Discrete Mathematics
Lecture 2
Logic and Validity of Arguments

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Introduction to lecture 2

This lecture is a continuation of lecture 1

Introduces the application of propositional logic in testing the validity of arguments.

Intended learning outcomes

At the end of this lecture, you will be able to;

- Define terms used in logic.
- Determine the validity of arguments.

Definition of terms

- Logic is the science of correct reasoning.
- It is fundamental to critical thinking and problem solving.
- Reasoning is defined as drawing of inferences or conclusions from known or assumed facts.

Types of Reasoning

- ❑Deductive reasoning is the type of logic that involves application of a general statement to a specific instance e.g., the use of a formula to solve a particular problem.
- ❑The quadratic formula is a general formula that is used to solve specific problems.

Types of Reasoning…contd…

- ❑Inductive reasoning is drawing conclusions from a specific example

e.g. the nature of a population drawn from a sample; when launching a new apps you need to pilot it with a few users.

Argument

- An argument is a discussion in which there is a disagreement.

- It is made up of premises and conclusions.

- A proof is an argument

Nature of an argument

- A valid argument is where the conclusion of argument is guaranteed, otherwise it is invalid.
- Argument is said to be valid if the conclusion follows logically from the assumptions or premises.
- An argument that is valid does not imply that the conclusion is true.

Example 1

Major premise *p: All lecturers are men.*
Minor premise *q: my mother is a lecturer.*
∴ *My mother is a man*

Note that the conclusion is valid though not true.

Example 2

- An argument being invalid does not mean the conclusion is false.

Major premise p: *All professional soldiers are snipers*
Minor premise *q: Kimani is a sniper*
∴ *Kimani is a professional soldier.*

- The argument is invalid, although the conclusion may not be false.

Testing the validity of an argument

Methods of testing the validity of an argument;

- ❑Venn diagram
- ❑Truth table
- ❑Rules of inferences

Venn Diagrams and Arguments

- ❑Venn diagrams can be used to determine if an argument is valid or invalid.
- ❑Three intersecting circles are needed to diagram a categorical syllogism.
- ❑A categorical syllogism is a deductive argument that consists of three categorical statements namely the two premises or assumptions, and the conclusion.
- ❑This works well when dealing with predicate logic.

Rules to follow

- ❑Draw the set representing the universal premise first (this will be explained in the next lecture on predicate logic).
- ❑Letter x is indicated on the line intersecting a section of the whole area is so designated in the premise.
- ❑You should <u>only</u> draw circles for premises/assumptions.
- ❑The argument or syllogism is only valid if the conclusion is self-evident in the Venn diagram drawn.

Example 3

Consider the following argument and use a Venn diagram to determine if it is valid

p: All drivers are men
<u>q: My mother is a driver</u>
∴My mother is a man

Solution:

Note that the set of drivers is contained in the set of men.

Since *my mother X* is contained in the set of *drivers,* it implies then that *my mother* is also in the set of *men*.

Hence the argument is valid.

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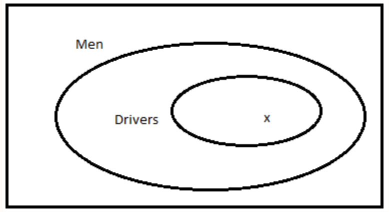


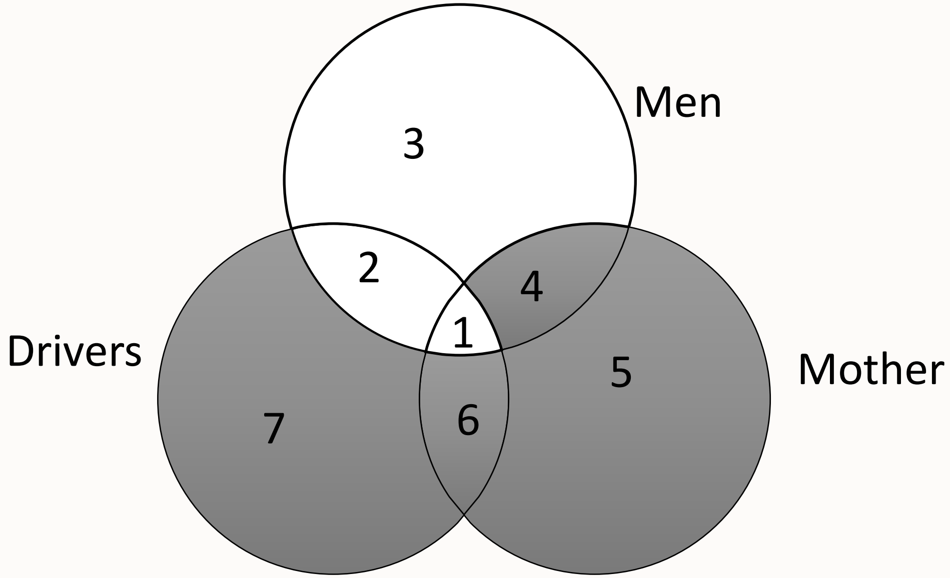
Alternatively

We first plot the universe: *All drivers are men* i.e., region 1 and 2.

Then for the premise *my mother is a driver* is region 1.

The conclusion *my mother is a man*

is self-evident from region 1.



Example 4

Use a Venn diagram to determine if it is valid.

p: All professional models are singers
q: Otieno is a singer
∴Otieno is a professional model.

Solution

If Y represents *Otieno as a singer* (Assumption *q*), X represents *Otieno is a professional model,* then the different positions means that the argument is invalid (though the conclusion is not necessarily false).

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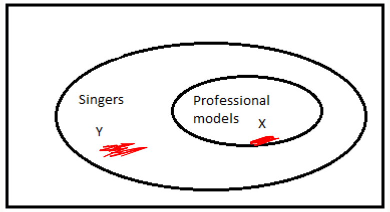
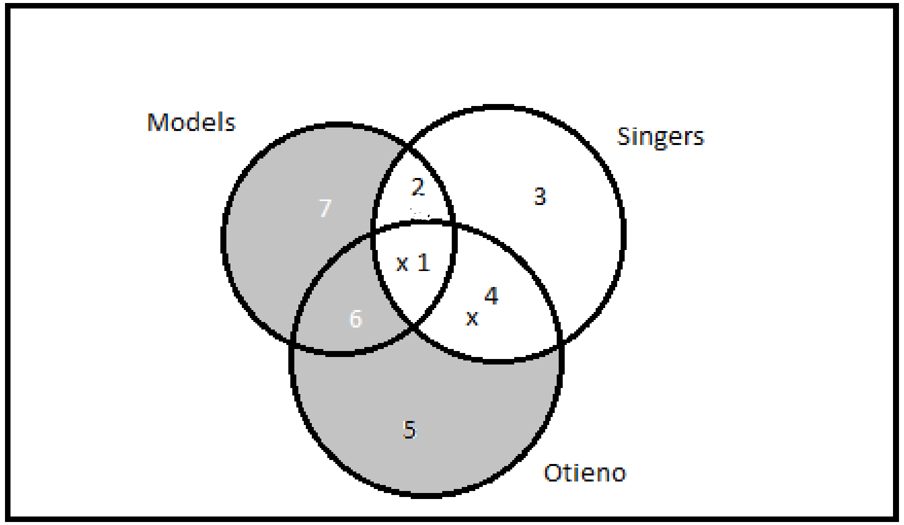


Alternatively

We first plot the universe i.e., All professional models are singer i.e. region 1 and 2.

Then *Otieno is a singer* is region 1 and 4.

The conclusion *Otieno is a professional model is not self-evident.*



Example 5

Consider the argument;

p: No gecko is warm-blooded
q: All mammals are warm-blooded
∴Geckos are not mammals

Let x be gecko, then as shown below the position is unique hence the argument is valid.

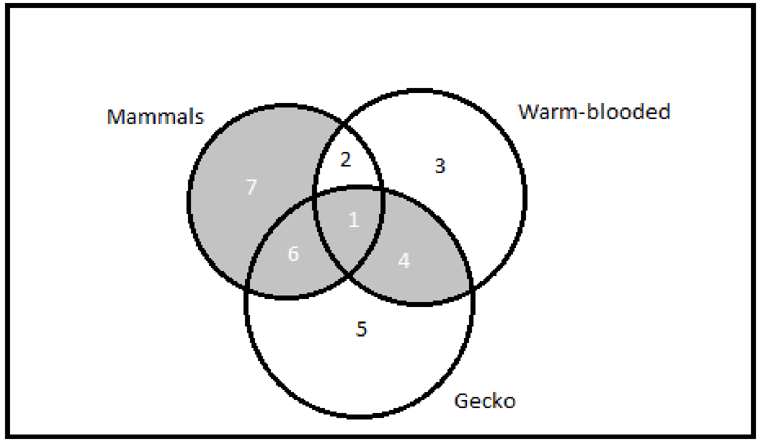
Alternatively

- We first plot the universe q: $All\ mammals\ are\ warm - blooded$ – this is region 1 and 2 where all mammals are warm-blooded.

- For p: $No\ gecko\ is\ warm - blooded$ –

we shade region 1 and 4 since no gecko is warm-blooded

- It is self-evident that geckos are not mammals (region 5)



Truth Tables and Validity of Arguments

Option 1: Finding the critical rows

- ❑ Construct the truth table for the argument.
- ❑ Determine the critical rows i.e., rows in which all the assumptions are true.
- ❑ Check the conclusion of all critical rows.

Then;

a. If in each critical row the conclusion is TRUE, then the argument is valid.
b. If there is a row in which the conclusion is FALSE, then the argument is invalid.

Example 6

Show that the argument; $p \to q, q \to p \therefore p \vee q$ is invalid.

From the table above, the last row is critical row i.e. it has all the assumptions true, but the conclusion is false. Hence, the argument is invalid.

p	q	$p \to q$	$q \to p$	$p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

Example 7

Option 2: Determining if an argument is a tautology. If the argument is a tautology, then it is valid otherwise it is invalid.

Show that the argument; $p \rightarrow q, q \rightarrow p \therefore p \vee q$ is invalid.

We use a truth table to check to see if $((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow (p \vee q)$ is a tautology.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$	$((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	F

From the last column, it is clear that $((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow (p \vee q)$ is NOT a tautology.

Hence the argument is invalid.

Example 8

Determine if the validity of the argument is valid: $p \rightarrow q; r \rightarrow q; \neg p \wedge r \therefore q$

Solution (by Option 1): Construct the truth table and determine the critical rows.

p	q	r	$\neg p$	$p \rightarrow q$	$r \rightarrow q$	$\neg p \wedge r$	q
T	T	T	F	T	T	F	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	F	F
T	F	F	F	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	F	T	F
F	F	F	T	T	T	F	F

Row 5 is the only critical row, since all the assumptions are TRUE.

Note that the conclusion is TRUE and hence the argument is valid.

Example 9

Option 2: We check if the argument is a tautology i.e. $[(p \to q) \land (r \to q) \land (\neg p \land r)] \to q$ is a tautology.

p	q	r	$\neg p$	$p \to q$	$r \to q$	$\neg p \land r$	$(p \to q) \land (r \to q) \land (\neg p \land r)$	$[(p \to q) \land (r \to q) \land (\neg p \land r)] \to q$
T	T	T	F	T	T	F	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	F	F	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	T	T	F	T	F	T
F	F	F	T	T	T	F	F	T

The argument is a tautology (the last row is all TRUE) and hence it valid.

Example 10

Rewrite the following argument in symbolic form, and hence determine its validity.

$$\begin{array}{l} \textit{If the carpenter is tall, the carpenter will not buy the flowers.} \\ \underline{\textit{The carpenter bought the flowers}} \\ \therefore \textit{The carpenter is tall} \end{array}$$

Solution

Let p: *the carpenter is tall*; q: *the carpenter bought the flowers*. The argument in logic connectives becomes;

$$\begin{array}{l} p \to \neg q \\ \underline{q \qquad\quad} \\ \quad \therefore p \end{array}$$

Alternatively, you can write the argument as follows; $p \to \neg q, q \therefore p$

Row three is the critical row, however the conclusion is false. Hence the argument is invalid.

Rules of inference

- Sometimes using truth tables to test for the validity of argument can be tedious.

- For example, to test the validity or argument with 5 different propositional variables it will require $2^5 = 32$ rows.

- Hence, we use the logic laws and rules of inference to establish the validity of such arguments.

Rules of inference…contd…

- A rule of inference is a compound proposition that is a tautology and involves an implication.

- Note that when using the critical rows, the assumptions must be all true and so must be the conclusion for the argument to be valid.

- In proving validity without using tables , it sufficient to assume the assumptions are true and then proceed to show if the conclusion is true.

Standard rules of inferences

Rule of inference	Tautology	Name
P, p → q ∴ q	[p ∧ (p → q)] → q	Modus ponens
¬q, p → q ∴ ¬p	[¬q ∧ (p → q)] → ¬p	Modus tollens
p ∴ p ∨ q	p → (p ∨ q)	Addition
p ∧ q ∴ p	(p ∧ q) → p	Simplification
p → q, q → r ∴ p → r	[(p → q) ∧ (q → r)] → (p → r)	Hypothetical syllogism
p ∨ q, ¬p ∴ q	((p ∨ q) ∧ ¬p) → q	Disjunctive Syllogism
p, q ∴p ∧ q	p ∧ q → p ∧ q	Conjunction

Example 11

Without use of tables, determine the validity of the argument; $p \rightarrow \neg q, q \therefore p$

Solution: We assume the premises are TRUE and proceed to determine the truth value of the conclusion.

1) *q is a premise, assume it is T.*
2) *$\neg q$ is F from (1) above.*
3) *$p \rightarrow \neg q$ is a premise we assume it is T.*
4) *P is false from (3) above (since (3) can only be true when both p and $\neg q$ are false.*

Since the premises are all true but conclusion is false then the argument is invalid.

Example 12

Consider the following argument:

It is not noisy, and it is hot.
I will go to class only if it is noisy.
If I don't go to class, then I will take a nap.
If I take a nap, then I will be at the party by 7pm.
Therefore, I will be at the party by 7pm.

Determine if the argument is valid.

Solution

Let *p: it is noisy; q: It is hot; r: I will go to class; s: I will take a nap; and t: I will be at the party by 7pm.* Then;

(1) $\neg p \wedge q$ *Premise*

(2) ¬p Simplification rule using (1)

(3) $r \rightarrow p$ *Premise*

(4) $\neg r$ *Modus Tollens using (2) (3)*

(5) $\neg r \rightarrow s$ *Premise*

(6) s Modus Ponens using (4) (5)

(7) $s \rightarrow t$ *Premise*

(8) t Modus Ponens using (6) (7).

The argument is valid.

References

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[Venn diagram and validity of arguments]