Discrete Mathematics

Lecture 1

Propositional logic

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Course description

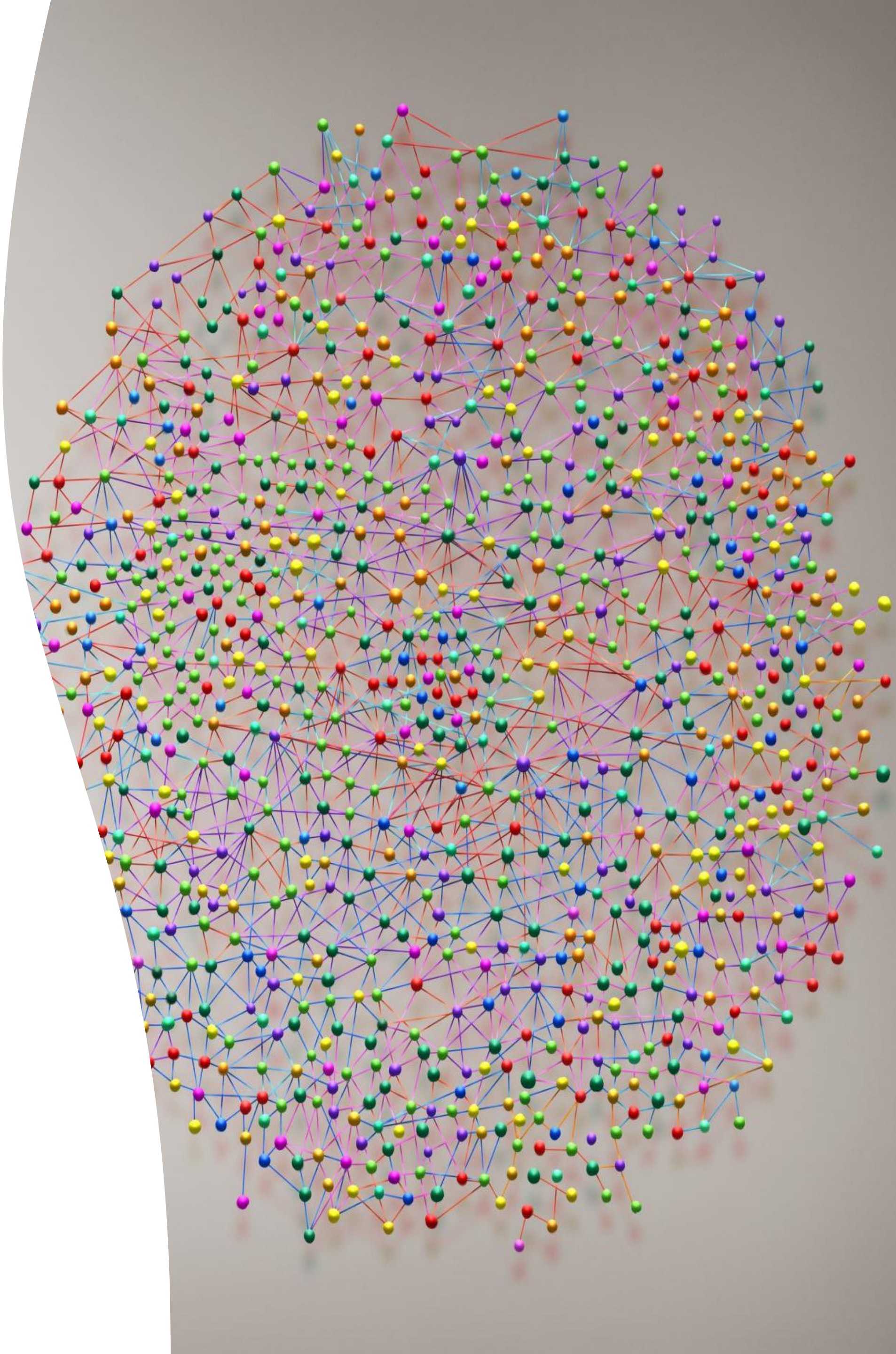
Discrete mathematics is the study of mathematical structures that are separated or distinct i.e., discrete.

Equip students with basic knowledge and skills in propositional and predicate logic, number theory, cryptography, graphs, relations, functions, and trees.

Discrete mathematics concepts form key mathematical foundation for understanding of computers.

Course goals and objectives

Equip students with knowledge and skills in discrete mathematical structures such as propositional and predicate logic, number theory, cryptography, graphs, relations, functions, and trees



Core textbook

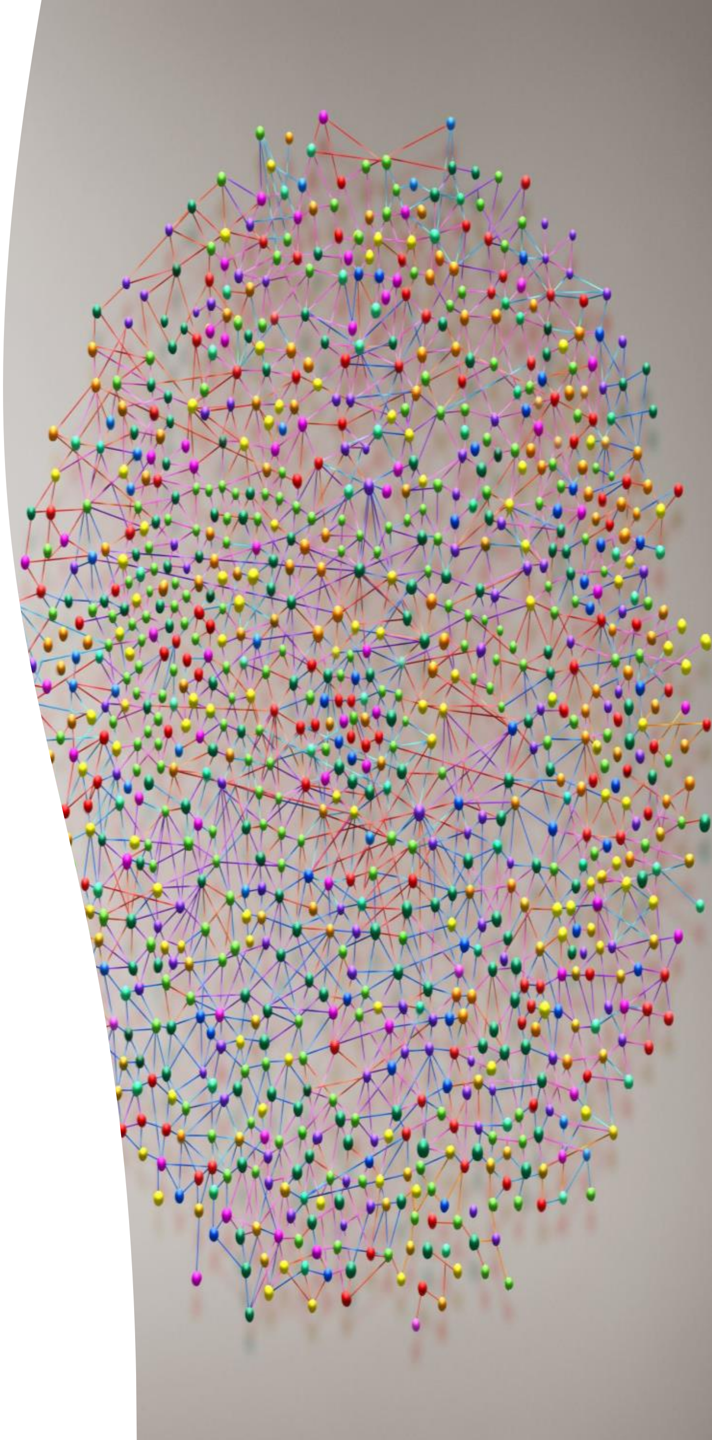
Discrete Mathematics and its Application 7th ed., Kenneth H. Rosen, McGraw-Hill, 2012

Topics

- ❑Propositional & Predicate logic
- ❑Logic and validity
- ❑Number theory; Linear congruence & Cryptography
- ❑Relations and functions
- ❑Graph theory & Trees
- ❑Mathematical proofs

Introduction to lecture 1

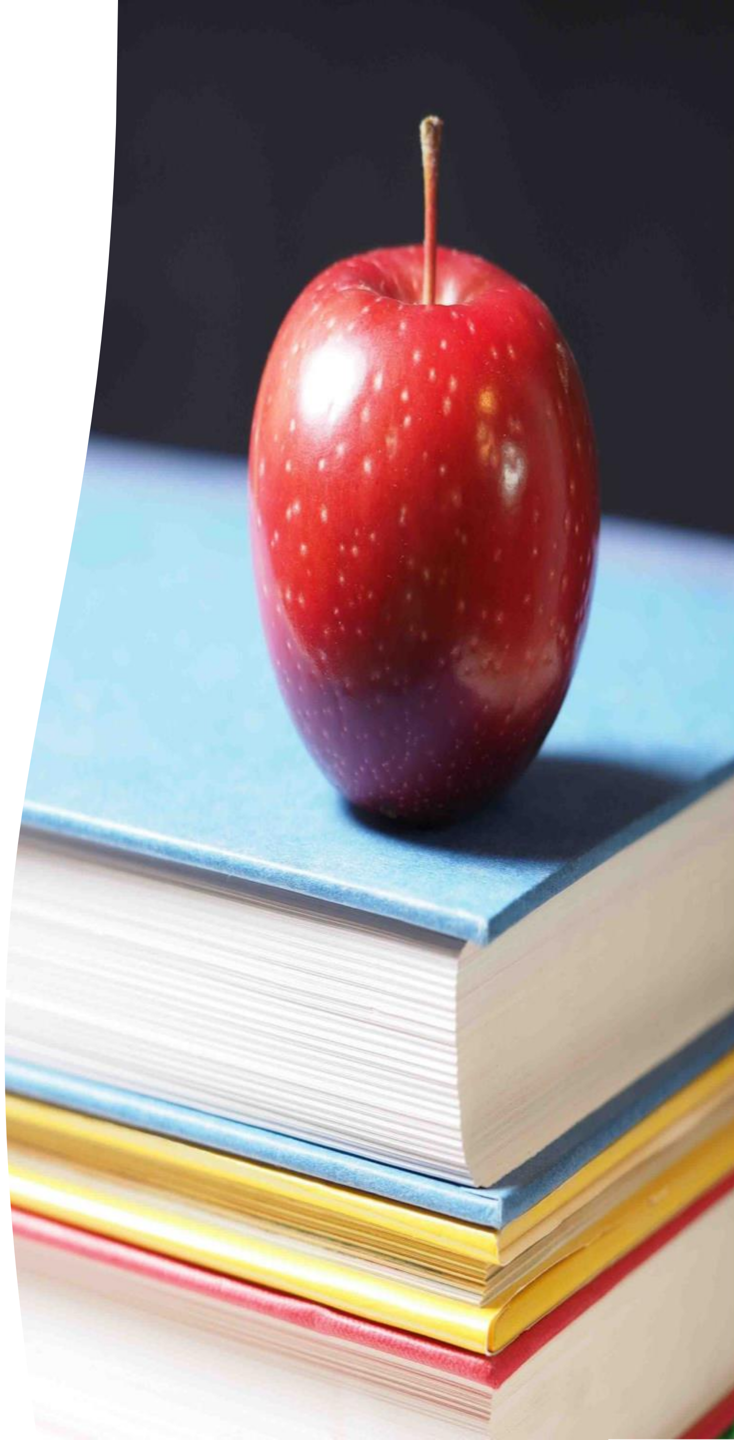
- This lecture introduces the concept of propositional logic, its application and importance to computing.
- logic automate reasoning.
- It is used to design digital circuits, construction, and verification of validity of computer programs.



References

These lecture notes have been derived from the following resources

- ❑Kenneth, 2012;
- ❑Koman et al., 2001;
- ❑ Lipschutz & Lipson, 2007;
- ❑ Susanna, 2003.



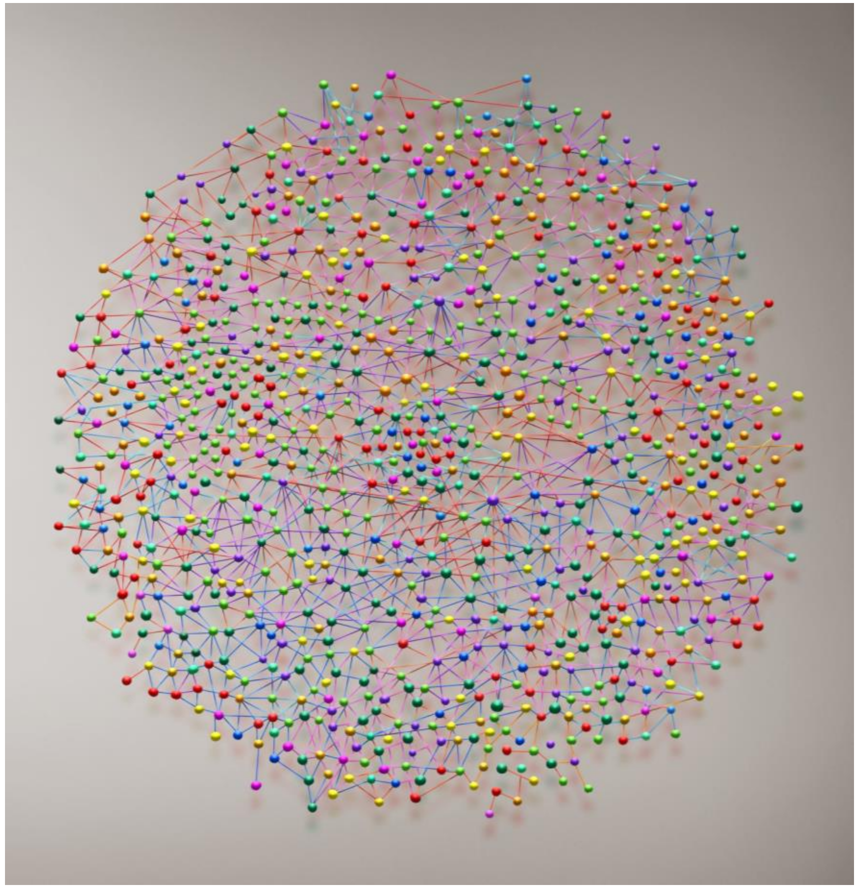
Intended Learning Outcomes

At the end of this lecture, you will be able to;

- Define terms used in propositional logic.
- Carry out operations involving propositional logic.
- Apply logic in circuit design.

Definition of terms

- A proposition is a statement that can be assigned a truth value.
- A statement that can either True (T) or False (F).
- For example, *12+7 = 19; 10-2=7; all cows have tail* etc.
- Some statements are not proposition since it is impossible to assign a truth value to them.
- For example; *come tomorrow; I am telling the truth* etc.



Negation of a proposition

We say that the proposition ¬p or Not p is true if the proposition p is false and is false if the proposition p is true. This is the negation of a proposition.

Consider the statement

p: the book cover is red.

Its negation is

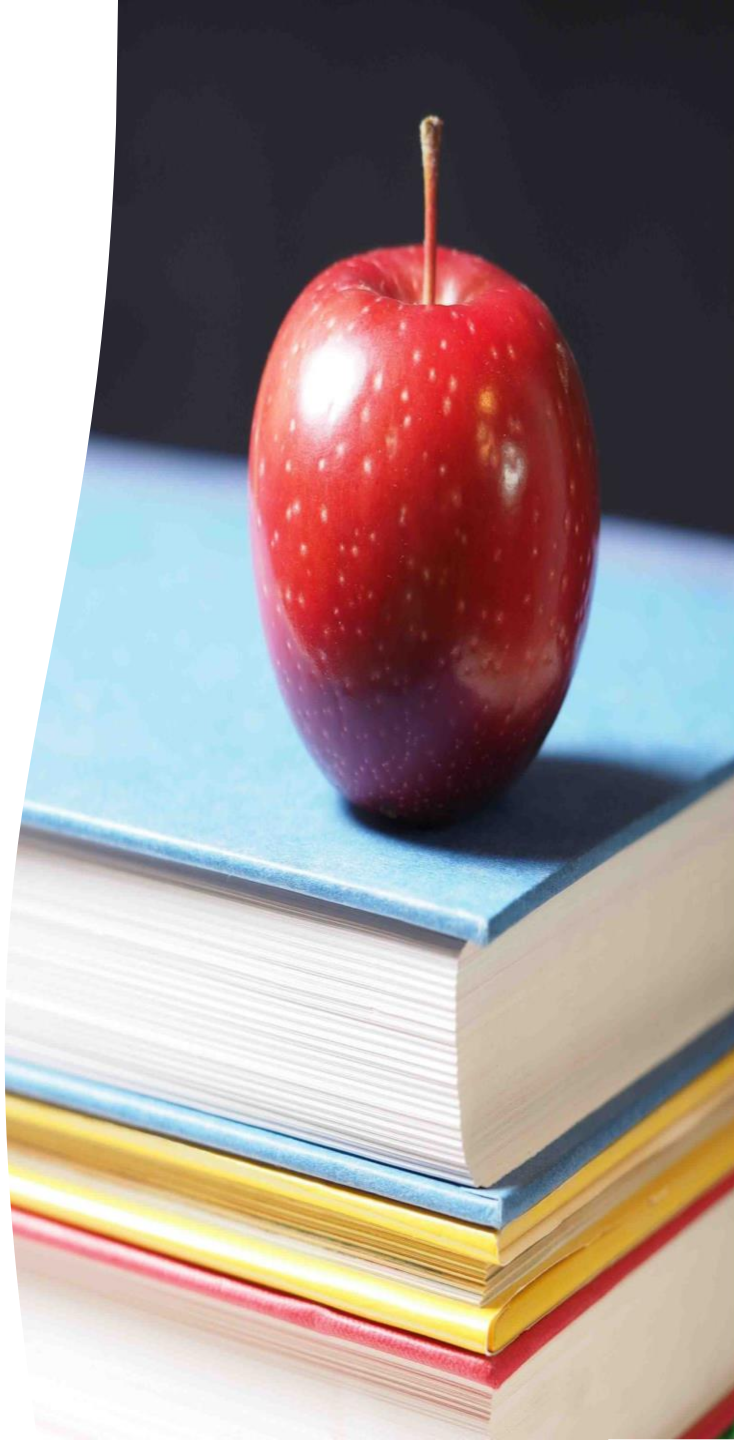
$\neg p$: *The book cover is not red.*

Conjunction

❑ A conjunction consists of two or more statements connected by the word '*AND*'.

❑ Suppose p and q are two simple statements then $p \wedge q$ is called the conjunction of p with q.

❑ Today is Friday and the elephant is sick.



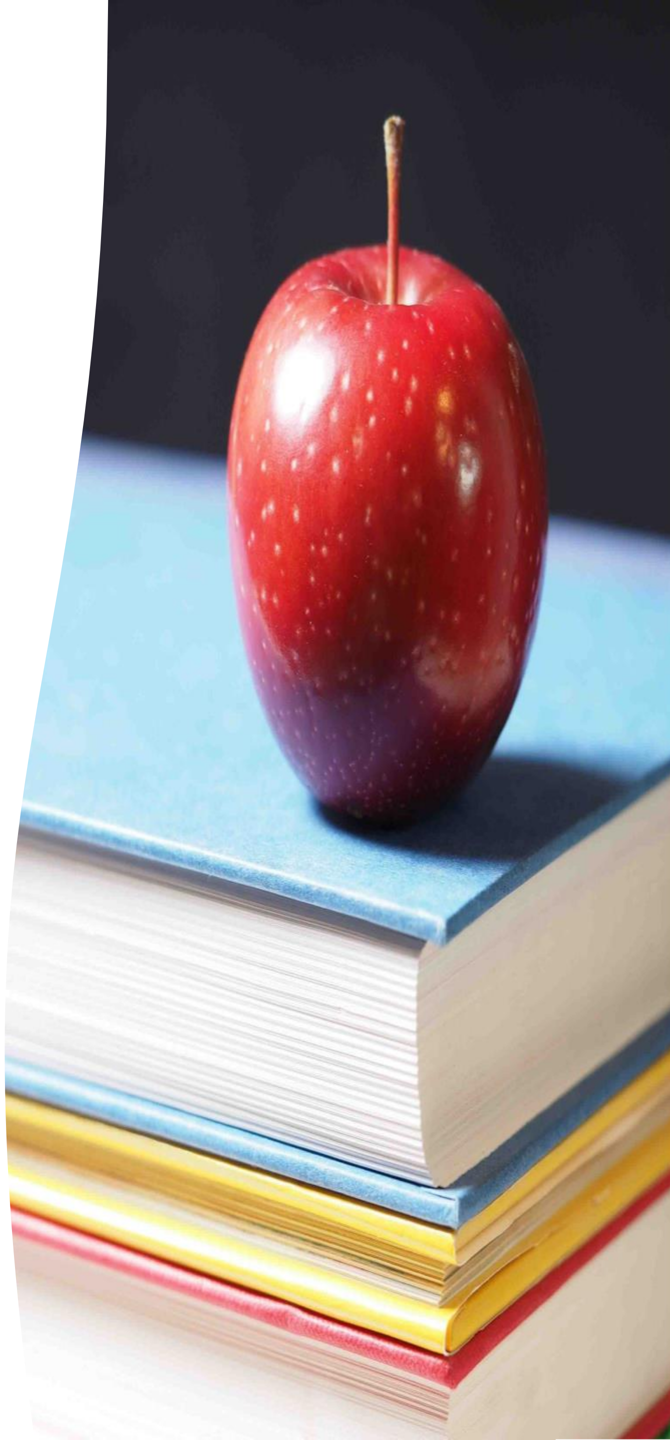
Disjunction

A disjunction consists of two or more statements connected with the word *‘OR’*.

Suppose p and q are two simple statements, then

p V q is called the disjunction of p with q.

The fruit is red OR the book cover is blue



Conditional proposition

A conditional proposition consists of two or more statements connected by the words, *'if · · ·then · · · '*

Suppose *p: I am happy* and *q: Today is Friday* are two simple statements, then; $p \rightarrow q$ is called the conditional of p with q.

Read as, *'If I am happy then today is Friday'.*

$p \rightarrow q$ can also be read as *'p only if q'* or *'p implies q'*

Converse

The **converse** of the conditional *'if p then q'* is

'if q then p' that is the converse of
$p \rightarrow q$ is $q \rightarrow p$.

For example, the converse of *'If I am happy, then today is Friday'* is

'If today is Friday, then I am happy'

Inverse

The **inverse** of the conditional *'If p then q'* is

'If not p then not q' i.e.,

The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

The inverse of the statement, *'If I am happy then today is Friday'* is

'If I am not happy, then today is not Friday'.

Contrapositive

The **contrapositive** of the conditional *'If p then q'* is *'If not q then not p'* i.e., $\neg q \rightarrow \neg p$.

The contrapositive of the conditional statement, *'If I am happy then today is Friday'* is *'If today is not Friday, then I am not happy'*.

Remarks

$p \rightarrow q \equiv \neg q \rightarrow \neg p$ i.e.,

the conditional is equivalent to contrapositive.

$q \rightarrow p \equiv \neg p \rightarrow \neg q$ i.e., converse is logically equivalent to inverse.

These equivalents are very useful in mathematical proofs.

Biconditional proposition

A biconditional is a statement of the form; $(p \to q) \land (q \to p)$ and is symbolized as; $p \leftrightarrow q$.

For example, *She dances if and only if it is a full moon*.

This statement is logically equivalent to, *if she dances then it is a full moon and if it is a full moon then she dances.*

Exclusive statement

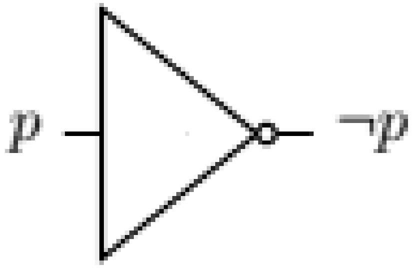
Exclusive statement denoted as p⊕q

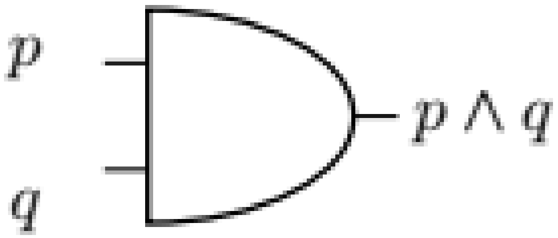
Read it as p 0-plus q

compound proposition that is true exactly when one atomic proposition is true and the other is false.

In maths OR is treated to be inclusive

Truth Tables & Gates





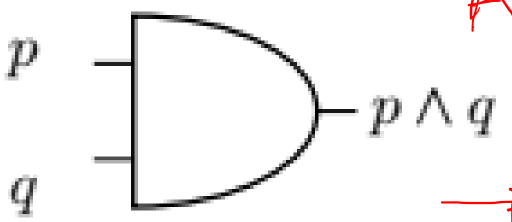
A truth table displays the relationships between the truth values of propositions.

Truth table for $\neg p$

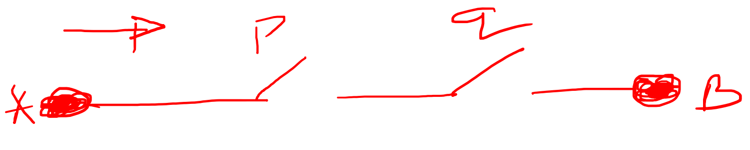
Truth Table for $p \wedge q$

Truth Tables & Gates/switches

AND Gate



Series

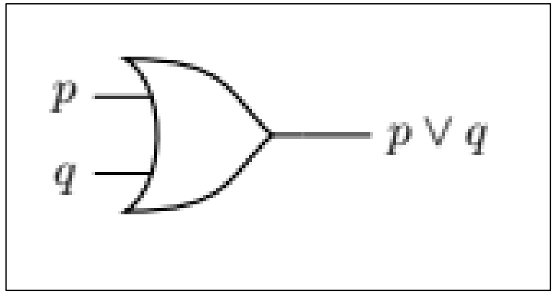


1 - closed

0 - open

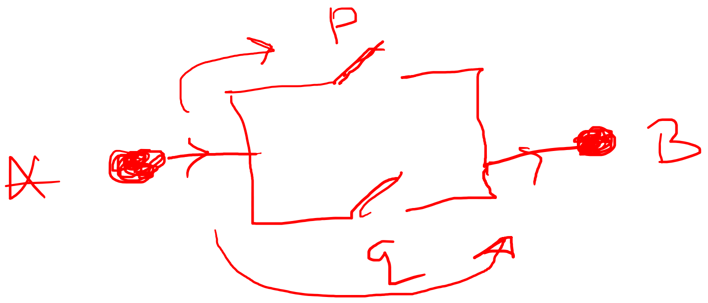
P	q	P ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction Proposition



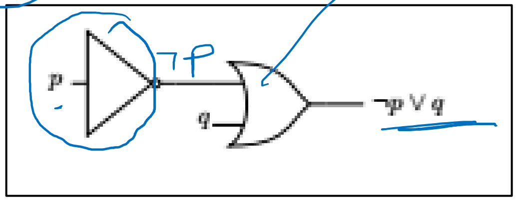
OR Gate

P	q	P ∨ q
1	1	1
1	0	1
0	1	1
0	0	0



Truth Tables & Gates

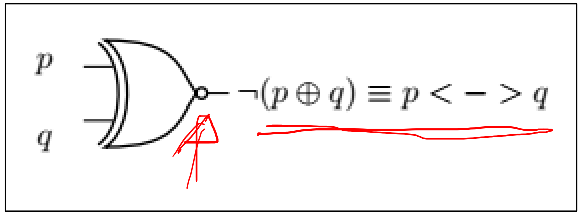
Truth Table for $p \to q$

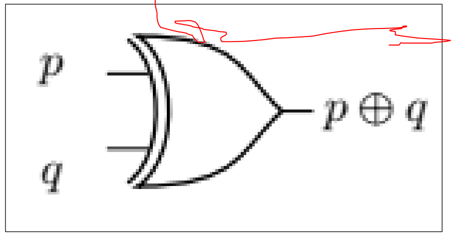


Truth Tables & Gates

Truth table for $p \leftrightarrow q$

Truth Table for $p \oplus q$





$p \leftrightarrow q$

XNOR Gate

XOR Gate

$p \oplus q$

p	q	$p \leftrightarrow q$	$p \oplus q$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	1	0

$$(p \leftrightarrow q) \equiv \overline{(p \oplus q)}$$

Today is Sunday or It is sunny

Laws of logic & logical equivalence

Double Negation $\neg\neg p \equiv p$

p: Today is Sunday

(¬p): Today is not Sunday

¬(¬p): Today is Sunday = p

implication law

$p \rightarrow q \equiv \neg p \lor q$

p	q	p↔q	p→q	q→p	(p→q) ∧ (q→p)
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

Biconditional

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

Laws of logic & logical equivalence

Identity laws

$p \land T \equiv p, \ p \lor F \equiv p$

De Morgan's laws for logic

$\neg(p \lor q) \equiv \neg p \land \neg q$

$\neg(p \land q) \equiv \neg p \lor \neg q$

Other laws

- Idempotent
- Commutative laws
- Associative laws
- Distributive laws
- Absorption

Example 1

Determine the negation of the following proposition; *If he cries, he will go home*.

Hence design a logic gate for the negation.

Solution: Suppose we let; *p: he cries*; *q: he will go home*. Then our statement can be written as; $p \rightarrow q$.

Then its negation is; $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$.

In words then we have, *He cries, and he will go home.*

Note that the negation: $\neg(p \rightarrow q)$ is read as; *It is not the case that if he cries, he will go home.*

Hence the logic gate for the negation is;

Example 2

Determine the negation of the following statement, *He sings if and only if the piano is white*. Hence design a logic gate for the negation.

Solution: Suppose we let; *p: he sings; q: the piano is white*.

Using logic connectives; $p \leftrightarrow q$.

Its negation is then; $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$.

In words this is;

He sings if and only if the piano is not white i.e., $p \leftrightarrow \neg q$.

He does not sing if and only if the piano is white i.e., $\neg p \leftrightarrow q$.

$$\neg(p \leftrightarrow q) \equiv p \oplus q$$

This corresponds to XOR gate i.e.

Example 3

Consider the following statement; *'It is not the case that both the virus and the parasites are not in the blood sample'.*

Compare the statement with the following statement; '*Either the virus or the parasites are in the blood sample'.*

Use a truth table to determine if the two statements are logically equivalent.

- **Proof:** Suppose we let *p: the virus is in the blood sample q: the parasites are in the blood sample.* Using logical connectives, we can have the statement;
- *'It is not the case that both the virus and the parasites are not in the blood sample'* as $\neg(\neg p \wedge \neg q)$.
- While the statement; '*Either the virus or the parasites are in the blood sample*' can be written as; $p \vee q$

Example 3…contd…

- ***'It is not the case that both the virus and the parasites are not in the blood sample'*** as

$$\neg(\neg p \land \neg q)$$

- While the statement; ***'Either the virus or the parasites are in the blood sample'*** can be written as;

$$p \lor q$$

The two statements are logically equivalent going by the 6th and 7th columns.

p	q	¬p	¬q	¬p ∧ ¬q	¬(¬p ∧ ¬q)	p ∨ q
1	1	0	0	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	0	0

Tautology

Tautology is a compound proposition that is always true, regardless of the truth values of the basic propositions which comprise it.

In other words, a tautology is a proposition that is true on logical ground only.

Consider the truth table for $[(p \rightarrow q) \land \neg q] \rightarrow \neg p$

p	q	$p \rightarrow q$	$\neg q$	$(p \rightarrow q) \land \neg q$	$\neg p$	$[(p \rightarrow q) \land \neg q] \rightarrow \neg p$
1	1					
1	0					
0	1					
0	0					

Contradiction

p	Q	¬p	p ∧ q	(p ∧ q) ∧ ¬p
1	1			
1	0			
0	1			
0	0			

A **contradiction** is a proposition that is always false regardless of the truth value of the basic propositions that comprise it.

For example: Show if the proposition, '*It is raining and sunny and it is not raining*' is a contradiction.

Solution: Suppose we let; *p: It is raining q: It is sunny.*

Then our statement using logical

connectives become; (p ∧ q) ∧ ¬p.

References

Kenneth, R. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.

Koman, B., Busby, R., & Ross, S. (2001). *Discrete Mathematical Structures*. Prentice-Hall.

Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.

Susanna, S. E. (2003). *Discrete Mathematics with Application* (3rd ed.). Brooks Cole.