ST. PAUL'S UNIVERSITY
DEPARTMENT OF ENGINEERING

CAT I

ANSWER ALL QUESTIONS

QUESTION ONE

The polar decomposition theorem. This is a central theorem in mechanics. To prove it we will use the square root theorem (without proof).

Thm*: If $\mathbf{S}$ is a positive definite, symmetric second-order tensor, then there exists a unique positive definite symmetric second-order tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{S}$.

Equipped with this result, the problem is to prove the following theorem.

Thm: (Polar decomposition). If $\mathbf{F}$ is a second order tensor such that $\det\mathbf{F} > 0$, then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{1}$$

SOLUTION

Show that $\mathbf{F}^T\mathbf{F}$ is symmetric and positive definite. Then you can apply the square root theorem $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$. Then define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. Show that $\mathbf{R}$ is proper orthogonal, establishing $\mathbf{R}$ as a rotation. Explain why the decomposition is unique.

To establish the left polar decomposition, let $\mathbf{Q} = \mathbf{V}^{-1}\mathbf{F}$. Then by arguments similar to those above deduce that $\mathbf{Q}$ is also a rotation. We then have $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{Q}$. Find an argument to show that $\mathbf{R} = \mathbf{Q}$.

QUESTION TWO - 10 Marks

Find the left and right polar decompositions of the matrices

$$(i) \begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} \quad (ii) \begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 3 \end{pmatrix}. \tag{2}$$

Steps: The key is to first compute $\mathbf{U}$ *as the square root of* $\mathbf{F}^{\mathrm{T}}\mathbf{F}$*. Once* $\mathbf{U}$ *is known, compute* $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$*. Once* $\mathbf{R}$ *is known, compute* $\mathbf{V}$ *as* $\mathbf{F}\mathbf{R}^{\mathrm{T}}$*. Once you have done the small one by hand, you may try a symbolic program (Mathematica or Maple).*

SOLUTION

The steps are already outlined in the solution so we only give the results. **Note that** to compute $\mathbf{U} = \sqrt{\mathbf{C}}$, you have to compute the eigenvalues of $\mathbf{C}$ which we call λ_i^2 and the normalised eigenvectors of $\mathbf{C}$. The you obtain $\mathbf{U} = \sum_i \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$. For (i),

$$\text{For (i)} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 5 & 0 \\ 0 & 45 \end{pmatrix} \quad \mathbf{U} = \sqrt{\mathbf{C}} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{pmatrix} \tag{3}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad \mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{pmatrix} \frac{7}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{13}{\sqrt{5}} \end{pmatrix} \tag{4}$$

$$\text{For (ii)} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 18 \end{pmatrix} \quad \mathbf{U} = \sqrt{\mathbf{C}} = \begin{pmatrix} 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}-1 & 0 \\ \frac{1}{\sqrt{2}}-1 & 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 3\sqrt{2} \end{pmatrix} \tag{5}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{pmatrix} 1+\frac{3}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}}-1 \\ 0 & \sqrt{2} & 0 \\ \frac{3}{\sqrt{2}}-1 & 0 & 1+\frac{3}{\sqrt{2}} \end{pmatrix} \tag{6}$$

If you want to practise your *Mathematica* skills, experiment with this program

```
F = {{1, -1, 3}, {1, 1, 0}, {-1, 1, 3}}
(* right Cauchy-Green strain tensor *)
CG = Transpose[F].F; CG // MatrixForm
λsq = Eigenvalues[CG]; v = Eigenvectors[CG];
(* normalise Eigenvectors *)
v[[1]] = Normalize[v[[1]]]; v[[2]] = Normalize[v[[2]]]; v[[3]] = Normalize[v[[3]]];
(* now evaluate U = ∑_i λ_i e_i ⊗ e_i *)
U = Sum[Sqrt[λsq[[i]]] TensorProduct[v[[i]], v[[i]]], {i, 1, 3}]; U // MatrixForm
R = F.Inverse[U] // FullSimplify; R // MatrixForm
V = F.Transpose[R] // FullSimplify; V // MatrixForm
```

QUESTION THREE - 10 Marks

Consider the simple shear

$$\mathbf{x}(\mathbf{X}) = (X_1 + \gamma X_2, X_2, X_3),\ \gamma \geq 0. \tag{7}$$

Calculate the principal stretches, and show that the right polar decomposition of the deformation gradient is given by

$$F = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \frac{1+\sin^2\theta}{\cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where $\tan\theta = \frac{\gamma}{2}$. Determine also the left polar decomposition. What are the Eulerian and Lagrangian axes?

SOLUTION

From the deformation map we can compute the deformation gradient $\mathbf{F}$ and right Cauchy-Green strain tensor $\mathbf{C}$

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Taking the square $\mathbf{U}$ in (8) and comparing with $\mathbf{C}$ (in which we substitute $\gamma = 2\tan\theta$) shows that (8) is indeed the polar decomposition. This determines $\mathbf{R}$ uniquely. The principal stretches are

$$\lambda_1 = 1 \qquad \lambda_2 = 2\tan\theta\,(\tan\theta - \sec\theta) + 1 \qquad \lambda_3 = 2\tan\theta\,(\tan\theta + \sec\theta) + 1 \tag{9}$$

The left polar decomposition is

$$\mathbf{F} = \mathbf{V}\mathbf{R} = \begin{pmatrix} \frac{1+\sin^2\theta}{\cos\theta} & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

You can confirm these results quite easily with *Mathematica*. Experiment with this:

```
F := {{1, γ, 0}, {0, 1, 0}, {0, 0, 1}}
CG = Transpose[F].F;
λsq = Eigenvalues[CG] /.  γ -> 2 Tan[θ];
v = Eigenvectors[CG] /.  γ -> 2 Tan[θ];
v[[1]] = v[[1]] // Normalize;
v[[2]] = v[[2]] // Normalize;
v[[3]] = v[[3]] // Normalize;
U = FullSimplify[ Sum[Sqrt[λsq[[i]]] TensorProduct[v[[i]], v[[i]]],
    {i, 1, 3}], Assumptions -> 0 < θ < Pi/2];
```