used for easy machine computation.

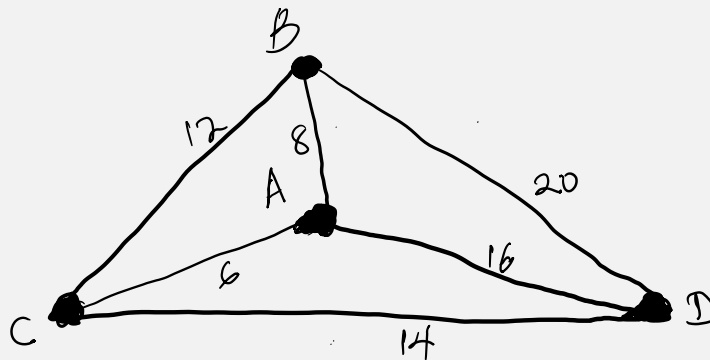
Theorem 1: A finite connected graph is said to be Eulerian if and only if each vertex has even degree.

Definition 2: A Hamiltonian circuit in a graph G is a closed path that passes every vertex in the graph exactly once.

Theorem 2: The complete graph K_n with $n \geq 3$ vertices, has $H = \frac{(n-1)!}{2}$ Hamiltonian circuits.

Definition 3: (Traveling-salesperson problem) Let G be a complete weighted graph, the traveling-salesman problem refers to finding a Hamiltonian circuit for G of minimum weight.

Example 1: Determine the Hamiltonian circuit of minimum weight for the complete weighted graph G .



The graph G has 4 vertices, by the above theorem 2 it has $H = \frac{(n-1)!}{2} = \frac{(4-1)!}{2} = \frac{3!}{2} = 3$ Hamiltonian circuits. Assume the circuits start at A then we have the following circuits;

$$|ABCD| = 8 + 12 + 14 + 16 = 50$$

$$|ACDB| = 6 + 14 + 20 + 8 = 48$$

$$|ACBDA| = 6 + 12 + 20 + 16 = 54$$

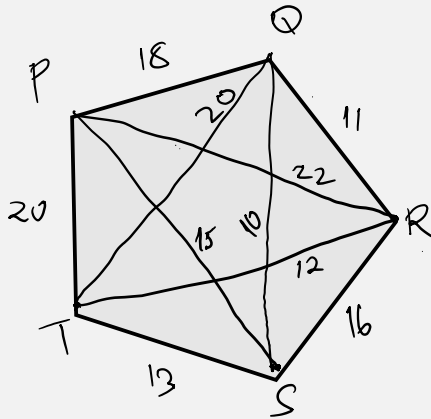
ACDBA with weight 48 is the Hamiltonian circuit of minimum weight.

Remark: This method is impractical for a graph with many vertices.

Nearest-neighbor Algorithm

The NNA starts at a random vertex, then chooses the edge with the least weight to the next closest vertex (neighbor). Proceed sequentially until a Hamiltonian circuit is completed. Note the NNA may miss shorter routes due to its 'greedy' nature.

Example 1: Consider the weighted graph below;



The graph has $\frac{(5-1)!}{2} = 12$ Hamiltonian circuits. We can apply NNA starting at (i) P (ii) Q (iii) R (iv) S (v) T to get the Hamiltonian circuits;

$$|PSQRTP| = 68$$

$$|QSTRPQ| = 75$$

$$|RQSTPR| = 76$$

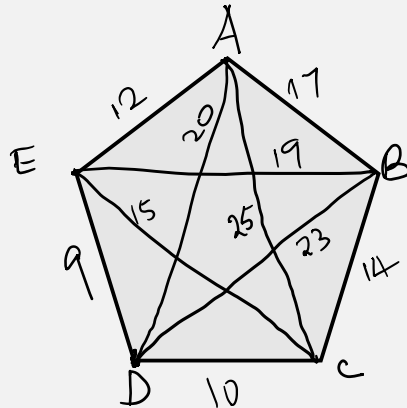
$$|SQRTPS| = 68$$

$$|TRQSPT| = 68$$

Exercise

- 1) Construct the tree of the algebraic expressions below
 - a) $(7 \times (3 - x)) \div \left(\left(2 + (7 - (y + 11)) \right) \times (5 - (x \div 2y)) \right)$
 - b) $(5 + (7 - 2)) - (x - (y + 9))$
- 2) Determine if R is a tree and determine its root. Where R is defined on set X.
 - a) $X = \{a, b, c, d, e\}, R = \{(a, d), (a, b), (a, c), (d, e)\}$
 - b) $X = \{a, b, c, d, e\}, R = \{(a, e), (a, b), (c, a), (a, d)\}$
- 3) Draw $K_5, K_6,$ and K_7 graphs.

- 4) Draw $K_{3,3}$, $K_{2,4}$, $K_{3,5}$.
- 5) Draw 3-regular, 4-regular, and 5-regular.
- 6) Consider the weighted graph below then determine;
 - a. Minimum spanning tree using the Kruskal's and prim's algorithm.
 - b. Determine the Hamiltonian circuits from the five vertices.



Further readings and exercises

- 7) Read chapter on Graph theory and attempt exercises in (Lipschutz & Lipson, 2007, pp. 154–200).
- 8) Read chapter on Trees and attempt exercises in (Rosen, 2012, p. 745).

References

- Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.
- Rosen, K. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.
- Wilson, R. J. (1998). *Graph Theory* (4th ed.). Addison Wesley Longman Ltd.