Discrete Mathematics
Lecture 10
Introduction to Graph Theory

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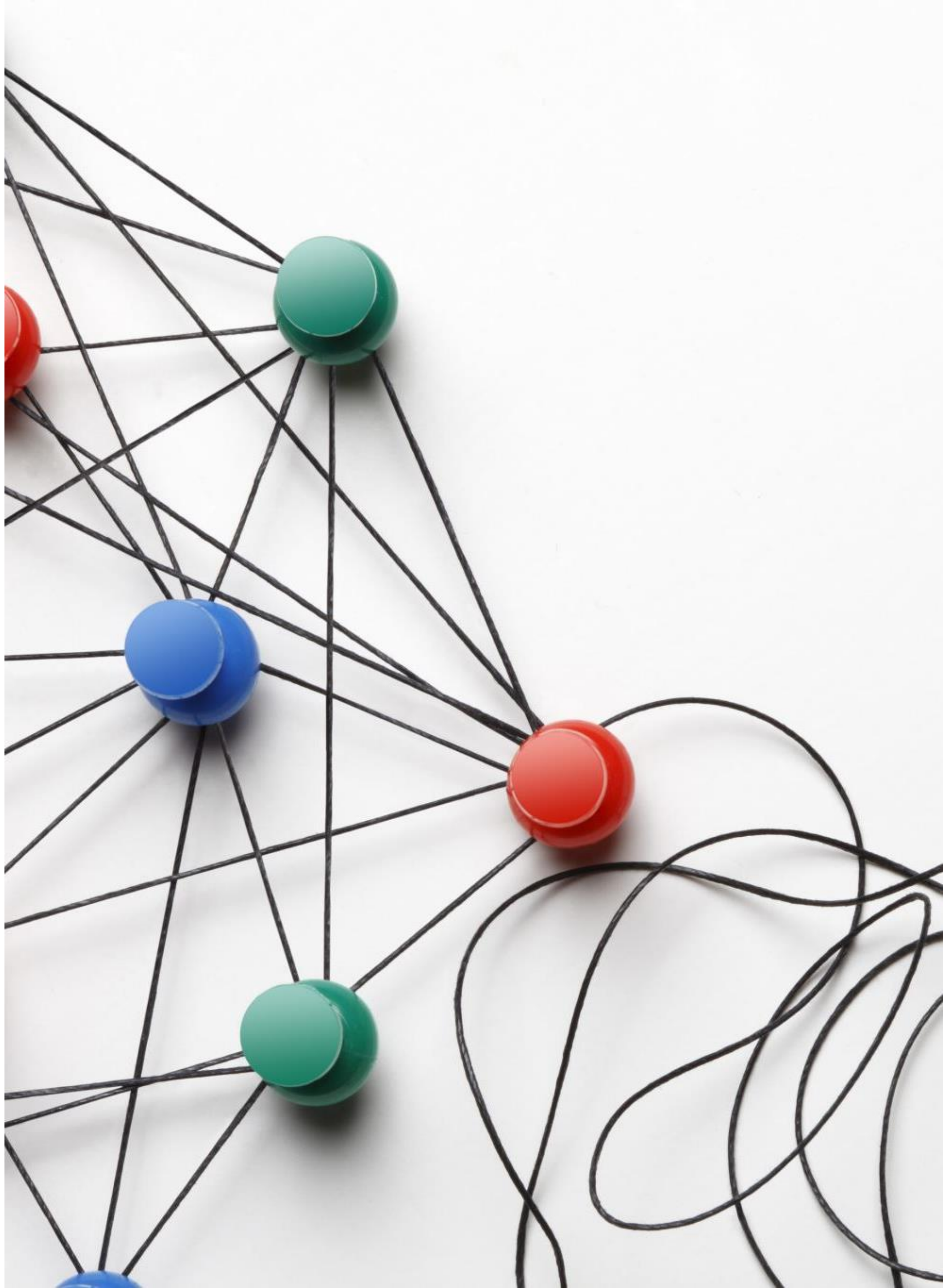
Introduction to lecture 10

- This lecture is a continuation of the lecture on Relations and functions.
- Graphs are associated with the diagraphs of a symmetric relations.
- Graph theory is applicable to data structures, understanding networks, and relationships.



References

These lecture notes have been derived from the following sources (Lipschutz & Lipson, 2007; Rosen, 2012).

Intended Learning Outcomes

Be	At the end of this lecture, you will be able to;
Define	Define terms used in graph theory.
Apply	Apply the concepts in solving problems involving graph theory.

Graph

Definition: A graph G consists of a finite set V of objects called the vertices, a finite set E of objects called edges, and a function f that assigns to each edge a subset $\{v, w\}$ where v and w are vertices.

The graph is denoted $G = (V, E, f)$.

Definition: If e is an edge and $f(e) = \{v_i, v_j\}$, we say that e is an edge between v_i and v_j and that e is determined by v_i and v_j. The vertices v_i and v_j are called the endpoints of e.

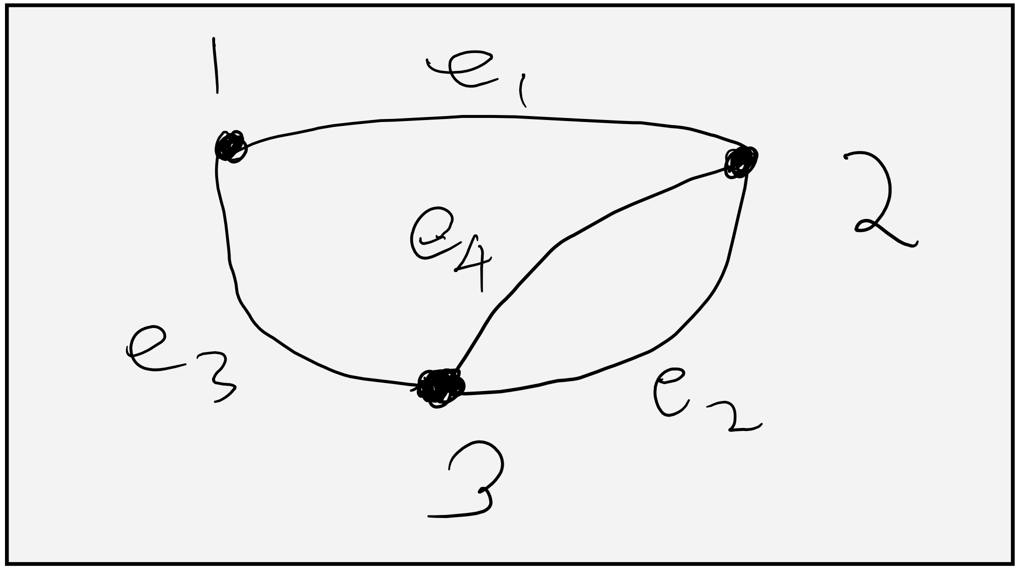
Example 1

Let the vertices $V = \{1,2,3\}$ and edges

$E = \{e_1, e_2, e_3, e_4\}$. Let f be defined by

$f(e_1) = (1,2), f(e_2) = f(e_4) = \{2,3\}, f(e_3) = \{1,3\}$.

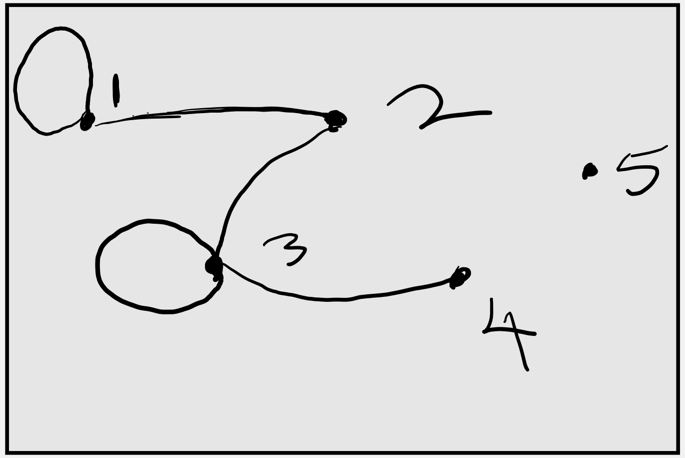
Then G is a graph represented by the picture.



Definitions

- ❑ **(Degree of a vertex)** The degree of a vertex is the number of edges having that vertex as an endpoint.
- ❑ The degree of a vertex is denoted $\deg(v)$.
- ❑ **(loop)** A graph may contain an edge from a vertex to itself referred to a s a loop.
- ❑ A loop contributes two to the degree of the vertex.
- ❑ **(Isolated vertex)** A vertex with degree 0 is referred to as an isolated vertex.

Theorem 1: Handshaking theorem



Let graph G be undirected graph with n edges. Then

$$2n = \sum_{v \in V} \deg(v)$$

For instance, consider the graph below

Theorem 2

Let the graph G (V, E) be a directed graph (with directed edges) then

$$\sum_{v\in V} \deg^{-}(v) = \sum_{v\in V} \deg^{+}(v) = |E|$$

Where E is the edges, and $\deg^{-}(v)$ and $\deg^{+}(v)$ are the in-degree and out-degree of a vertex v respectively.

For example

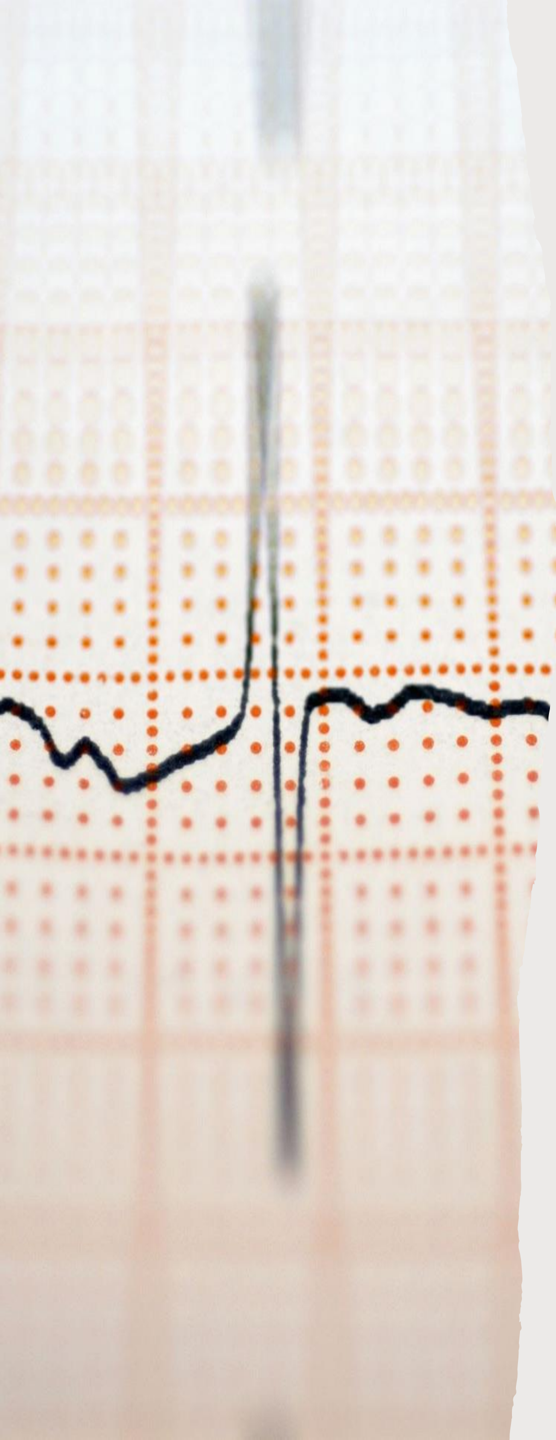
More Definitions

(A path) A path in a graph is a sequence $\{x\} = v_1, v_2, \cdots v_k$ of vertices, each adjacent to the next, having an edge between each v_i and v_{i+1} such that no edge is chosen more than once.

(Circuit) A circuit is a path that begins and ends with the same vertex.

Euler path) A path in a graph G is called a Euler path if it includes every edge exactly once. Similarly, a Euler circuit is a Euler path that is a circuit.

(Hamiltonian path) A path that contains each vertex exactly once. Similarly, a circuit that contains each vertex exactly once except for the first vertex, which is also the last.



Families of graphs

Discrete graphs: Let D_n denote the graph with n vertices and no edges, for each integer $n \geq 1$, then D_n is a discrete graph with n vertices e.g. D_2 · · D_5 · · · · ·

Complete graph: Let K_n denote the graph with vertices $\{v_1, \cdots, v_n\}$ and with edge $\{v_i, v_j\}$ for every i and $j, \forall n \geq 1$ i.e., every vertex in K_n is connected to every other vertex.

Then K_n is called a complete graph. T

he degree of each vertex of K_n is $(n-1)$.

Families of graphs…contd…

Linear graph: Let L_n denote the graph with n vertices $\{v_1, v_2, \cdots v_n\}$ and with edges $\{v_i, v_{i+1}\}, \forall n \geq 1$ and $1 \leq i \leq n$,then L_n is called a linear graph of n vertices.

Regular graph: A graph G is regular of degree k or k-regular if every vertex has degree k i.e. a graph is regular if every vertex has the same degree.

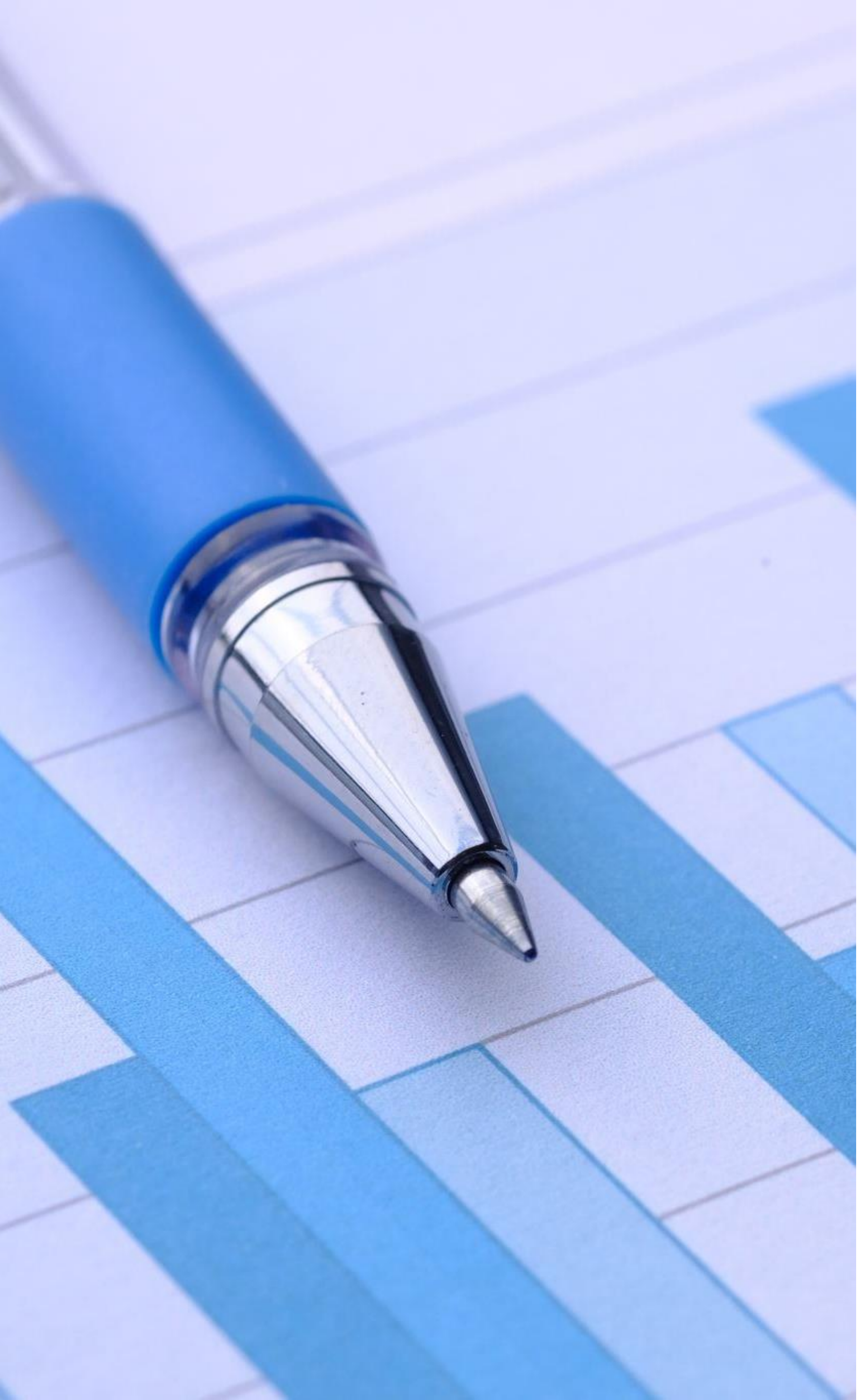
Remark: A complete graph K_n with n vertices, is a regular graph of degree $(n - 1)$.

Bipartite graphs

A graph G is said to be bipartite if its vertices can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N denoted K_{mn}, where $m \leq n$.

Bipartite graphs are used to model problems that require application of matching.

For example, in assigning different tasks to a set of workers or users.



Planar graphs

A graph that can be drawn on a plane that its edges do not cross is said to be planar e.g., tree graphs, K_4,

Maps

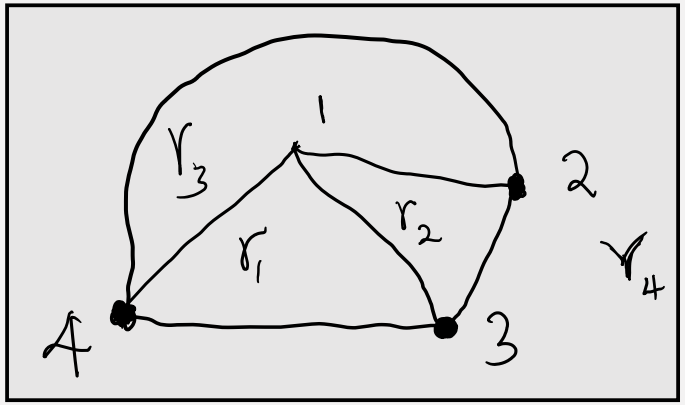
A map is a finite planar multigraph.

A map divides the plane into various regions.

A degree of a region denoted $\deg(r)$ is the length of the cycle which borders r.

Theorem 1: The sum of the degrees of the regions of a map equal to twice the number of edges.

Example 1



Consider the K_4 below. The map as 4 regions r_1, r_2, r_3, r_4

$$deg(r_1) = 3; deg(r_2) = 3; deg(r_3) = 3; deg(r_4) = 3$$

Sum of degrees is 12.

The graph has 6 edges therefore $(\text{sum of degrees}) = 2(\text{No. of edges})$. Straightening the edges will give rise to a tetrahedron.

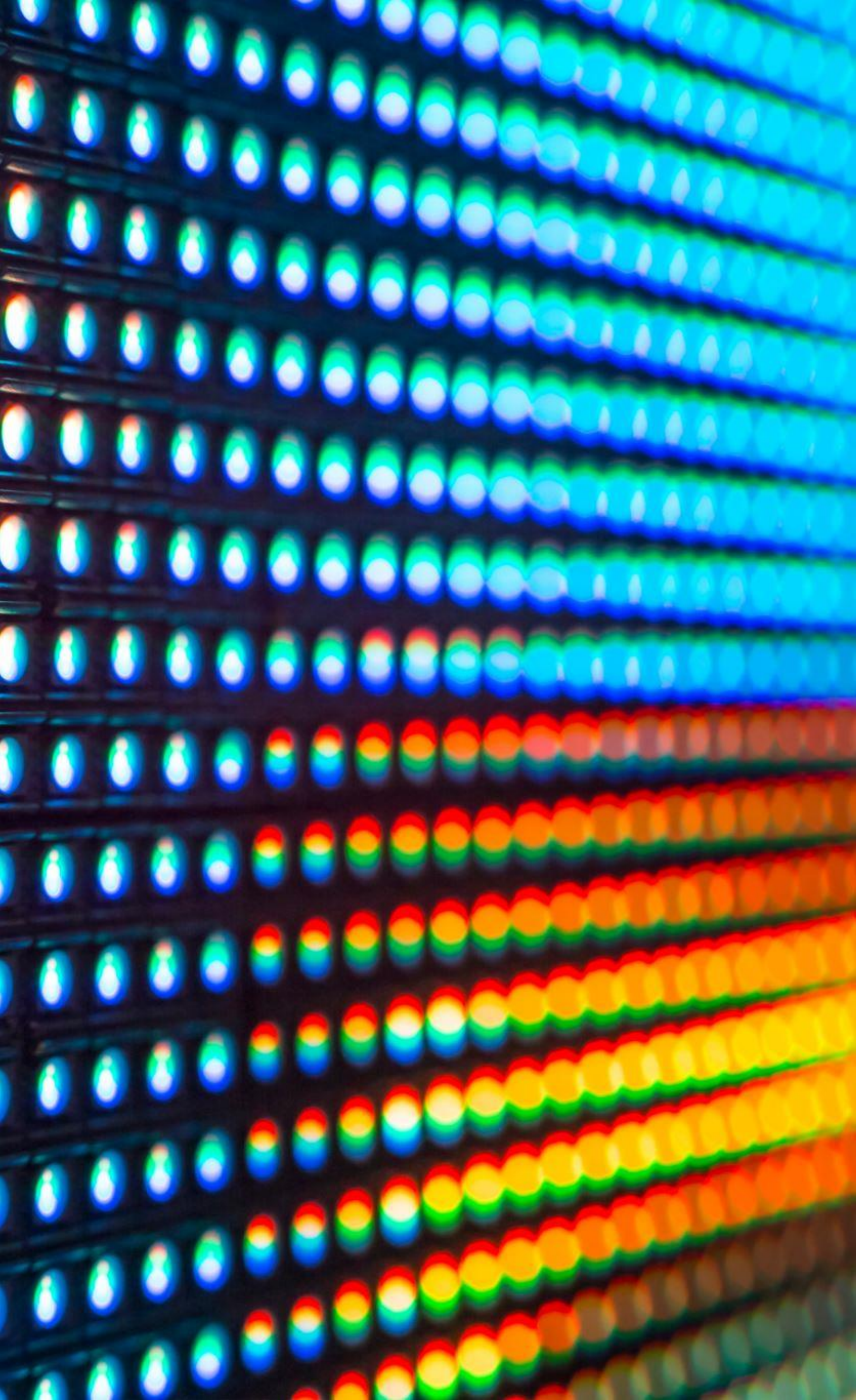
Euler's Formula

- $R + V - E = 2$ where V is the number of vertices, E is the number of edges and R is the number of any connected regions or faces.
- For objects , we use faces, however, in some instances it does not sum to 2.
- It may sum to 1 or zero e.g., the mobius strip and torus.
- Hence the more general formula is the Euler's characteristic i.e., $F + V - E = \chi$

Representing Graphs in Computer Memory

We have two ways of maintaining a graph G in the memory of a computer.

- Sequential representation of G i.e., using the graph adjacency matrix and incidence matrices.
- Linked representation i.e., using the linked list of neighbors.



Adjacency matrix

Suppose G is a graph with n vertices and suppose the vertices have been ordered say $v_1, v_2, \cdots v_n$.

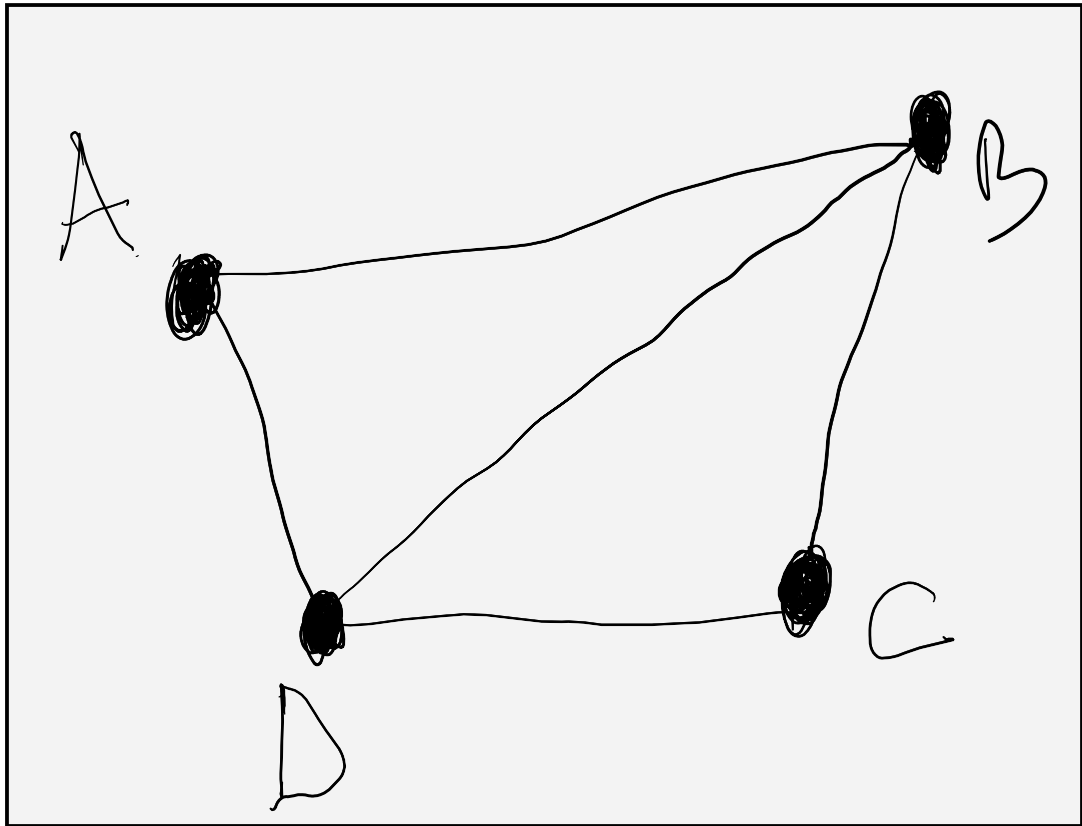
Then the adjacency matrix $A = [a_{ij}]$ of the graph G is the matrix defined by;

$$a_{ij} = \begin{cases} 1 & \text{if } \mathrm{v_i} \text{ is adjacent to } \mathrm{v_j} \\ 0 & \text{otherwise} \end{cases}$$

Example 1

Consider the graph below;

Adjacency matrix



Remarks

Graphs can be represented by matrices that are based on incidence of vertices and edges.

The above adjacency matrix is for a simple graph (that has no loops).

Such matrices are symmetric and have only zero and ones as entries.

Example 2

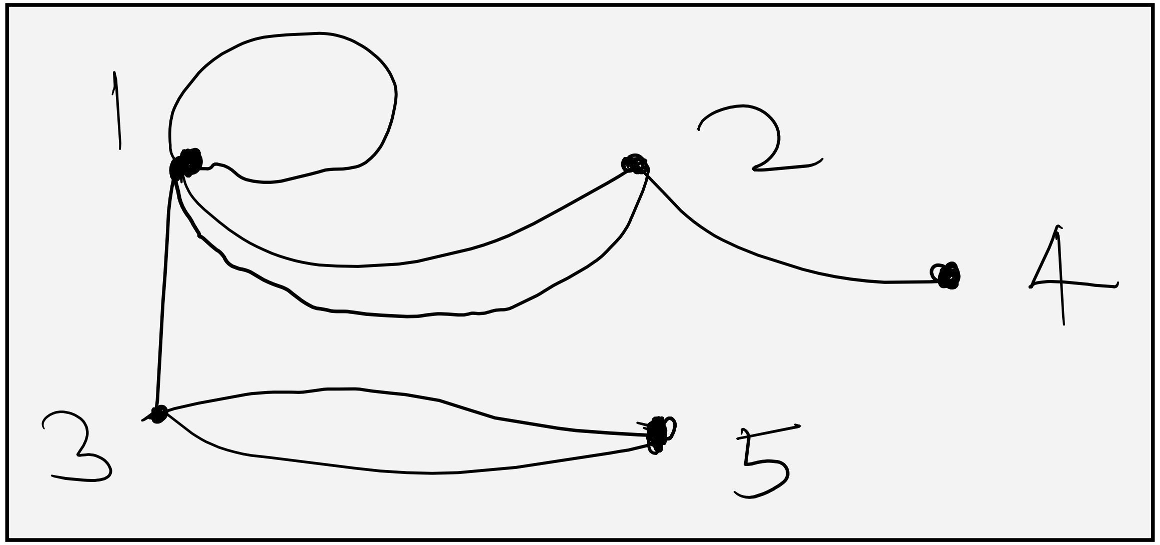
Consider the adjacency matrix below of a graph with four vertices.

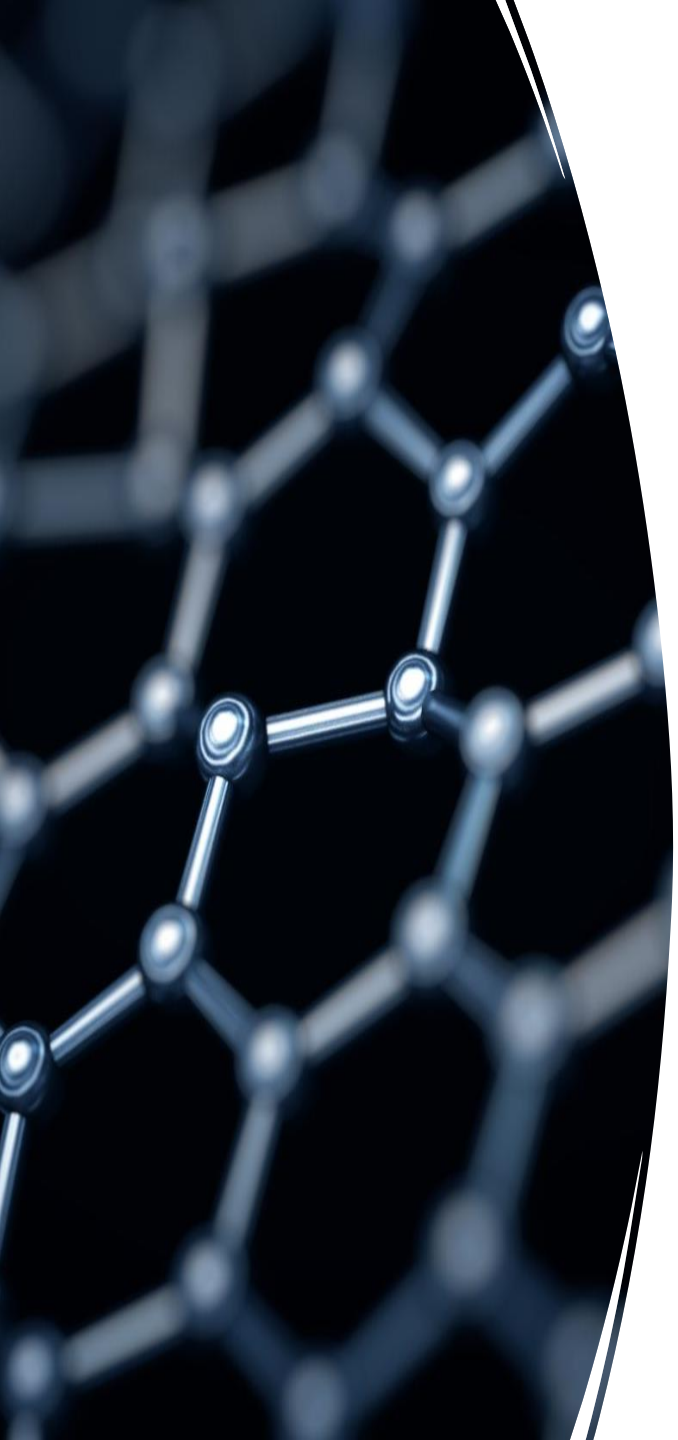
Plot the graph (Assumes the vertices are labelled 1,2,3,4).

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Example 3

Consider the following pseudograph with multiple edges and loops. Determine its adjacency matrix.





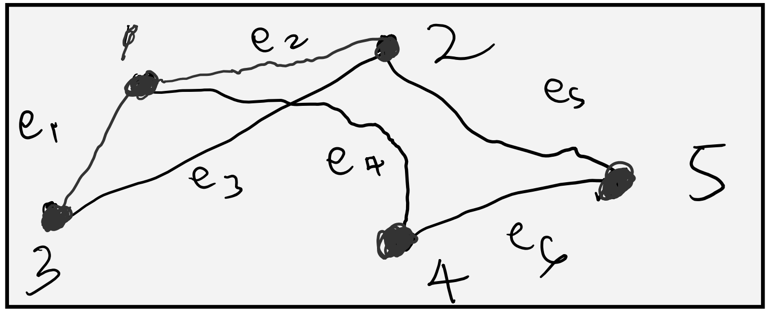
Incidence matrices

Let G be undirected graph with vertices $v_1, v_2, \cdots, v_n$ and edges $e_1, e_2, \cdots, e_n$, then the incidence matrix for G with respect to the vertices and edges is the matrix $M = [a_{ij}]$ where;

$$a_{ij} = \begin{cases} 1 & \text{where edge } e_j \text{ is incident with } v_j \\ 0 & \text{otherwise} \end{cases}$$

Example 1

Consider the undirected graph below



$$
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}
\begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

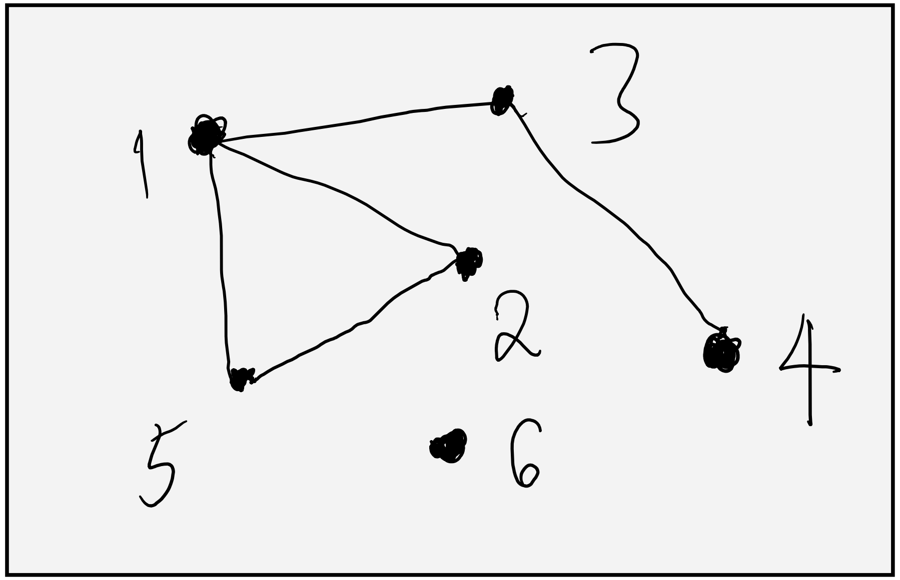
Linked representation/ Adjacency structure

Adjacency matrix system consumes more memory.

In this method, each vertex of G is followed by its adjacency list i.e., its list of adjacent vertices or neighbors.

If a vertex has no neighbor, we denote it with Ø in the list.

Example 1



Consider the graph below:

Vertex **Adjacency list**

Example 2

Plot the graph represented by the following adjacency list

Vertex	Adjacency list
1	3,5
2	3,4
3	1,2
4	2
5	1

References

Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.

Rosen, K. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.

Wilson, R. J. (1998). *Graph Theory* (4th ed.). Addison Wesley Longman Ltd.