Mathematics For Information Technology

Week 15: Financial mathematics: Percentages, Inverse Percentages, Interests (Simple, Compound), Loan Amortization

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outline

- Intended learning outcome
- Percentages:
- Inverse Percentages:
- Interests:
- Loan Amortization:

Learning outcomes

- Understand the concept of percentages and be able to perform calculations involving percentages
- Apply inverse percentage calculations to find the original value given a new value and the percentage change.
- Understand the concepts of simple interest and compound interest, including the formulas for calculating interest.
- Understand the concept of loan amortization, including the components of loan payments.

Financial mathematics

- Financial mathematics is a branch of mathematics that deals with various concepts and calculations related to finance.
- The following concepts will be discussed in this section i.e. percentage, inverse percentage etc.

Percentages

- A **percentage** is a number which is expressed as a fraction of 100.
- Percent means "number of parts per hundred" and the symbol we use for percent is the percent sign % e.g.
- $43\% = \frac{43}{100} = 0.43$
- $1\% = \frac{1}{100} = 0.01$

Types of percentages

Percentage of an amount

- This is where we are asked to find a certain percentage of an amount e.g.
- Find 25% of 32.
- This is the same as finding a ¼ of 32.

$$32 \div 4 = 8$$

- The answer is 8

Percentage as operator

- In order to calculate a percentage of an amount, a percentage increase or a percentage decrease we can use a **percentage multiplier.**
- To do this we change the percentage that we want into a decimal, and then multiply the amount by that decimal to calculate the answer.

For example,

$$34\%\ of\ 58$$

- Here we **want 34%** which as a **decimal is $\mathbf{0.34}$**
- Therefore the calculation is

$$58 \times 0.34 = 19.72$$

Percentage increase

- This is where we are asked to increase (make bigger) a value by a certain amount e.g. Increase $40g\ by\ 10\%$
- We can find 10% and add it on.

$$10\%\ of\ 40\ is\ 4$$

$$40 + 4 = 44$$

- The answer is 44g

Percentage decrease

- This is where we are asked to decrease (make smaller) a value by a certain amount e.g.
- Decrease $60kg\ by\ 10\%$
- We can find 10% and subtract it.

$$10\%\ of\ 60\ is\ 6$$

$$60 - 6 = 54$$

- The answer is 54kg

Percentage multipliers

- This is where we can find a decimal number and use it as a multiplier to make calculating percentages more efficient e.g. Find 41% of 800.
- We can use 0.41 as a multiplier to find the amount needed.

$$41\% = \frac{41}{100} = 0.41$$

$$800 \times 0.41 = 328$$

- The answer is 328

Reverse Percentages

- **Reverse percentages** (or inverse percentages) means working backwards to find an original amount, given a percentage of that amount.
- We can do this using a calculator by taking the percentage we have been given, dividing to find 1% and then multiplying by 100 to find 100%.
- We can also do this without a calculator by using factors of the percentage we have been given.
- Sometimes we are given a percentage of an amount and we need to work out what the original value was.

Reverse percentages (calculator method)

- In order to find the original amount given a percentage of the amount :
- Write down the percentage and put it equal to the amount you have been given.
- Divide both sides by the percentage (e.g. if you have 80%, divide both by 80). This will give you 1%.
- Multiply both sides by 100.This will give you 100%.

Example

45% of a number is 36. Find the original number.

solution

- Put the percentage equal to the amount $45\% = 36$
- Divide both sides by the percentage to find 1%.

$$1\% = 0.8$$

- Multiply by 100 to find 100%.
- The original number was 80.

Example

150% of a number is 690. Calculate the original number.

solution

- Put the percentage equal to the amount.

$$150\% = 690$$

- Divide both sides by the percentage to find 1%. In this case the percentage is 150%, so divide by 150.
- Multiply by 100 to find 100%.
- The original number was 460.

Non-calculator method

- In a situation where we do not have a calculator, we can often simplify the problem by using common factors.
- Write down the percentage and put it equal to the amount you have been given.
- Identify a common factor of the percentage and 100%.
- Use division to find that percentage of your amount.
- Use multiplication to find 100%.

Example

70% of an amount is 56. Find the original amount.

solution

- Put the percentage equal to the amount.

$$70\% \ = \ 56$$

- Identify a common factor of 70% and 100%.
- Factors of 70: $1, 2, 5, 7, 10, 14, 35, 70$
- Factors of 100: $1, 2, 4, 5, 10, 20, 25, 50, 100$

- Here 1, 2, 5 and 10 are all factors of both 70 and 100 (HCF).
- As 10% is a factor of both 70% and 100%, we need to find 10% of our amount.

$$70\% \div 7 = 10\%.$$

- As we now have 10%, we need to multiply by 10 to find 100%.
- The original amount was 80.

Simple and Compound Interest

- Money is not free to borrow!
- We will refer to money in terms of **present value P**, which is an amount of money at the present time, and **future value F,** which is an amount of money in the future.
- Usually, if someone loans money to another person in present value, and are promised to be paid back in future value, then the person who loaned the money would like the future value to be more than the present value.

- That is because the value of money declines over time due to inflation. Therefore, when a person loans money, they will charge interest.
- They hope that the interest will be enough to beat inflation and make the future value more than the present value.

Simple Interest

- **Simple interest** is interest that is only calculated on the initial amount of the loan (present value, P).
- This means you are paying the same amount of interest every year.

- To calculate simple interest only,

$$I = Prt$$

- where,
- P is the Present value
- r is the Annual percentage rate (APR) changed to a decimal
- t is the Number of years

- To calculate the future value based on simple interest,

$$F = P(1 + rt)$$

- where,
- F is the Future value
- P is the Present value
- r is the Annual percentage rate (APR) changed to a decimal
- t is the Number of years

Example

Sue borrows 2000 at 5% annual simple interest from her bank. How much does she owe after five years?

solution

Year	Interest Earned	Total Balance Owed
1	$2000 \times .05 = 100$	$2000 + 100 = 2100$
2	$2000 \times .05 = 100$	$2100 + 100 = 2200$
3	$2000 \times .05 = 100$	$2200 + 100 = 2300$
4	$2000 \times .05 = 100$	$2300 + 100 = 2400$
5	$2000 \times .05 = 100$	$2400 + 100 = 2500$

After 5 years, Sue owes 2500.

Example

Chad got a student loan for 10,000 at 8% annual simple interest. How much does he owe after one year? How much interest will he pay for that one year?

Solution

$$P = 10{,}000, r = 0.08, t = 1$$

$$F = P(1 + rt)$$

$$F = 10000\big(1 + 0.08(1)\big) = 10{,}800$$

$$= 10000(1 + 0.08(1)) = 10{,}800$$

- Chad owes 10,800 after one year.
- He will pay $10800 - 10000 = 800$ in interest.

Example

Carlos deposits 20,000 into a savings account earning 7.25% annual simple interest. How much does he have in the account after 6 years? What was the total interest earned?

Solution

$$P = 20{,}000, r = 0.0725, t = 6$$

$$F = P(1 + rt)$$

$$F = 20000(1 + 0.0725(6)) = 28{,}700$$

Carlos has 28,700 in his account after 6 years.

He earned $28{,}700 - 20000 = 8{,}700$ in interest.

Compound Interest

- **Compound interest** is interest paid both on the original principal and on all interest that has been added to the original principal.
- Most banks, loans, credit cards, etc. charge you compound interest, not simple interest. This is interest paid on the principal AND the interest accrued. Interest on a savings account can be compounded quarterly (four times a year).
- Interest on a credit card can be compounded weekly or daily!

Compounding type	**Number of compounding periods per year, m**
Annually	1
Semi annually	2
Quarterly	4
Monthly	12
Daily	365

Example

Suppose you invest 3000 into an account that pays you 7% interest per year for four years. Using compound interest, after the interest is calculated at the end of each year, then that amount is added to the total amount of the investment. Then the following year, the interest is calculated using the new total of the loan.

Year	Interest Earned	Total of Loan
1	$3000 \times 0.07 = 210$	$3000 + 210 = 3210$
2	$3210 \times 0.07 = 224.70$	$3210 + 224.70 = 3434.70$
3	$3434.70 \times 0.07 = 240.43$	$3434.70 + 240.43 = 3675.13$
4	$3675.13 \times 0.07 = 257.26$	$3675.13 + 257.26 = 3932.39$
Total	3932.39	

So, after four years, you have earned 932.39 in interest for a total of 3932.39.

Compound Interest Formula

$$F = P\left(1 + \frac{r}{m}\right)^{mt}$$

- where

F = Future value

P = Present value

r = Annual percentage rate (APR) changed into a decimal

t = Number of years

m = Number of compounding periods per year

Example

Let's compare a savings plan that pays 6% simple interest versus another plan that pays 6% annual interest compounded quarterly. If we deposit $8,000 into each savings account, how much money will we have in each account after three years?

Solution

6% Simple Interest:

$$P = \$8{,}000, r = 0.06, t = 3$$

$$F = P(1 + rt)$$

$$F = 8000\left(1 + 0.06(3)\right)$$

$$= 8000(1 + 0.06(3))$$

$$F = 9440$$

$$F = 9440$$

- Thus, we have $9440 in the simple interest account after three years.

Example

6% Interest Compounded Quarterly: $P = 8{,}000, r = 0.06, t = 3, m = 4$

solution

$$F = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$F = 8000\left(1 + \frac{0.06}{4}\right)^{4(3)}$$

$$F = 8000\left(1 + \frac{0.06}{4}\right)^{12}$$

$$F = 9564.95$$

So, we have 9564.95 in the compounded quarterly account after three years.

Example

Suppose you know that you will need 40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

$$F = \$40{,}000, r = 0.04, t = 18, m = 4$$

In this case, we're going to have to set up the equation, and solve for P.

$$40000 = P\left(1 + \frac{0.04}{4}\right)^{18(4)}$$

$$40000 = P(1.01)^{72}$$

$$40000 = P(2.0471)$$

$$P = \frac{40000}{2.0471}$$

$$= 19539.84$$

- So you would need to deposit 19,539.84 now to have 40,000 in 18 years.

Amortized Loans

- If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

- They receive the loan amount now, called the principal, *P*, (or present value), and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments, *PMT*, plus interest.

- Consider the following scenario:
- Two people, Mr. Cash and Mr. Credit, go to buy the same car that costs \$15,000. Mr. Cash pays cash and drives away, but Mr. Credit wants to make monthly payments for five years.
- Our job is to determine the amount that Mr. Credit needs to pay each month for 5 years. We reason as follows:
- If Mr. Credit pays PMT dollars per month, then the PMT dollar payment deposited each month at 9% for 5 years should yield the same amount as the \$15,000 lump sum deposited in an annuity for 5 years.

- Again, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like them to be the same.
- Since Mr. Cash is paying a lump sum of $15,000, the future value F is given by the lump sum compound interest formula, and it is

$$F = 15{,}000\left(1 + \frac{.09}{12}\right)^{60}$$

$$= 15{,}000\left(1 + \frac{.09}{12}\right)^{60}$$

- Mr. Credit wishes to make a sequence of payments of PMT dollars per month, and the future value is given by Future Value, F, of an ordinary annuity is the amount in the account, including interest, after making all payments.

$$F = \frac{PMT\left[\left(1+\frac{.09}{12}\right)^{60} - 1\right]}{\frac{.09}{12}}$$

We set the two future amounts equal to each other and solve for the unknown value, *PMT*.

$$15{,}000\left(1+\frac{.09}{12}\right)^{60}$$

$$= \frac{PMT\left[\left(1+\frac{.09}{12}\right)^{60} - 1\right]}{\frac{.09}{12}}$$

$$15{,}000(1.5657)$$

$$= PMT(75.4241)$$

$$311.3792 = PMT$$

- Therefore, the monthly payment needed to repay the loan is 311.38 for five years.

Amortization Formula

$$PMT = P \cdot \frac{\frac{r}{m}}{\left[1 - \left(1 + \frac{r}{m}\right)^{-mt}\right]}$$

- P is the principal, or amount of the loan
- r is the annual interest rate in decimal form
- t is the length of the loan, in years
- m is the number of compounding periods in one year
- PMT is the loan payment (monthly payment, annual payment, etc.)

You want to take out a 340,000 mortgage (home loan). The interest rate on the loan is 3.5%, and the loan is for 30 years. How much will your monthly payments be? How much interest will you pay over the life of the loan?

Solution

- We're looking for PMT.
- $P = 340{,}000$ the starting loan amount
- $r = 0.035$ annual rate
- $t = 30$ since we're making monthly payments for 30 years
- $m = 12$ since we're doing monthly payments, we'll compound monthly

$$PMT = P \cdot \frac{\left(\frac{r}{m}\right)}{\left[1 - \left(1 + \frac{r}{m}\right)^{-mt}\right]}$$

$$PMT = 340000 \cdot \frac{\left(\frac{0.035}{12}\right)}{\left[1-\left(1+\frac{0.035}{12}\right)^{-12(30)}\right]}$$

$$PMT = 1526.7519$$

- You will make payments of 1526.76 per month for 30 years.
- You will pay a total of 1526.76 per month for 360 months which equals 549,633.60 to the loan company.
- The total paid over the life of the loan is

$$549,633.60 - 340,000 = 209,633.60.$$

Present Value of an Annuity

- The **present value of an annuity** is the amount of money we would need now in order to be able to make the annuity payments in the future.
- Often, we know how much we can afford to pay for each regular payment, so we need to find how much money we can borrow.

Jordan can afford 400 per month as a car payment. The car dealership offers an auto loan at 12% interest for 4 years. What is the present value of the car? In other words, what loan amount can Jordan afford at 400 per month?

Solution

- PMT= 400the monthly loan payment
- r =0.12 annual rate
- t=4 since we're making monthly payments for 4 years
- m= 12 since we're doing monthly payments, we'll compound monthly

$$PMT = P \cdot \frac{\left(\frac{r}{m}\right)}{\left[1 - \left(1 + \frac{r}{m}\right)^{-mt}\right]}$$

$$400 = P \cdot \frac{\frac{0.12}{12}}{\left[1 - \left(1 + \frac{0.12}{12}\right)^{-12(4)}\right]}$$

$$P = 15189.5838$$

- Jordan will pay a total of 15,189.58

Present Value of an Annuity Formula

- Use this formula when you know the payment and you want to find the present value, *P*.

$$P = PMT \cdot \frac{\left[1 - \left(1 + \frac{r}{m}\right)^{-m(t)}\right]}{\frac{r}{m}}$$

- P is the principal, or amount of the loan
- r is the annual interest rate in decimal form
- t is the length of the loan, in years
- m is the number of compounding periods in one year
- PMT is the loan payment (monthly payment, annual payment, etc.

❖ Grace buys an iPad from a rent-to-own business on credit with payments of $30 a month for four years at 14.5% APR compounded monthly. If Grace had bought the iPad from Best Buy or Amazon it would have cost $500. What is the price that Grace paid for the iPad at the rent-to-own business? How much interest was paid over the life of the loan? What is the better option?

$$PMT = \$30, r = 0.145, t = 4, m = 12$$

$$P = PMT \cdot \frac{\left[1 - \left(1 + \frac{r}{m}\right)^{-m(t)}\right]}{\left(\frac{r}{m}\right)}$$

$$P = 30 \cdot \frac{\left[1 - \left(1 + \frac{0.145}{12}\right)^{-12(4)}\right]}{\left(\frac{0.145}{12}\right)}$$

$$P = 1087.8254$$

- That's a lot more that \$500!
- Also, the total amount paid over the course of the loan was $\$30 \times 12 \times 4 = \$1440$.
- Therefore, the total amount of interest paid was

$$\$1440 - \$1087.83 = \$352.17.$$

References

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Thank you for listening to me