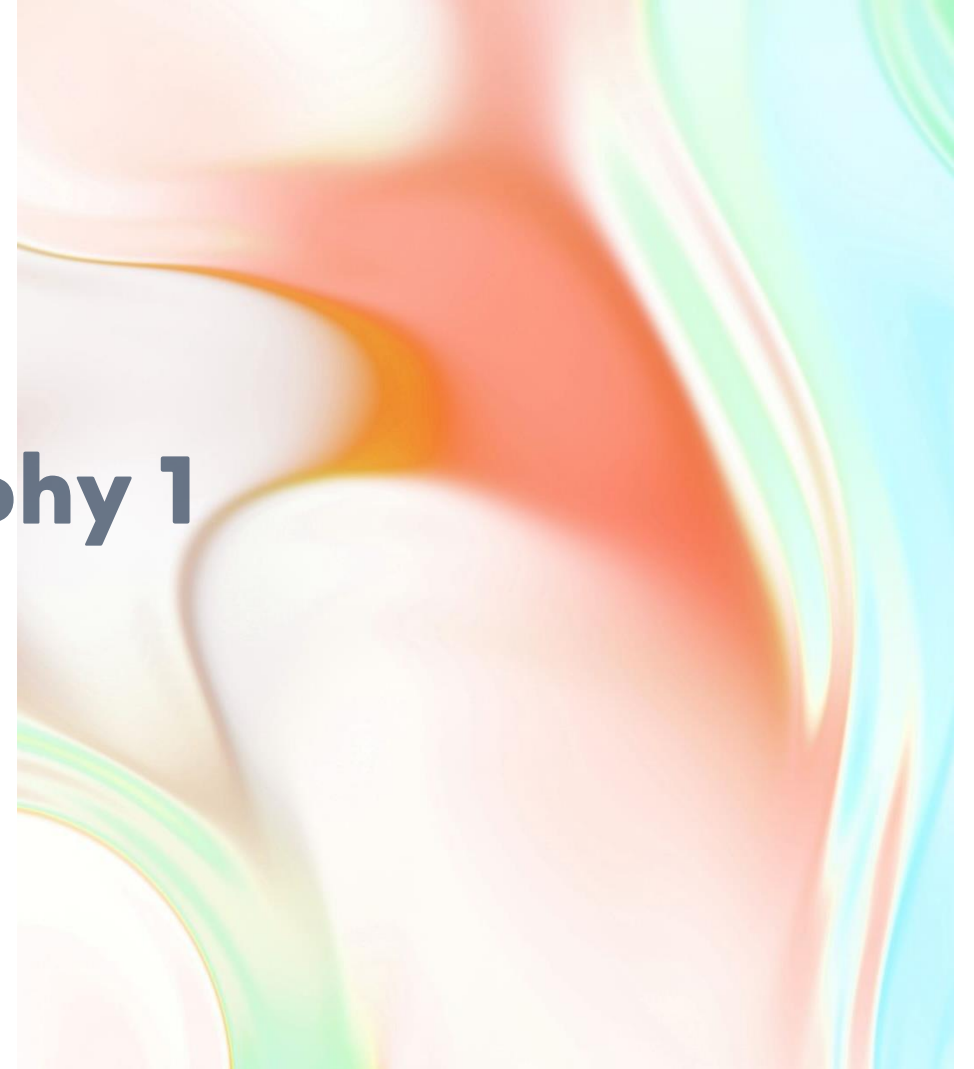


**Discrete Mathematics**  
**Lecture 7**  
**Introduction to Cryptography 1**

**Lecturer: Kahenya, N.P**



# Introduction to lecture 7

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Lecture 7 is a continuation of lecture on Number theory.

This lecture will introduce the concept of cryptography, classical cryptography, and key terms in cryptography.



# References

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These lecture notes have been derived from the following

Sources:

(Koman et al., 2001; Rosen, 2011, 2012).



# Intended Learning Outcomes

Be	At the end of this lecture, you will be able to;
Define	Define cryptography.
Solve	Solve problems in shift ciphers.

# Definition of Terms



Cryptography is the science that deals with the design and implementation of secrecy systems.



It is a technique of securing data and communication during transmission such that only the authorized recipient of the data can comprehend it.



The prefix *crypt* means secret or hidden while the suffix *graphy* means writing.



mathematical concepts or ideas and set of laws or rules known as algorithms are used to convert messages into a form that cannot be read or understood by unauthorized persons.

# Application

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These algorithms are applicable in day-to-day activities such as:

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digital signatures

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verification processes to protect data privacy

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protecting confidential conversations

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Chats

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transactions

# Terminologies



Plaintext is a message that is to be altered into a secret form.



Encryption is the process of altering a message into secret form.



Ciphertext is the secret form of the message.



Cipher is a method of altering a message into a secret message.

The receiver needs to understand the encrypted message.



Hence the message needs to be decrypted.



Decryption is the process of changing a ciphertext back to a plaintext.



A cryptosystem is a 5-tuple  $(p, c, k, e, d)$

- $p$  - the set of plaintext strings,
- $c$  is the set of ciphertext strings,
- $k$  is the set of possible keys,
- $e$  is the set of encryption functions, and
- $d$  is the set of decryption functions.

We have public and private cryptosystems.



Cryptoanalysis is the discipline that deals with breaking such secrecy systems.



It is the process of recovering plaintext from ciphertext without the knowledge of both the encryption method or the key.



Cryptology is a combination of cryptography and cryptanalysis.

# Classical cryptography : Shift ciphers

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The earliest known uses of cryptography was by Julius Caesar (Rosen, 2012).

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The Caesar cipher is a type of shift cipher.

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The method involved moving or shifting an alphabet a few letters or steps forward.

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For example, letter D is moved to letter G.



The Caesar's encryption method can be represented by the function  $f$  that assigns to the nonnegative integer  $p$ ,

$p \leq 25$ , the integer  $f(p) = c$  in the set  $\{0,1,2, \dots , 25\}$ , that is, integers modulo 26,  $\mathbb{Z}_{26}$  by;

$$f(p) = c = (p + 3) \bmod 26$$


# For example,

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An encrypted message the letter  $p$  is replaced with;

$f(p) = (p + 3) \bmod 26$  where 3 is the shift key.

The receiver of the message will use the decryption function  $f(p)^{-1}$  to change the ciphertext  $c$  to plaintext  $p$ .

The decryption function is given as;

$$f(p)^{-1} = p = (c - 3) \bmod 26$$

In general, the shift cipher with shift key  $k$  can be represented by the function

$$f(p) = (p + k) \bmod 26$$

This function is the encrypting function, where  $p$  is the plaintext.

To recover the plaintext, we use the inverse function  $f(p)^{-1}$  i.e.,

$$f(p)^{-1} = (p - k) \bmod 26$$

# Example 1

Otieno wants to send a secret message to Michael.

He encrypts the plaintext using the shift cipher with key 3.

Determine the secret message he will send for the message, 'MEET AFTER THE PARTY'.

The encrypted message sent is *PHHWDIWHUWKHSDUWB*.

Plaintext P	M	E	E	T	A	F	T	E	
	12	4	4	19	0	5	19	4	
$(p + 3) \bmod 26$	15	7	7	22	3	8	22	7	
Ciphertext	P	H	H	W	D	I	W	H	
Plaintext	R	T	H	E	P	A	R	T	Y
	17	19	7	4	15	0	17	19	24
$(p + 3) \bmod 26$	20	22	10	7	18	3	20	22	1
ciphertext C	U	W	K	H	S	D	U	W	B



# Example 2

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Juma wants to send a secret message to Mutiso.

The secret message is to be encrypted with the shift cipher with key  $k = 7$ .

Determine the ciphertext for the plaintext;

TWO ZEBRA GRAZING ALONG THE HIGHWAY

The ciphertext sent is *ADVGLIYHNYHGGQUNHSVUNAOLOPNOCHF*

p	T	W	O	Z	E	B	R	A	G	R	A	Z	I	N	G
$(p + 7) \bmod 26$	19	22	14	25	4	1	17	0	6	17	0	25	9	13	6
	0	3	21	6	11	8	24	7	13	24	7	6	16	20	13
c	A	D	V	G	L	I	Y	H	N	Y	H	G	Q	U	N
p	A	L	O	N	G	T	H	E	H	I	G	H	W	A	Y
	0	11	14	13	6	19	7	4	7	8	6	7	22	0	24
$(p + 7) \bmod 26$	7	18	21	20	13	0	14	11	14	15	13	14	2	7	5
c	H	S	V	U	N	A	O	L	O	P	N	O	C	H	F

# Example 3

Ciphertext C	X	F	D	S	T	Y	T	Q
	23	5	3	18	19	24	19	16
$(c - 5) \bmod 26$	18	0	24	13	14	19	14	11
Plaintext P	S	A	Y	N	O	T	O	L
Ciphertext	F	E	N	S	J	X	X	
	5	4	13	18	9	23	23	
$(c - 5) \bmod 26$	0	25	8	13	4	18	18	
Plaintext P	A	Z	I	N	E	S	S	

Akot sent a decrypted message *XFDSTYTQFENSJXX* to his friend Akur.

He had encrypted the plaintext message using the encrypting function

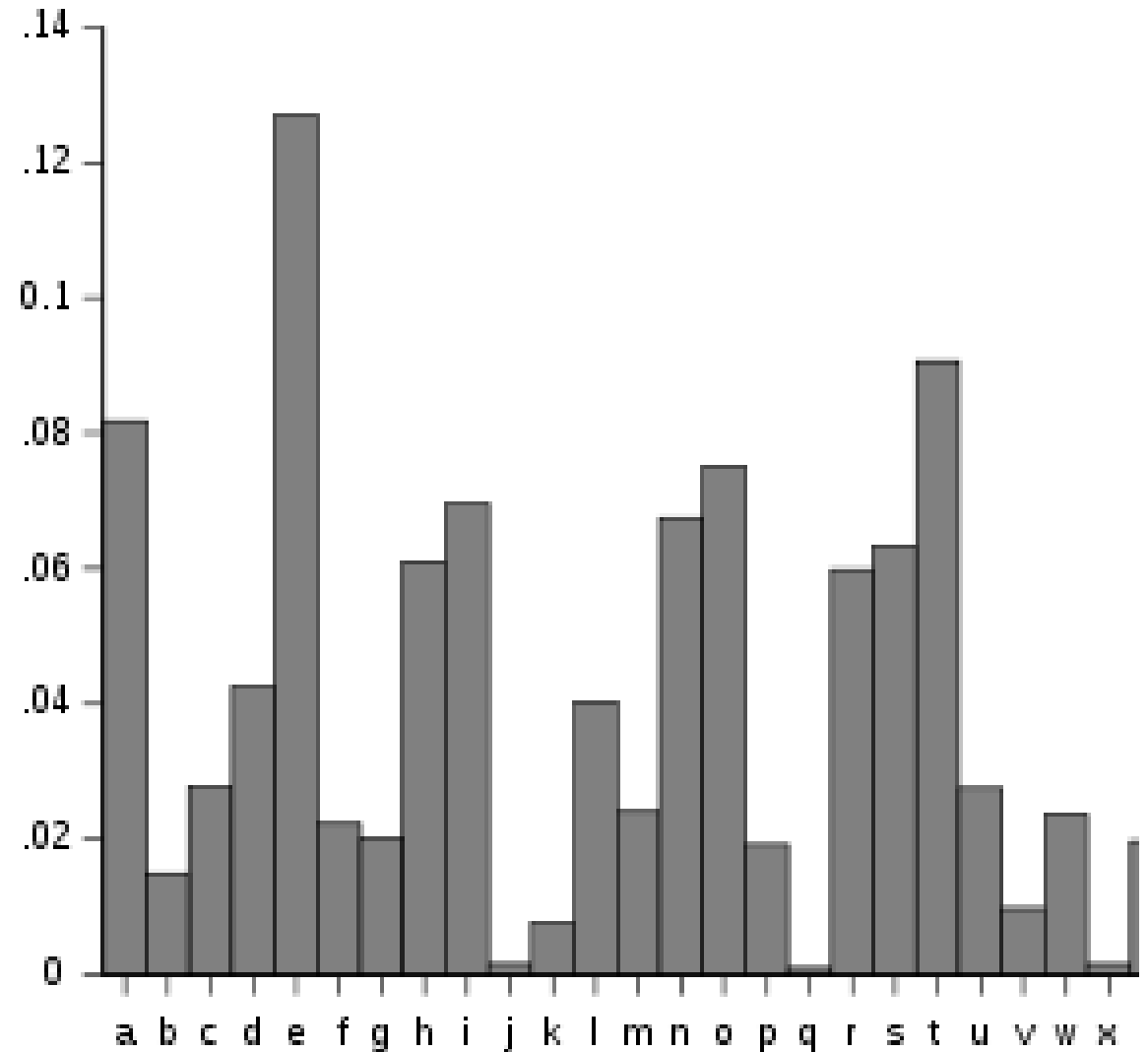
$$f(p) = (p + 5) \bmod 26.$$

Help Akur determine the plaintext message.

The original message sent by Akot is SAY NO TO LAZINESS.

# Relative frequency of letters

- ❑ The shift ciphers are weak and vulnerable to attacks.
- ❑ One can easily do an analysis of letters frequently appearing in a ciphertext to determine the key.
- ❑ In the English words the approximate relative frequency of letters put the most common letters as
- ❑ E (13%), T(9%), A (8%), O (8%), I (7%), S (7%) and so on (Rosen, 2012).
- ❑ The rarest are Z, Q, X, and J.
- ❑ The most common pairs of letters are TH, ER, ON and AN
- ❑ SS, EE, TT, and FF are the most common repeats.
- ❑ These patterns pose areas of weakness that are exploited by ciphertext-only attacks.



Source: [Frequency analysis - Wikipedia](#):

# Example 5

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Decrypt the ciphertext;

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*YFXMPCESPZCJTDFDPQFWQZCPYNTASPCTYRXPDDLRPD*

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**Solution:** We want to use brute-force attack to determine the shift key that was used to encrypt this message.

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First, we determine the frequency of each letter.

Letter	Y	F	X	M	P	C	E	Z	J	T	D	Q	W	N	A	S	R	L
Frequency	3	3	2	1	7	4	1	2	1	3	5	2	1	1	1	2	2	1

- ❑ In the ciphertext the most frequent letter is P.
- ❑ Hence, we can assume P with a value of 15 is equivalent to E with a value of 4.
- ❑ E is the plaintext that was encrypted to P (the ciphertext).
- ❑ Taking our encrypting function to be
- ❑  $c = (p + k) \bmod 26$  where p is the plaintext and c the ciphertext, we have,
- ❑  $15 = (4 + k) \bmod 26$ , thus  $k = 11$

Therefore, the encrypting function is;  $c = (p + 11) \bmod 26$  and hence the decrypting function is;

$$p = (c - 11) \bmod 26$$

<b>Ciphertext C</b>	<b>Y</b>	<b>F</b>	<b>X</b>	<b>M</b>	<b>P</b>	<b>C</b>	<b>E</b>	<b>S</b>	<b>P</b>	<b>Z</b>	<b>C</b>	<b>J</b>	<b>T</b>	<b>D</b>
	24	5	23	12	15	2	4	18	15	25	2	9	19	3
$(p - 11) \bmod 26$	13	20	12	1	4	17	19	7	4	14	17	24	8	18
<b>Plaintext P</b>	<b>N</b>	<b>U</b>	<b>M</b>	<b>B</b>	<b>E</b>	<b>R</b>	<b>T</b>	<b>H</b>	<b>E</b>	<b>O</b>	<b>R</b>	<b>Y</b>	<b>I</b>	<b>S</b>
<b>Ciphertext C</b>	<b>F</b>	<b>D</b>	<b>P</b>	<b>Q</b>	<b>F</b>	<b>W</b>	<b>Q</b>	<b>Z</b>	<b>C</b>	<b>P</b>	<b>Y</b>	<b>N</b>	<b>T</b>	<b>A</b>
	5	3	15	16	5	22	16	25	2	15	24	13	19	0
$(p - 11) \bmod 26$	20	18	4	5	20	11	5	14	17	4	13	2	8	15
<b>Plaintext P</b>	<b>U</b>	<b>S</b>	<b>E</b>	<b>F</b>	<b>U</b>	<b>L</b>	<b>F</b>	<b>O</b>	<b>R</b>	<b>E</b>	<b>N</b>	<b>C</b>	<b>I</b>	<b>P</b>

# Affine ciphers

The shift ciphers can easily be cracked by brute force attack, that is, a method that uses trial-error to determine the encrypting key.

To improve on the shift ciphers, one can generalize them to a function of the form;

$$f(p) = c = (\alpha p + \beta) \bmod 26$$

with  $\alpha$  and  $\beta$  are integer and are such that the function  $f$  is a bijection.

For decrypting we use the inverse function;

$$f(p)^{-1} = p = \bar{\alpha}(c - \beta) \bmod 26$$



<b>Plaintext</b>	<b>W</b>	<b>H</b>	<b>I</b>	<b>S</b>	<b>P</b>	<b>E</b>	<b>R</b>	<b>I</b>	<b>N</b>	<b>G</b>
Value	22	7	8	18	15	4	17	8	13	6
$f(p) = (3p + 5) \bmod 26$	$71 \bmod 26 = 19$									
Ciphertext	T									

## Example 1

Encrypt the following plaintext message *WHISPERING* using the affine cipher

$$f(p) = (3p + 5) \bmod 26.$$

# Example 2

Decrypt the message *PHYQBPHYVRTNHOT* that the original message was encrypted with the function

$$c = (5p + 7) \bmod 26.$$

**Solution:** The decrypting function is the inverse function

$$p = \bar{5}(c - 7) \bmod 26.$$

We first need to find the inverse of  $5 \bmod 26$  (discussed in a previous lesson for week 5).

Let the inverse of 5 be  $x$  then  $5x \equiv 1 \bmod 26$

$\Rightarrow \gcd(5, 26) = 1$  that is;

$$26 = 5(5) + 1$$

$$\therefore 1 = 26 - 5(5)$$



# Example 2...contd...

Introducing mod 26 we get;

$$26 \bmod 26 - 5 \cdot 5 \bmod 26 = 1 \bmod 26$$

$$\Rightarrow -5 \cdot 5 \bmod 26 = 1 \bmod 26$$

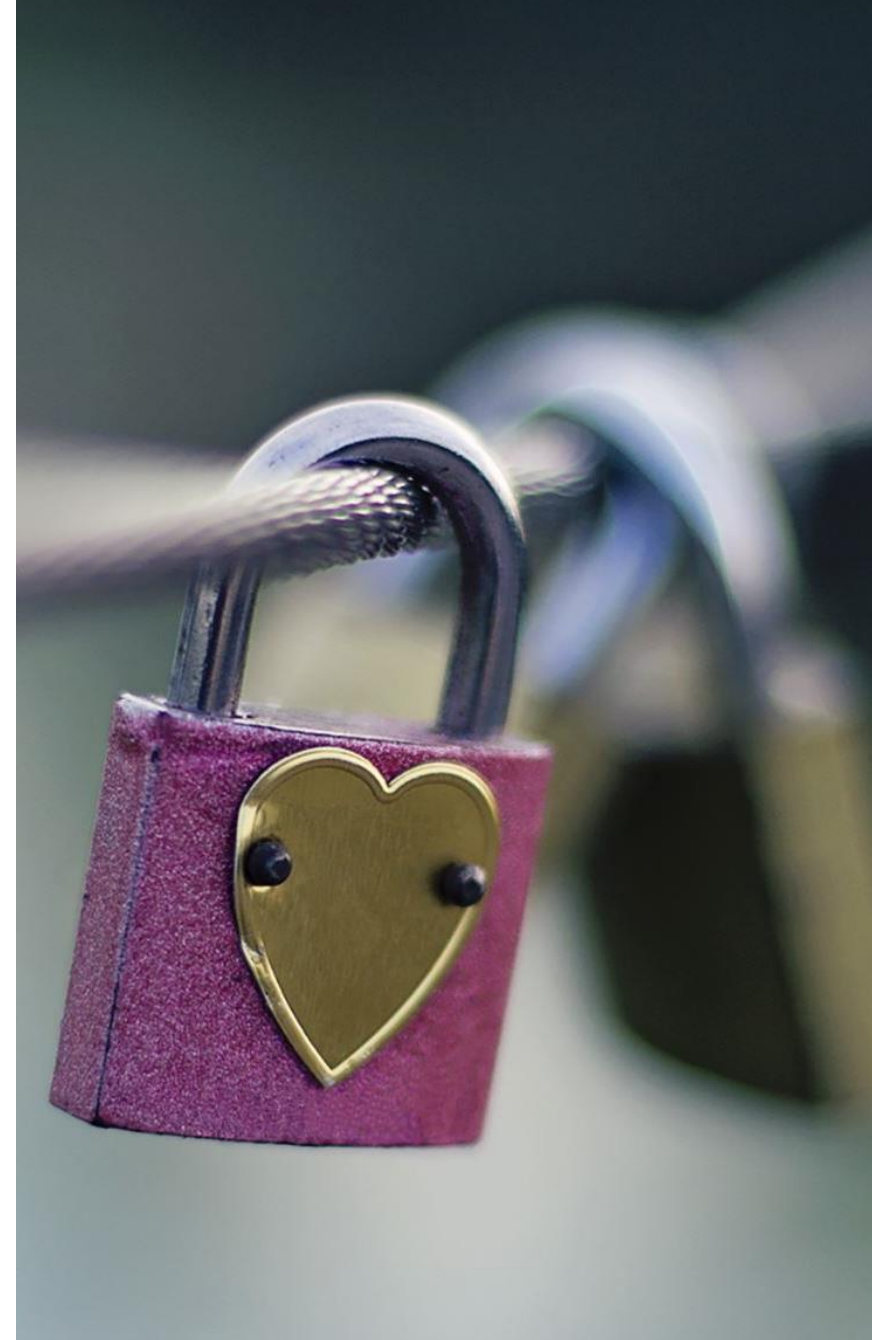
$$(26 - 5)5 \bmod 26 \equiv 1 \bmod 26$$

$$21 \cdot 5 \bmod 26 \equiv 1 \bmod 26$$

$$\therefore \bar{5} = 21$$

Therefore, the decrypting function is;

$$p = 21(c - 7) \bmod 26$$



The decrypting function is

$$p = 21(c - 7) \bmod 26$$



	P	H	Y	Q	B	P	H	Y	V	R	T	N	H	O	T
	15	7	24	16	1	15	7	24	21	17	19	13	7	14	19
	<b>M</b>	<b>A</b>	<b>T</b>	<b>H</b>	<b>E</b>	<b>M</b>	<b>A</b>	<b>T</b>	<b>I</b>	<b>C</b>	<b>S</b>	<b>W</b>	<b>A</b>	<b>R</b>	<b>S</b>

# Theorem

Let  $a, b, c, d, e, f,$  and  $m$  be integers with  $m \geq 0$

such that  $(\Delta, m) = 1$  where

$\Delta = ad - bc$  then the systems of linear congruences

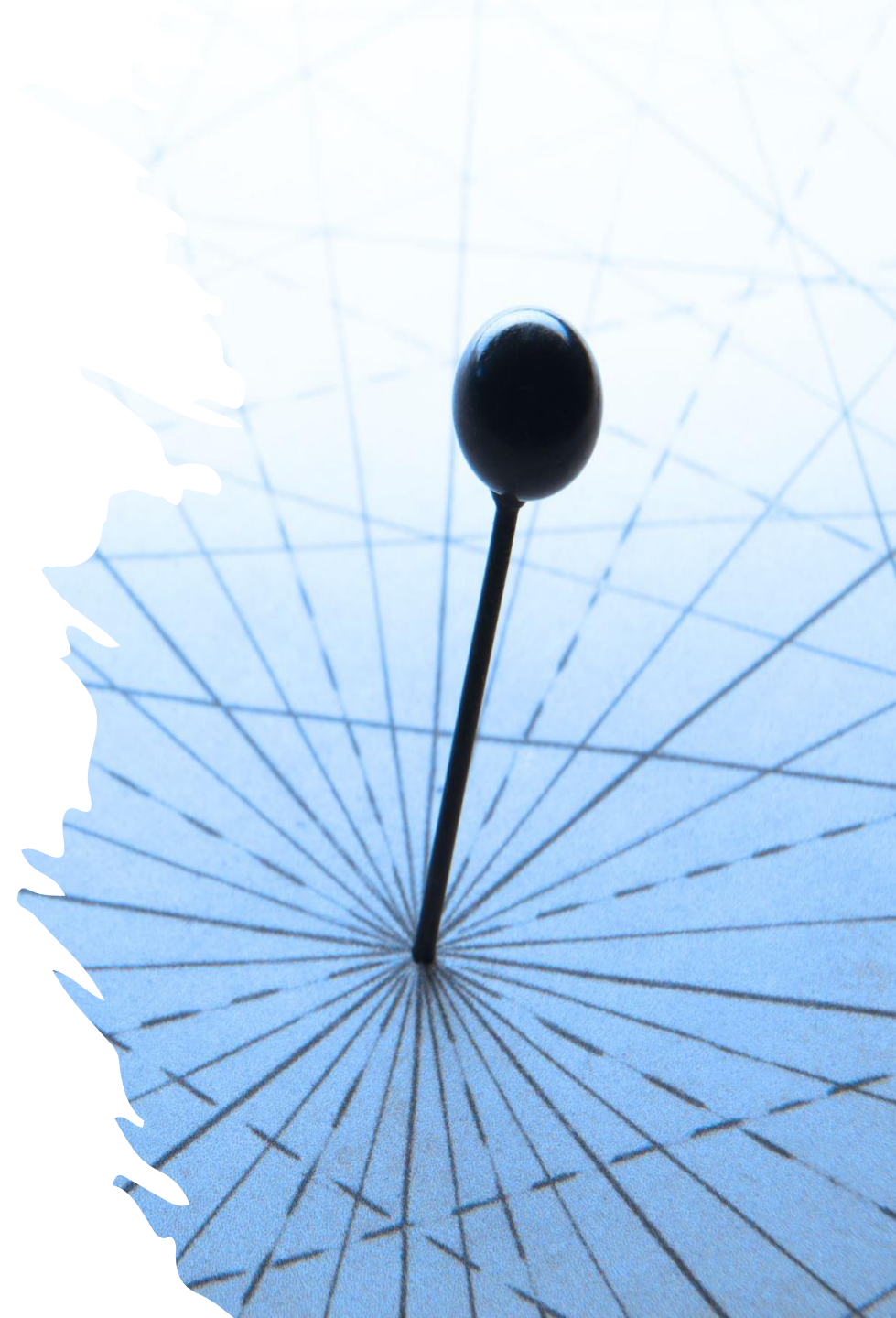
$$ax + by = e \pmod{m},$$

$$cx + dy = f \pmod{m}$$

has a unique solution modulo  $m$  given by;

$$x = \bar{\Delta}(de - bf) \pmod{m} \text{ and } y = \bar{\Delta}(af - ce) \pmod{m}$$

where  $\bar{\Delta}$  is the inverse  $\Delta$  modulo  $m$ .



# Example 1

Solve the system of linear congruences

$$19 = (24x + y) \pmod{26}$$

$$22 = (13x + y) \pmod{26}$$

**Solution:** Recall from a previous lecture that

$a \equiv b \pmod{m}$  is the same as  $b \equiv a \pmod{m}$ .

Hence, we can have;

$$24x + y \equiv 19 \pmod{26} \dots (i)$$

$$13x + y \equiv 22 \pmod{26} \dots (ii)$$



Note that  $\Delta = ad - bc = 24 - 13 = 11$  and  $(\Delta, m) = (11, 26) = 1$  implying that the system has a unique solution.

Subtracting (ii) from (i) we have;  $11x \equiv -3 \pmod{26} = 23 \pmod{26}$

Let another parameter  $t$  be such that;

$$11t = 1 \pmod{26} \Rightarrow 11tx = 23t \pmod{26} \therefore x = 23t \pmod{26} \dots \text{(iii)}$$

From  $11t = 1 \pmod{26}$  we have;

$$26 = 2(11) + 4$$

$$11 = 2(4) + 3$$

$$4 = 1(3) + 1$$

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Hence

$$1 = 4 - 1(3) = 4 - 1[11 - 2(4)] = 3(4) - 1(11) = 3[26 - 2(11)] - 1(11)$$
$$\therefore 1 = 3(26) - 7(11)$$

Introducing mod 26 we have;

$$-7(11) \equiv 1 \pmod{26}$$

$$15 \cdot 11 \equiv 1 \pmod{26} \therefore 11^{-1} = 15 = t$$

Therefore equation (iii) becomes  $x = 23 \times 15 \pmod{26} = 7$

Next, we determine  $y$  by multiplying equation (i) and (ii) by 13 and 24 respectively (to eliminate  $x$ ) and subtracting to get;

$$\begin{array}{r} 312x + 13y = 247 \pmod{26} \\ 312x + 24y = 528 \pmod{26} \\ \hline 11y = 281 \pmod{26} \cdots \text{(iv)} \end{array}$$

Again, we introduce a parameter  $t$  such that  $11t = 1 \pmod{26}$ .

Thus equation (iv) becomes  $11ty = 281t \pmod{26} \Rightarrow y = 281t \pmod{26}$

From the previous working  $t = 15$  and therefore;

$$y = 281 \times 15 \pmod{26} = 4215 \pmod{26} = 3$$

# Example 2

Decrypt the message below given that an affine cipher of the form

$C = (aP + b) \bmod 26$  was used with  $(a, 26) = 1, 0 \leq b \leq 25$ .

*RVRQQCBERFRWRYJ**W**ABGORBFWLY*  
*KW*  
*ABYNYWPHPOOPVRY**W**ABL**S**W**BO**YP*  
***PY***

	R	V	Q	C	B	E	F	W	Y	J	A	G	O	L	K	N	P	H	S
	7	2	2	1	6	1	2	7	7	1	3	1	4	2	1	1	5	1	1

**Solution:** Make a frequency table for the ciphertext

# Example 2...contd...

RVRQQCBERFRWRYJ**W**ABGORBFW**LY**KW ABYNYWPHPOOPVRY**W**ABL**S****W****BO**Y**PPY**

R, W, Y, and B have high frequency. We must determine which will assume the values E, T, A, O, I, N etc.

Recall that The most common pairs of letters are TH, ER, ON and AN.

*In our ciphertext the common pair is WA with 2 counts and BO and PY with 1 count each.*

*We can let W= T and A=H we can also let B = E (the pair ER correspond to BO). Also, O = R (R is a popular element).*

	<b>R</b>	<b>V</b>	<b>Q</b>	<b>C</b>	<b>B</b>	<b>E</b>	<b>F</b>	<b>W</b>	<b>Y</b>	<b>J</b>	<b>A</b>	<b>G</b>	<b>O</b>	<b>L</b>	<b>K</b>	<b>N</b>	<b>P</b>	<b>H</b>	<b>S</b>
	<b>7</b>	<b>2</b>	<b>2</b>	<b>1</b>	<b>6</b>	<b>1</b>	<b>2</b>	<b>7</b>	<b>7</b>	<b>1</b>	<b>3</b>	<b>1</b>	<b>4</b>	<b>2</b>	<b>1</b>	<b>1</b>	<b>5</b>	<b>1</b>	<b>1</b>

## Example 2...contd...

We can then investigate the following;

$$\begin{aligned} 22 &= (19a + b) \bmod 26; & 1 &= (4a + b) \bmod 26 \\ 0 &= (7a + b) \bmod 26 & 14 &= (17a + b) \bmod 26 \end{aligned}$$

Suppose we solve the congruences;

$$22 = (19a + b) \bmod 26 \text{ and } 1 = (4a + b) \bmod 26$$

We can see that

$$\begin{aligned} 22 &= 19a + b \cdots \text{(i)} \\ 1 &= 4a + b \cdots \text{(ii)} \end{aligned}$$



## Example 2...contd...

Subtracting (ii) from (i) we get  $21 \bmod 26 \equiv 15a$

We let another parameter  $t$  such that  $15t = 1 \bmod 26$

$$\therefore a = 21t \bmod 26$$

$$\Rightarrow 26 = 15(1) + 11$$

$$15 = 1(11) + 4$$

$$11 = 2(4) + 3$$

$$4 = 1(3) + 1$$

Therefore  $1 = 7 \cdot 15 \bmod 26 \Rightarrow t = 7$

$$\therefore a = 21 \cdot 7 \bmod 26 = 17$$

From equations (i) and (ii) we can eliminate  $a$  to get  $15b = 9 \bmod 26$



## Example 2...contd...

Again, we let a parameter  $t$  is such that  $15t = 1 \pmod{26}$  hence  $b = 9t \pmod{26}$ .

This congruence is the same as the above working and therefore;

$$t = 7 \Rightarrow b = 9 \cdot 7 \pmod{26} = 11$$

We now have our encrypting function  $c = (17p + 11) \pmod{26}$ .

The decryption function is then  $p = \overline{17}(c - 11) \pmod{26}$

It can be shown that  $\overline{17} = 23$

The decrypting function is  $p = 23(c - 11) \pmod{26}$

# Example 2...contd...

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The plaintext message is;

I WILL BE VISITING THE PRIEST AND THE NUN  
TOMORROW IN THE AFTERNOON

# References



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Rosen, K. (2011). *Elementary Number Theory and Its Application* (6th ed.). Person.



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