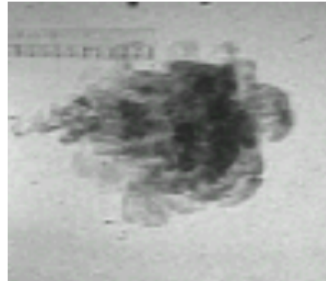


## 3. Diffusion of an Instantaneous, Point Release



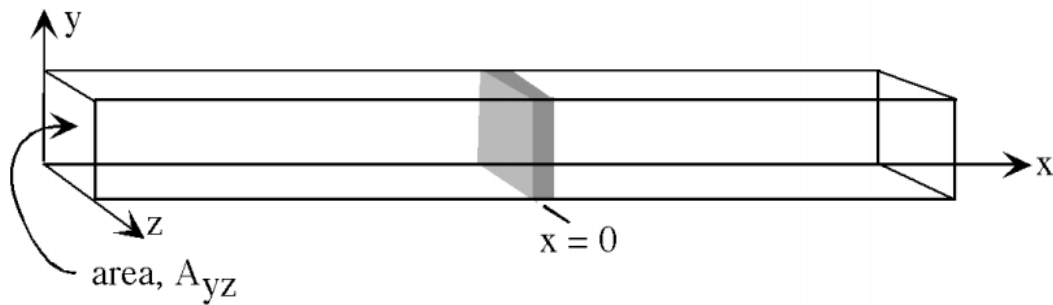
This section follows on from Section 1, exploring the Gaussian concentration profile that develops after an instantaneous, point release. Several important properties of the Gaussian distribution (and their utility) are emphasized. The spread of a point release in all 3 dimensions is discussed, noting that the profile of concentration along any straight line cut through the patch will have a Gaussian distribution. The animation shows the spread of an instantaneous, point release with anisotropic diffusivity, highlighting the dependence of length scales of the Gaussian distribution on the square root of  $D$ .

### 3. Diffusion of an Instantaneous Point Source

The equation of conservation of mass is also known as the transport equation, because it describes the transport of scalar species in a fluid systems. In this and subsequent sections we consider analytical solutions to the transport equation that describe the fate of mass under different boundary and initial conditions. For incompressible flow with isotropic and homogeneous diffusion coefficients, we can begin with the following form of the transport equation. At this point we will also ignore the source/sink term.

$$(1) \quad \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \left[ \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right]$$

The first solution describes the evolution of a slug of mass,  $M$ , released instantaneously at time  $t = 0$  into a stationary fluid ( $u, v, w = 0$ ). We consider a one-dimensional system in the  $x$ -direction, i.e. the system is uniform in  $y$  and  $z$ , such that  $\partial/\partial y = 0$  and  $\partial/\partial z = 0$ . A model for such a system is shown below. A slab of mass (gray) is released into a long, narrow tube. The mass is distributed uniformly in the  $y$ - $z$  plane and with negligible dimension in  $x$ , such that the initial concentration is  $C(x) = M \delta(x)$ , where  $\delta()$  is the Dirac delta function. The transport equation for this system is given in (2).



$$(2) \quad \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}, \quad M \text{ released at } t = 0, x = 0.$$

The solution,  $C(x,t)$ , can be found as follows. From (2) we anticipate that the concentration will be a function of  $M$ ,  $D$  [ $L^2T^{-1}$ ],  $x$ [ $L$ ], and  $t$ [ $T$ ], with dimensions expressed in brackets in terms of mass [ $M$ ], length [ $L$ ], and time [ $T$ ]. In a one-dimensional system, concentration has units of mass per length,  $C$ [ $M L^{-1}$ ]. From dimensional analysis (e.g. Buckingham Pi Theorem) the above parameter set forms the following dimensionless groups.

$$(3) \quad \frac{C}{M/\sqrt{4\pi Dt}} = f\left(\frac{x}{\sqrt{4Dt}}\right)$$

The factors  $4\pi$  and  $4$  are added now for convenience, because they appear in the final solution. Originally, these factors were found through trial and error. Defining  $\eta = x/\sqrt{4Dt}$ , and then evaluating (2) with (3),

$$(4) \quad \frac{d}{d\eta} \left( \frac{df}{d\eta} + 2f\eta \right) = 0$$

Setting the terms within the bracket to zero provides one solution to (4),

$$(5) \quad \frac{df}{d\eta} + 2f\eta = 0,$$

which in turn has the solution

$$(6) \quad f = A_0 \exp(-\eta^2),$$

with  $A_0$  an arbitrary constant. (6) gives us a specific form for (3) that is a solution for (2).

$$(7) \quad C(x,t) = \frac{A_0}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right)$$

Since the total mass must be conserved over time, for all time

$$(8) \quad M = \int_{x=-\infty}^{\infty} C dx,$$

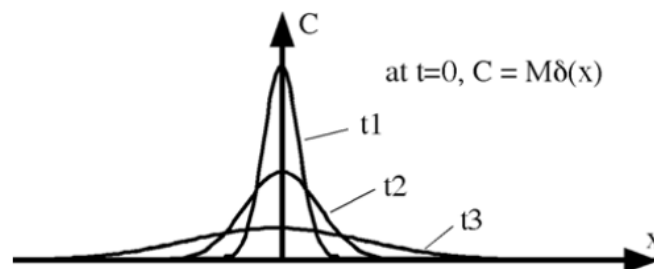
so that  $A_0 = M$ . In summary we have the following solution.

*One-dimensional, Instantaneous Point Source*

$M$  released at  $t = 0$  and  $x = 0$ .

Initial condition,  $C(x) = M \delta(x)$ .

$$(9) \quad C(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) = [M/L]$$



Note that the above solution is correct for a one-dimensional mathematical system, in which  $C[M/L]$ . In real (three-dimensional) systems concentration must have units of  $M/L^3$ . To convert the mathematical solution given in (9) to real space, divide by the neglected dimensions, here the cross-sectional area of the system in the  $y$ - $z$  plane,  $A_{yz}$ .

$$(10) \quad C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) = [M/L^3]$$

Recall that the same solution was derived from a statistical model of diffusion, as described in Conceptual Model of Diffusion.

### Moments of the Concentration Distribution.

It is useful to characterize the geometry of the evolving distribution of mass based on its moments. The zeroth moment gives the total mass,  $M$ , within the cloud,

$$\text{zeroth moment} = M_0 = M = \int_{-\infty}^{\infty} C(x,t) dx .$$

Higher moments indicate the centroid and variance of the mass distribution.

$$\text{first moment} = M_1 = \int_{-\infty}^{\infty} xC(x,t) dx$$

$$\text{second moment} = M_2 = \int_{-\infty}^{\infty} x^2 C(x,t) dx$$

The mean indicates the centroid of mass for the cloud,

$$\text{Mean} = \mu = \frac{M_1}{M_0} .$$

The variance indicates the width of the distribution.

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} (x - \mu)^2 C(x,t) dx}{\int_{-\infty}^{\infty} C(x,t) dx} = \frac{M_2}{M_0} - \mu^2 .$$

Evaluating the mean and variance for the distribution given in (10), we find that the mean,  $\mu = 0$ , i.e. the centroid of the patch remains at  $x = 0$  for all time. The variance is

$$(11) \quad \sigma^2 = 2Dt .$$

If the coefficient of diffusion,  $D$ , is constant, the variance of the mass distribution grows linearly in time. Taking the derivative of (11),

$$(12) \quad D = \frac{1}{2} \frac{d\sigma^2}{dt} = \frac{1}{2} \frac{\sigma_2^2 - \sigma_1^2}{t_2 - t_1}.$$

This provides a useful tool for estimating the diffusion coefficient by tracking the variance of a diffusing tracer over time. It is useful to note that for a Gaussian distribution, the concentration at one standard deviation from the center of mass is,

$$(13) \quad C(x = \sigma = \sqrt{2Dt}) = \left[ \frac{M}{A_{yz} \sqrt{4\pi Dt}} \right] \exp\left(-\frac{2Dt}{4Dt}\right) = C_{\max}(t) \exp(-1/2) = 0.61 C_{\max},$$

where  $C_{\max}$  is the maximum concentration in the patch at time  $t$ , and is given by the bracketed coefficient to the exponential term in (13),  $C_{\max} = M / (A_{yz} (4\pi Dt)^{1/2})$ .

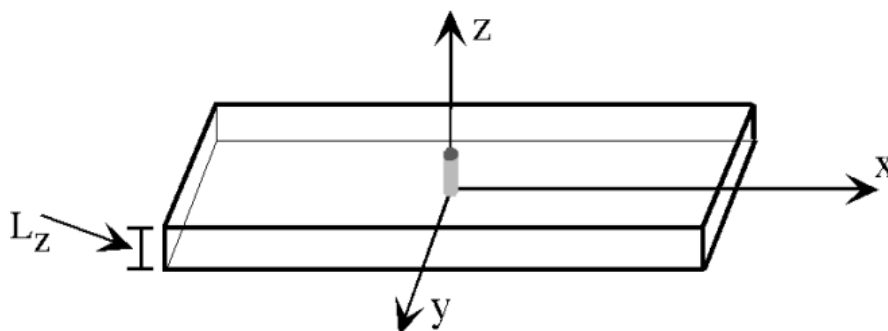
Finally, as described in Chapter 1, the standard deviation,  $\sigma$ , provides a useful length-scale for the diffusing patch. Noting that 95% of the total mass is contained within  $\pm 2\sigma$  of the centroid, a conventional length-scale for a patch is

$$(14) \quad L = 4\sigma.$$

With this definition for patch size, the edges of the patch are defined by the concentration  $C(x = \pm 2\sigma) = 0.14 C_{\max}$ .

### Instantaneous Point Source in Two-Dimensions

A slug of mass,  $M$ , is released at  $x = y = z = 0$  in a stationary fluid ( $u = v = w = 0$ ). The flow conditions are uniform in  $z$  ( $\partial/\partial z = 0$ ). The mass slug is initially distributed uniformly in  $z$  and with negligible dimension in  $y$  and  $x$ , such that the initial concentration is  $C(x, y, t=0) = M \delta(x)\delta(y)$ , where  $\delta()$  is the Dirac delta function.



For generality we will assume anisotropic diffusion,  $D_x \neq D_y$ . The transport equation for this system is then,

$$(15) \quad \frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2}.$$

From Fick's Law and by inspection of (15), the diffusion in x (1<sup>st</sup> term on right hand side) depends only on the distribution in x and the diffusion in y (2<sup>nd</sup> term on right hand side) depends only on the distribution in y. This leads us to propose that the solution to (15) may be the product of two uncoupled solutions describing the distribution in x and y.

$$(16) \quad C(x,y,t) = MC_1(x,t)C_2(y,t),$$

where, explicitly,  $C_1$  is not a function of y and  $C_2$  is not a function of x. Inserting the proposed solution form into (15),

$$(17) \quad \frac{\partial}{\partial t}(C_1C_2) = C_1 \frac{\partial C_2}{\partial t} + C_2 \frac{\partial C_1}{\partial t} = D_x C_2 \frac{\partial^2 C_1}{\partial x^2} + D_y C_1 \frac{\partial^2 C_2}{\partial y^2}$$

The above equation can be rewritten to isolate the functions  $C_1$  and  $C_2$ .

$$(18) \quad C_2 \left[ \frac{\partial C_1}{\partial t} - D_x \frac{\partial^2 C_1}{\partial x^2} \right] + C_1 \left[ \frac{\partial C_2}{\partial t} - D_y \frac{\partial^2 C_2}{\partial y^2} \right] = 0.$$

The trivial, but useless, solution is  $C_1 = C_2 = 0$ . Alternatively each bracketed set of terms could be zero. For example, setting the first bracketed set to zero yields,

$$(19) \quad \frac{\partial C_1}{\partial t} = D_x \frac{\partial^2 C_1}{\partial x^2},$$

which is the 1-D, diffusion equation (2). From (7), we can write the solution to (19).

$$(20) \quad C_1(x,t) = \frac{A_1}{\sqrt{4\pi D_x t}} \exp\left(-\frac{x^2}{4D_x t}\right)$$

A similar form is found for  $C_2$  by setting the second bracketed term in (18) to be zero. The full solution will then have the form,

$$(21) \quad C(x,y,t) = C_1(x,t)C_2(y,t) = \frac{A_1 A_2}{4\pi t \sqrt{D_x D_y}} \exp\left(-\frac{x^2}{4D_x t} - \frac{y^2}{4D_y t}\right),$$

with  $A_1$  and  $A_2$  constants. The product  $A_1 A_2 = M$  is found by applying the condition

$$(22) \quad M = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} C dx dy .$$

Finally, the two-dimensional solution (21) has units  $[M L^{-2}]$ , because the third dimension has been neglected. To recover the correct concentration in real, three-dimensional space, divide the two-dimensional solution by the length of the missing dimension, e.g.  $L_z$  in the figure given above. In summary we have the following solution.

*Instantaneous Point Source in Two-Dimensions*

M released at  $t = 0$  and  $x = y = 0$ .

Initial condition,  $C = M \delta(x) \delta(y)$

$$(23) \quad C(x,y,t) = \frac{M}{L_z 4\pi t \sqrt{D_x D_y}} \exp\left(-\frac{x^2}{4D_x t} - \frac{y^2}{4D_y t}\right) = [ML^{-3}].$$

From (23) the profile of concentration,  $C(x,y)$  along any straight line cut through the patch will have a Gaussian distribution. Obvious examples are the profiles  $C(x, y = 0)$  and  $C(x = 0, y)$ . This is the direct result of Fick's Law, which requires that the flux in any direction be proportional only to the gradients in that direction. If the diffusion coefficients are anisotropic, the cloud will grow anisotropically, increasing in length more quickly along the axis of maximum diffusion rate. The length scale of the cloud along any axes will be proportional to the diffusion coefficient along that axes.

$$(24) \quad \begin{aligned} L_x &= 4\sigma_x = 4\sqrt{2D_x t} \\ L_y &= 4\sigma_y = 4\sqrt{2D_y t} \end{aligned}$$

With this length scale definition, the edge of the patch is located  $2\sigma$  from the centroid along a concentration contour  $C = 0.14 C_{\max}$ , similar to the one-dimensional case described above. The basic characteristics of a two-dimensional slug release are depicted in the following animation.

Animation - Anisotropic Diffusion in Two-Dimensions - see chapter homepage

This animation depicts the diffusion of a discrete mass released at  $(x=0, y=0, t=0)$ . The diffusion is anisotropic,  $D_x = 4 D_y$ . As indicated in (24), the length scales grow in proportion to the square root of the diffusion, such that the dimensions of the cloud are anisotropic, with  $L_x/L_y = \sqrt{D_x/D_y}$ . In this system,  $L_x = 2L_y$ . Note that the profiles of concentration along the x- and y-axes are Gaussian in shape. Finally, noting that the  $2\sigma$  contour reaches  $x = \pm L$  at  $t = 600s$  provides an estimate for the diffusion coefficient,  $D_x$ . Specifically, use the constraint  $L_x = 2L$  at  $t = 600s$  in (24) to estimate  $D_x$ .

The above analyses are easily extended to three dimensions.

*Instantaneous, Point Release in Three Dimensions*

M released at  $t = 0$  and  $x = y = z = 0$ .

Initial Condition,  $C = M \delta(x)\delta(y)\delta(z)$ .

$$(25) \quad C(x, y, z, t) = \frac{M}{(4\pi t)^{3/2} \sqrt{D_x D_y D_z}} \exp\left(-\frac{x^2}{4D_x t} - \frac{y^2}{4D_y t} - \frac{z^2}{4D_z t}\right) = [ML^{-3}]$$

**Dilution Rate:**

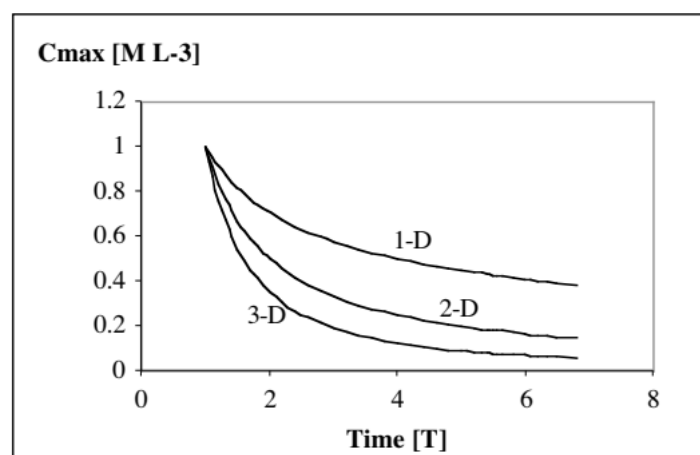
The dilution rate of a patch may be described by the rate of change in the maximum concentration within the patch. For a slug release the maximum concentration,  $C_{\max}(t)$ , occurs at the centroid and corresponds to the coefficient leading the exponential in (10), (23), and (25).

$$1-D: \quad C_{\max} = \frac{M}{A_{yz}(4\pi D t)^{1/2}}$$

$$(26) \quad 2-D: \quad C_{\max} = \frac{M}{L_z 4\pi t \sqrt{D_x D_y}}$$

$$3-D: \quad C_{\max} = \frac{M}{(4\pi t)^{3/2} \sqrt{D_x D_y D_z}}$$

The three-dimensional system provides the greatest dilution rate,  $C_{\max} \sim t^{-3/2}$ , because diffusion can occur in all directions. In the 1-D and 2-D systems, diffusive flux can only occur in 1 or 2 directions, respectively, such that the rate of dilution is diminished to  $t^{-1/2}$  and  $t^{-1}$ , respectively. The difference in dilution rate is depicted below. A slug of mass is released into a 1-D, 2-D, and 3-D system, such that at some small time the maximum concentration in each cloud is the same. Subsequently, the maximum concentration decays most rapidly in the least constrained system, i.e. the 3-D system.



**CLASS EXERCISE WITH SOLUTIONS****Problem 1**

Two airplanes are conducting tests to estimate the coefficient of diffusion in a stagnant atmosphere. The first airplane, flying 1 km above the earth's surface, instantaneously releases 1 kg of trace gas. The second airplane flies through the cloud 60 minutes later and measures a maximum concentration of 0.03 mg/L within the cloud.

Estimate the coefficient of diffusion within this region of the atmosphere.

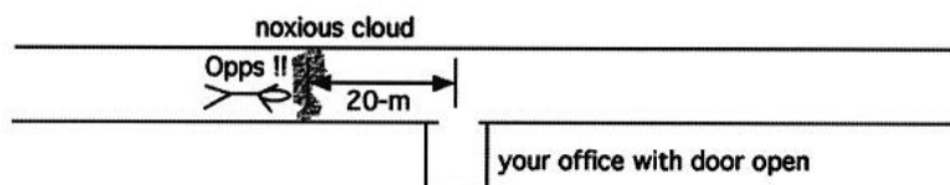
Hint 1 Can you assume uniform concentration in any direction?

Hint 2 What assumption can you make about the air currents?

Hint 3 What assumption must you make about the coefficient of diffusion?

**Problem 2**

A friend of yours is coming to meet you at your office, which is located mid-way along a very long (100 m) hallway. In preparation for a small prank, he is carrying a vial containing 10-g of a noxious smelling gas. He trips 20-m before reaching your office door, and the vial breaks. The gas rapidly mixes vertically and horizontally within the hallway, which is 2-m wide and 3-m high. The human nose will detect the gas at concentrations greater than  $10 - \mu\text{g}/\text{l}$ . Assume an isotropic diffusion,  $D = 0.05 \text{ m}^2/\text{s}$ .



- What governing equation describes the evolution of gas concentration in the hall?
- At what time after the spill do you smell the gas?
- When does the smell, as perceived by humans, disappear from the hallway?

**Problem 3**

A drop of red and a drop of blue dye, each 1 mg, are released 10 cm apart into a layer of stagnant fluid between two plates. The plates are  $1 \text{ m} \times 1 \text{ m}$  in area and are 5 mm apart. The dye drops are released at the center of the plate area and mix rapidly across the fluid layer, i.e. between the plates. The molecular diffusion of the dyes are:  $D_{red} = 10^{-5} \text{ cm}^2/\text{s}$  and  $D_{blue} = 4 \times 10^{-5} \text{ cm}^2/\text{s}$ .

The human eye can detect the color of the dye at concentrations of 10 g/L. No reactions occur between the two dyes, but at locations where the two dyes coexist and are both above the visible threshold, the mixture will appear purple.

- (a) While both clouds are fully visible ( $C > 10 \text{ g/L}$ ), which cloud appears larger, and by how much?

Hint 1 Make a sketch that defines the diameter of each cloud

- (b) At what time and at what location will the two dye clouds first appear to touch?

Hint 2 Simplify the governing equations with assumptions

Hint 3 Write the mathematical criterion for the condition when clouds first touch

- (c) At what time will the line connecting the release points be completely purple?

Hint 4 Define a mathematical criterion for this to occur

## **CLASS EXERCISE – SOLUTIONS**

### **Answer 3.1**

**Estimate the coefficient of diffusivity within this region of atmosphere.**

#### **Hint 1 - Can you assume uniform concentration in any direction?**

The gas is effectively released into an unbounded domain, so that one cannot expect it to mix rapidly to uniform conditions in any direction, i.e.  $\partial C/\partial y \neq 0$ ,  $\partial C/\partial z \neq 0$ ,  $\partial C/\partial x \neq 0$ . All three dimensions must be retained in the governing equation.

#### **Hint 2 - What assumption can you make about the air currents?**

The atmosphere is stagnant, which implies that there are no ambient air currents. You may assume that  $u=v=w=0$ . The passage of each plane will create air movement. This movement may locally and temporarily enhance the dilution of the cloud. However, this effect dies out within a few minutes, so that over an hour time frame, the effect is negligible.

#### **Hint 3 - What assumption must you make about the coefficient of diffusion?**

Since you are only given one value of concentration, you can solve uniquely for only one value of diffusion. Thus, to make the problem tractable, you must assume that the diffusion is isotropic.

With the above assumptions, the transport equation reduces to,

$$\frac{\partial C}{\partial t} = D \left[ \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right].$$

Letting the initial release point be at  $(x, y, z)=0$ , the initial condition can be written,

$$C(x,y,z,t=0) = M \delta(x) \delta(y) \delta(z).$$

The solution to the above equation and initial condition is [Equation 25](#) in Chapter 3.

$$C(x,y,z,t) = \frac{M}{(4\pi D t)^{3/2}} \exp\left(-\frac{x^2 + y^2 + z^2}{4Dt}\right).$$

The maximum concentration within the cloud is given in [Equation 26](#) in Chapter 3.

$$C_{\text{MAX}} = \frac{M}{(4\pi D t)^{3/2}}.$$

Using the information given in the problem statement, we can find D,

$$D = (M/C_{\text{MAX}})^{2/3} / 4\pi t = (10\text{kg}/3 \times 10^{-5} \text{kgm}^{-3})^{2/3} / (4\pi \cdot 3600 \text{ s}) = 0.1\text{m}^2\text{s}^{-1}$$

### Answer 3.2

**What governing equation describes the evolution of the gas concentration in the hall?**

For isotropic diffusion, the governing equation is:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \left[ \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right]$$

Align the coordinate x to the length of the hallway and define the y-z plane as the cross-section. Since the gas "mixes rapidly" in the vertical and horizontal, we assume  $\partial C/\partial y = \partial C/\partial z = 0$ . The problem statement gives no information about air currents in the hallway, so we assume they are negligible,  $u = 0$ . With these assumptions, the governing equation is reduced to,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}.$$

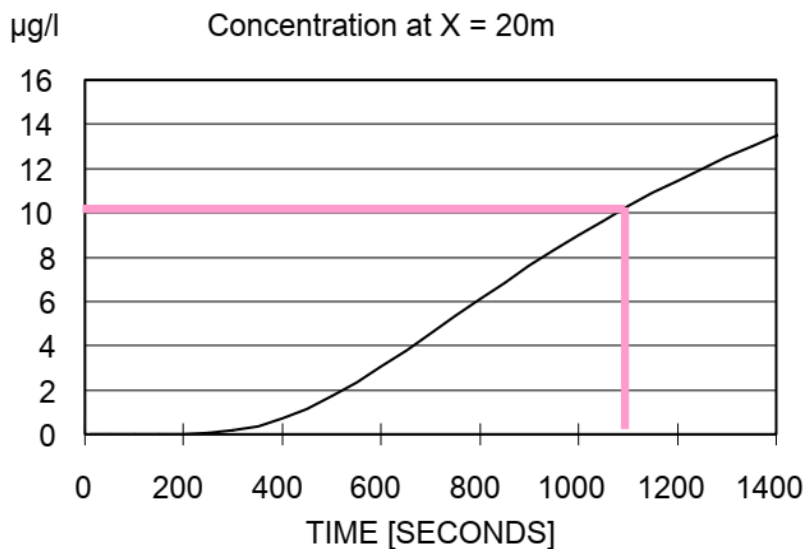
If the spill location is taken as  $x = 0$ , the initial condition is then,  $M = \delta(x)$

**At what time after the spill do you smell the gas?**

The governing equation and initial condition above describe an instantaneous, point release diffusing in one-dimension. The concentration field is described by Equation 10 in chapter 3.

$$3.10 \quad C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) = [M/L^3]$$

Use 3.10 to find the time at which the concentration at your door,  $C(x=20m, t)$  is  $10 \mu\text{gl}^{-1}$ . As time appears in both the exponential and leading terms, it is simpler to use a graphical solution.



You will smell the gas at your office approximately 1100 seconds after the spill.

**When does the smell, as perceived by humans, disappear from the hallway?**

To answer this question we need information on the end conditions of the hallway. Lets first consider that the hallway is open at both ends, so that odor can diffuse beyond the end of the hall. Then, we need only consider the evolution of the maximum concentration, located at the site of the spill,  $x = 0$ . We seek the time for which

$$C_{\max}(t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}} < 10\mu\text{gl}^{-1},$$

or simply

$$t > \left(\frac{10\text{g}}{2\text{m} \times 3\text{m} \times 0.01\text{gm}^{-3}}\right)^2 / (4\pi \times 0.05\text{m}^2\text{s}^{-1}) = 44,232 \text{ seconds} = 12.3 \text{ hrs}$$

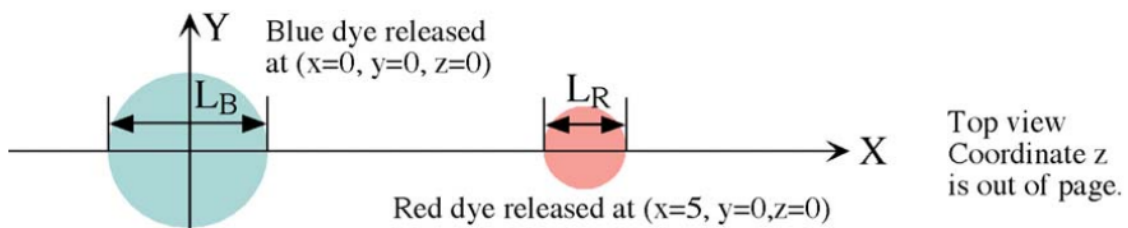
At this time, the length of the cloud will be  $4\sigma = 4\sqrt{2Dt} = 266\text{m}$ , indicating that the cloud has diffused beyond the length of the hallway. The above time scale is correct, only if the hallway is open at both ends. If the hallway is shut off by fire doors at both ends, then in reality the cloud cannot diffuse beyond the length of the hall. Under these conditions, the final concentration in the hallway set the maximum possible dilution, which is determined by distributing the total mass released over the total volume of the hallway (2m x 3m x 100m).

$$C_{\text{final}} = 10\text{g} / (2\text{m} \times 3\text{m} \times 100\text{m}) = 0.016\text{gm}^{-3} = 16\mu\text{gl}^{-1}.$$

Since the final concentration is above the detection limit, the smell will not disappear until the fire doors are opened.

**Answer 3.3**

**Hint 1: Make sketch that defines the diameter of each cloud**



**a. While both clouds are fully visible ( $C > 10\text{-g l}^{-1}$ ), which cloud will appear larger, and how much?**

Since the size of the cloud increases in proportion to the diffusion coefficient, the blue cloud grow more rapidly, and thus appear bigger, than the red cloud. Specifically, the length scale each cloud, as defined in chapter 3, equation 26, is

$$L_B = 4\sigma_B = 4\sqrt{2D_B t} \quad \text{and} \quad L_R = 4\sigma_R = 4\sqrt{2D_R t}.$$

The ratio of dye cloud diameters is then,  $L_B/L_R = \sqrt{D_B/D_R} = 2.$

**b. At what time and at what location will the two dye clouds first appear to touch?**

**Hint 2: Simplify governing equation with assumptions.**

Begin with the full transport equation that governs the evolution of both dye drops.

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = \frac{\partial}{\partial x} D_x \frac{\partial C}{\partial x} + \frac{\partial}{\partial y} D_y \frac{\partial C}{\partial y} + \frac{\partial}{\partial z} D_z \frac{\partial C}{\partial z} \pm S$$

(1)      (2)      (3)      (4)      (5)      (6)      (7)      (8)

Simplifying assumptions:

- The dyes do not interact, so we can drop the source/sink term 8.
- The fluid is stagnant, thus  $u=v=w=0$ , and we can drop terms 2,3,4
- Molecular diffusion is homogeneous and isotropic,  $D_x=D_y=D_z=D$ , and  $D \neq f(x,y,z)$ .  
This reduces the diffusion terms to:  $D (\partial^2 C/\partial x^2 + \partial^2 C/\partial y^2 + \partial^2 C/\partial z^2)$ .
- The dyes mix rapidly between the plates, such that  $\partial C/\partial z = 0$ , and we can drop term 7.

Simplified governing equation: 
$$\frac{\partial C}{\partial t} = D \left[ \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right].$$

Initial conditions:

Blue Dye:  $C_B(x,y,t=0) = M \delta(x) \delta(y)$

Red Dye:  $C_R(x,y,t=0) = M \delta(x-5) \delta(y)$

The diffusion of each dye is thus described by the two-dimensional, instantaneous, point release solution. See equation 23 in Chapter 3, with  $D_x=D_y=D$ . Note, here  $L_z$  is the plate gap, 5mm.

$$C_B(x,y,t) = \frac{M}{L_z 4\pi D_B t} \exp\left(-\frac{x^2 + y^2}{4D_B t}\right)$$

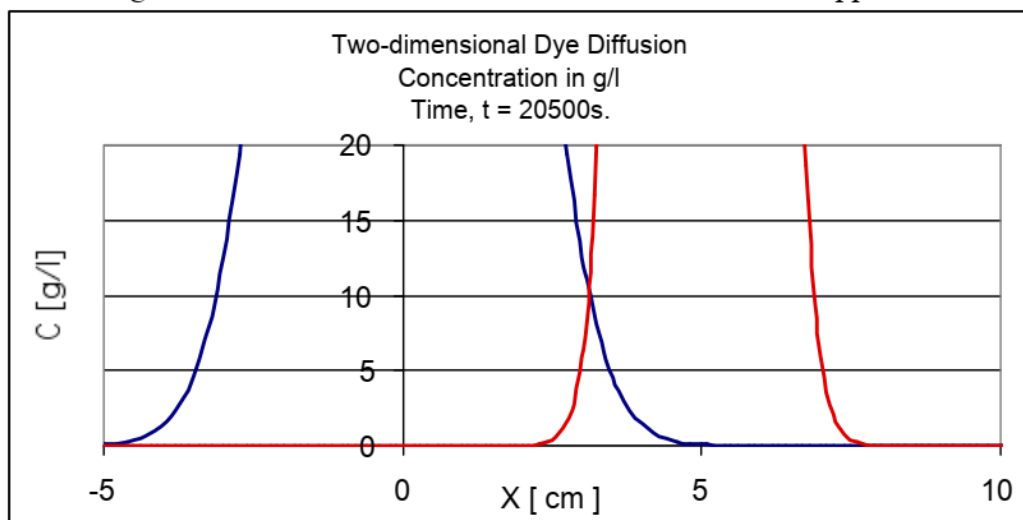
$$C_R(x,y,t) = \frac{M}{L_z 4\pi D_R t} \exp\left(-\frac{(x-5)^2 + y^2}{4D_R t}\right)$$

Note that the exponential is written to be one at the center of the red cloud, i.e. at  $(x=10,y=0)$ , with the position given in units of centimeters.

### Hint 3: Write mathematical criteria for condition when clouds first touch

From geometry, the two clouds will first touch along the line  $y = 0$ . We wish to find the time,  $t$ , and location,  $x$ , for which  $C_B(x, y=0, t) = C_R(x, y=0, t) = 10 \text{ g/l}$ . This could be tackled analytically by first finding the position  $x_{BR}(t)$  at which  $C_B=C_R$ . Then solve for  $t$  using this position to constrain  $x$ , e.g. solve  $C_B(x = x_{BR}, y=0, t)$ . However, a simpler approach is to graph  $C_B$  and  $C_R$  in an interactive graphing package, such as Excel, and then vary time until the intersection of the two concentration curves lies at  $C = 10 \text{ g/l}$

**Solution:** - At  $t = 20500\text{s}$ , the intersection of the blue and red concentration curves corresponds to  $C = 10 \text{ g/l}$  and is located at  $x = 3.1 \text{ cm}$ . The clouds will first appear to touch and  $x = 3.1 \text{ cm}$ .



**Make a rough estimate of the location** using your result from part a?

Based on the definition used in a. and the definition sketch shown above, the two clouds first appear to touch when,  $(L_B/2)+(L_R/2) = 5$ . Additionally,  $L_R = (L_B/2)$ , so the edge of the blue cloud will be at  $x = (L_B/2) = 5/1.5=3.3\text{cm}$ , when it first touches the red cloud.

**c. At what time will the line connecting the release points be completely purple?**

**Hint 4:** - Define a mathematical criteria for this to occur?

This condition requires that  $C_B$  and  $C_R > 10$  g/l in the region  $x = 0$  to  $10$  cm. Use the spreadsheet created in part b to interrogate the concentration field over a range of time.

**Solution** - Graph  $C_B$  and  $C_R$  in an interactive graphing package, such as Excel and vary until the above criteria is met. You will find that the criteria is never met. Between  $0 <$  when  $C_R \geq 10\text{g/l}$  then  $C_B < 10$  g/l, and when  $C_B \geq 10\text{g/l}$ ,  $C_R < 10$  g/l.

