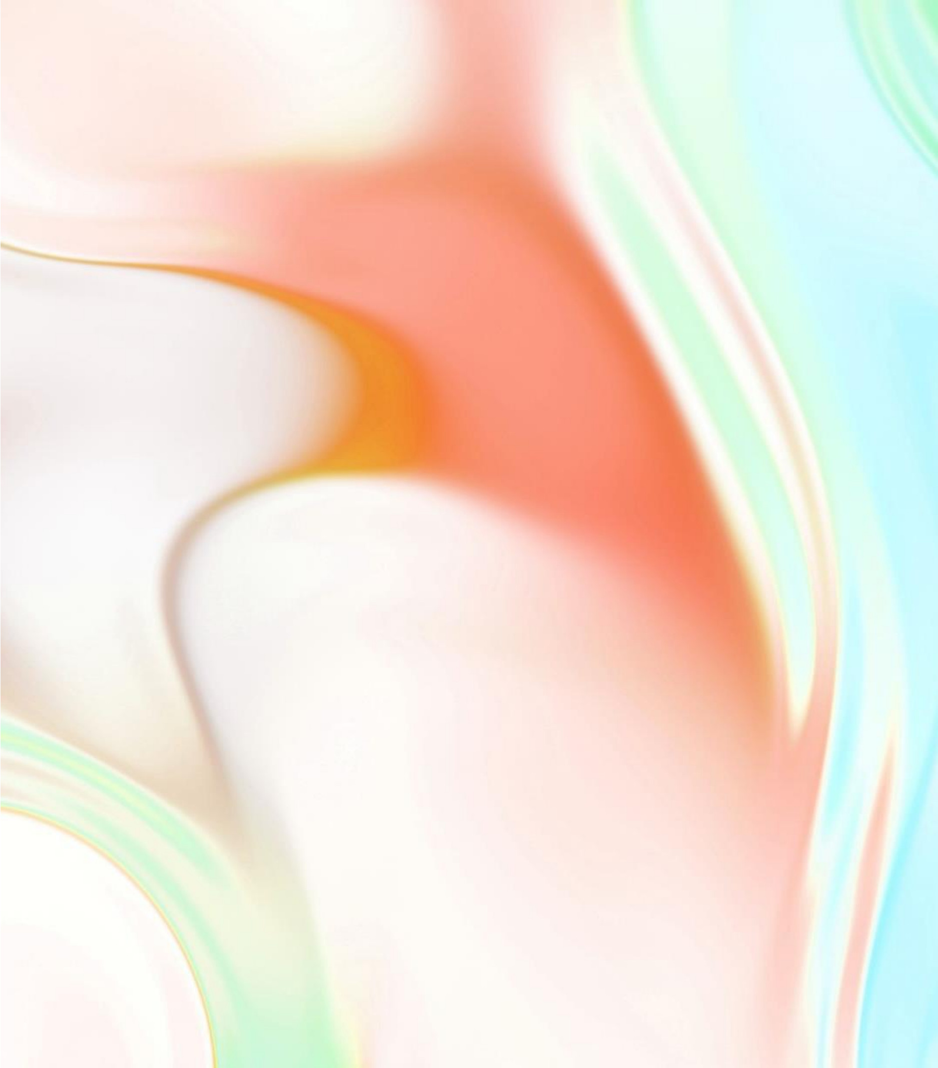
Discrete Mathematics
Lecture 11
Trees



Lecturer: Kahenya, N.P

Introduction to lecture 11

- This lecture builds up on the lecture on introduction to graph theory.
- Trees are connected graphs and have many applications in computer science, communication, sociology, and many other fields.
- The idea of trees in a computer can be seen in the file system that consists of directories, subdirectories, and files.
- The administrative structure of an institution, family lineage etc. adopts the structure of a tree.
- Trees are used to model the process of solving mathematical expressions.

References for further reading

These lecture notes have been derived from the following sources (Lipschutz & Lipson, 2007; Rosen, 2012; Wilson, 1998).

Intended Learning Outcomes

Be	At the end of this lecture, you will be able to;
Define	Define terms used in trees.
Apply	Apply the concepts in solving problems involving trees.

Definition of Terms

A graph G (V,E) is connected if there is a path from any vertex to any other vertex in the graph, otherwise it is disconnected.

A cycle is a path in a graph that has the following properties:

1) It includes at least one edge.

2) There are no repeated edges.

3) The first and last vertices coincide i.e., are the same but there are no other repeated vertices.

Tree

A tree is a connected graph with no cycles.

Properties of a tree

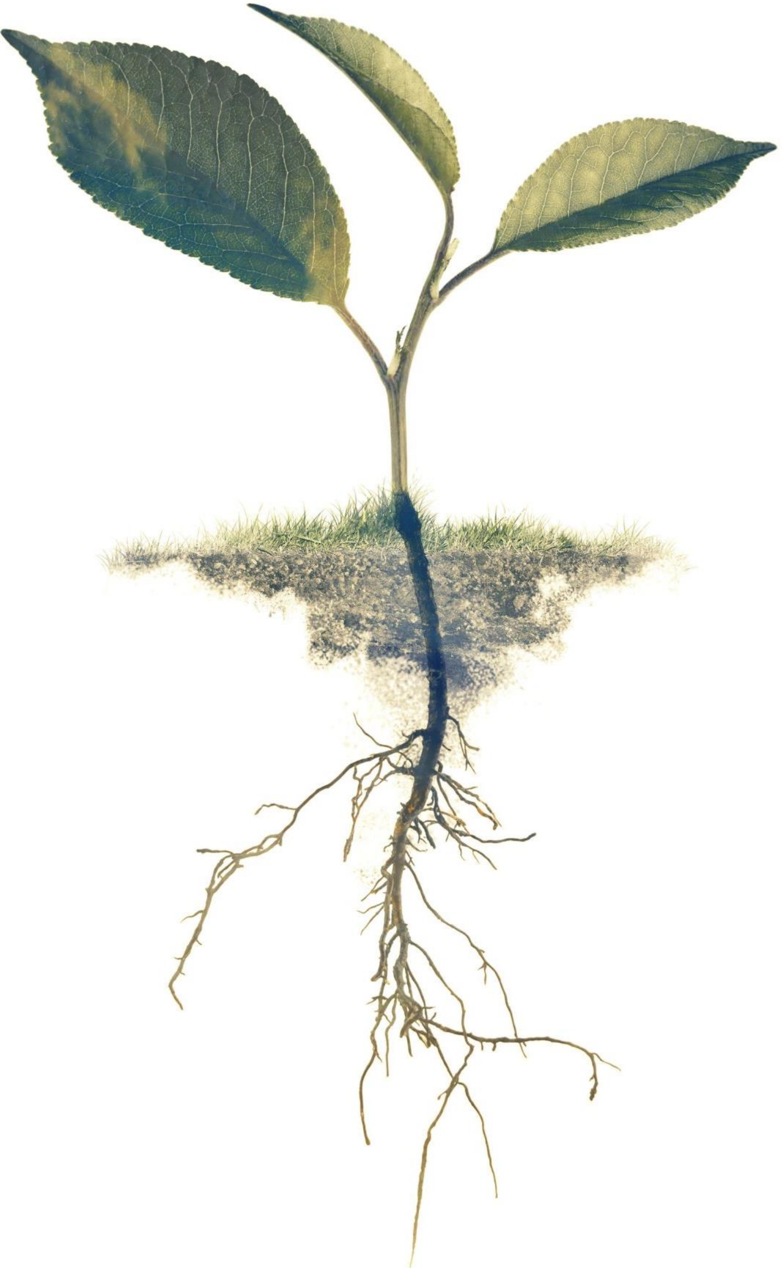
(i) For each pair of vertices u and v in a tree, there is exactly one path from u to v that does not repeat any vertices or edges.

(ii) Inserting an edge between any two vertices of a tree produces a graph containing a cycle.

(iii) Removing any edge from a tree produces a disconnected graph

A Rooted tree

- A rooted tree (T, V_0) is a tree in which one of the vertices is specified as the root V_0.
- Rooted trees are usually drawn with the root at the top.



Theorem 1: Let $(\mathrm{T}, \mathrm{V}_0)$ be a rooted tree then;

i) There are no cycles in T.

ii) V_0 is the only root of T.

iii) Each vertex in T, other than V_0 has in-degree one, and V_0 has in-degree 0.

Example 1

Let $\mathrm{A} = \{\mathrm{v}_1, \mathrm{v}_2, \cdots \mathrm{v}_{11}\}$ and let the tree $\mathrm{T} =$
$\{(\mathrm{v}_1, \mathrm{v}_2), (\mathrm{v}_1, \mathrm{v}_4), (\mathrm{v}_3, \mathrm{v}_6), (\mathrm{v}_4, \mathrm{v}_{10}), (\mathrm{v}_5, \mathrm{v}_8), (\mathrm{v}_6, \mathrm{v}_7), (\mathrm{v}_3, \mathrm{v}_5), (\mathrm{v}_4, \mathrm{v}_{11}), (\mathrm{v}_5, \mathrm{v}_9), (\mathrm{v}_3, \mathrm{v}_1)\}$

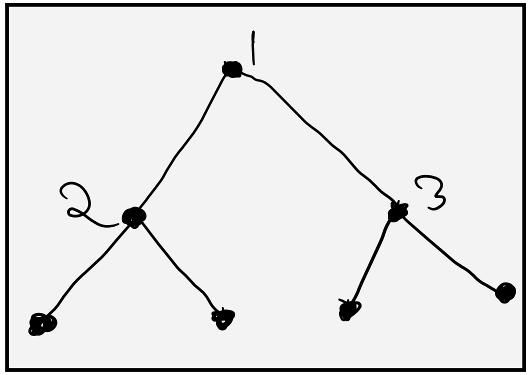
Show that T is a rooted tree and determine the root.

Complete tree

A tree T is said to be complete if all its levels except possibly the last, have the maximum number of possible nodes, and if all the nodes at the last level appear as far left as possible.

A node that has no child is called a leaf node.

Any two nodes v_i and v_j which are connected by an edge are referred to as adjacent nodes. For example;



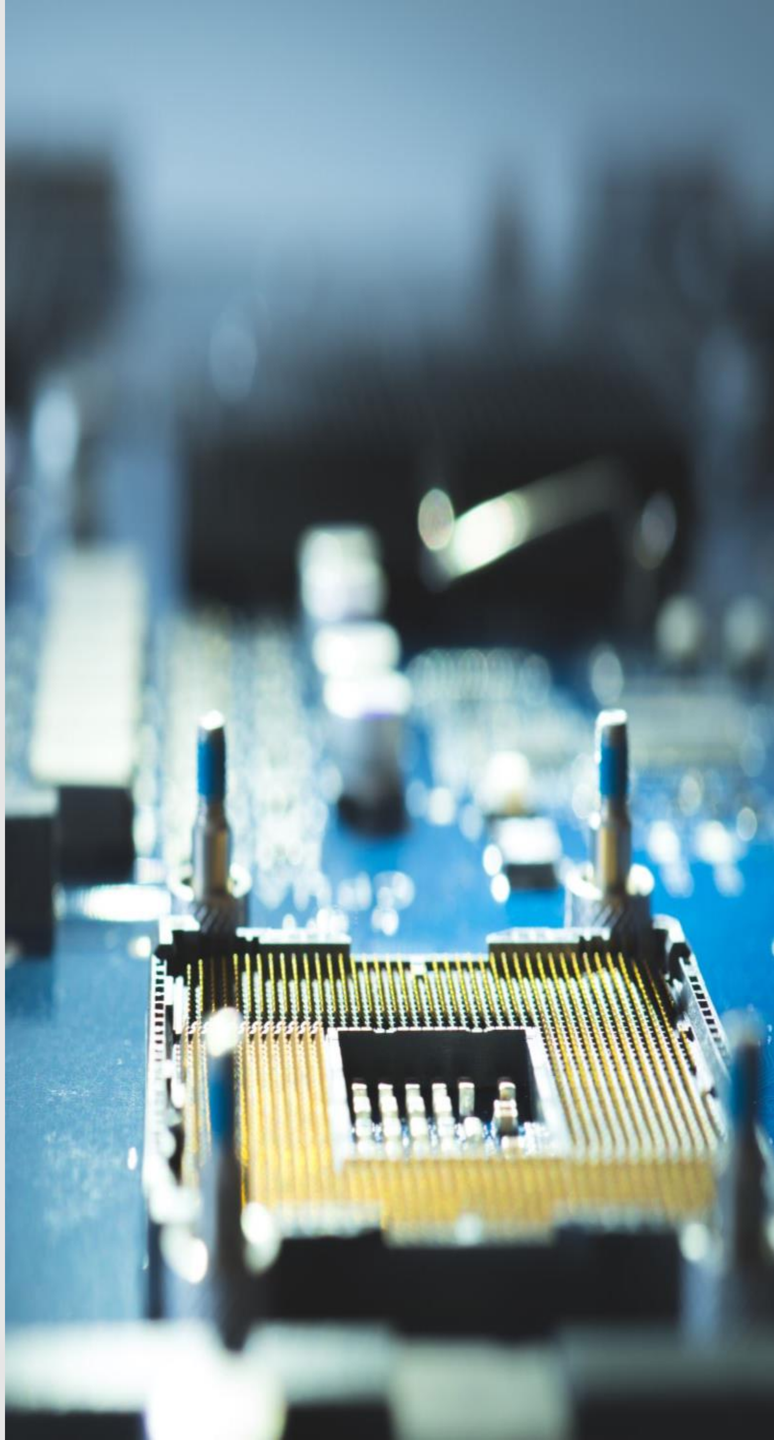
Binary tree

A 2-tree is called a binary tree, that is, if each node has either 0 or 2 children.

The nodes with 2 children are called internal and nodes with 0 children are called external nodes.

Labelled trees

In computer science, the set of vertices is not important compared to the utility of the tree which is emphasized by the label of the vertices.



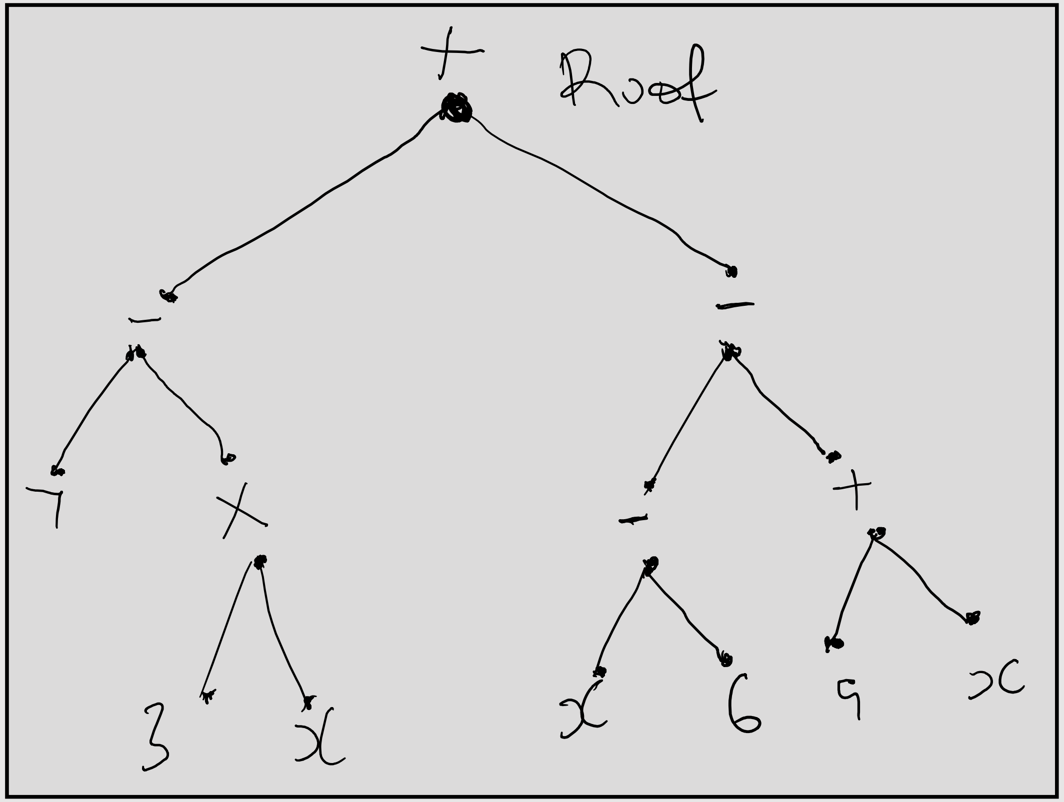
Example 1

Consider the algebraic expression

$$\big(7 - (3 \times x)\big) + \big((x - 6) - (9 + x)\big)$$

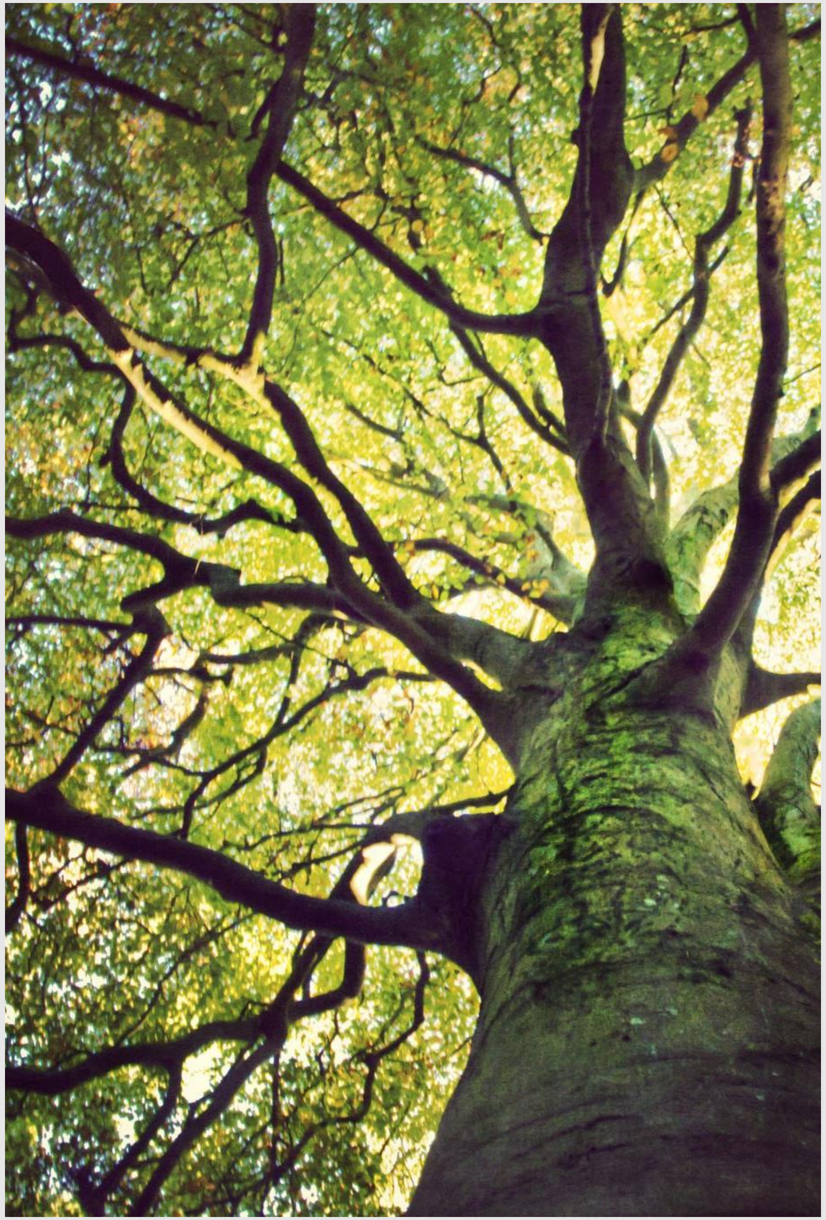
Note that '+' is the central operator.

The expression can be graphically represented by a labelled binary tree, where the root is the central operator i.e.



Spanning tree

A subgraph T of a connected graph G is called a spanning tree of G if T is a tree and T include all the vertices of G.



Minimum spanning tree

A minimal spanning tree of G is a spanning tree whose total weight is as small as possible.

The weight of a minimum spanning tree is unique but the minimum spanning tree itself is not.

A graph G may have more than 1 spanning trees.

Given a connected graph G with n vertices then the minimum spanning tree T must have edges.

Algorithms for finding the minimum spanning tree

Greedy algorithms such as Prim-Jarnik’s and Kruskal’s algorithms are used to determine the minimum spanning tree.

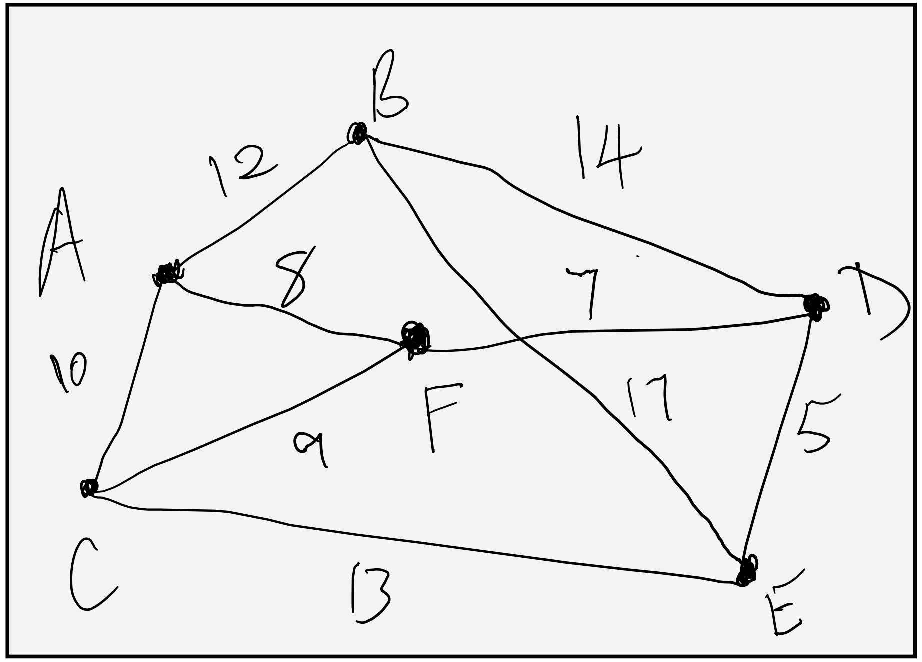
Prim-Jarnik’s algorithm

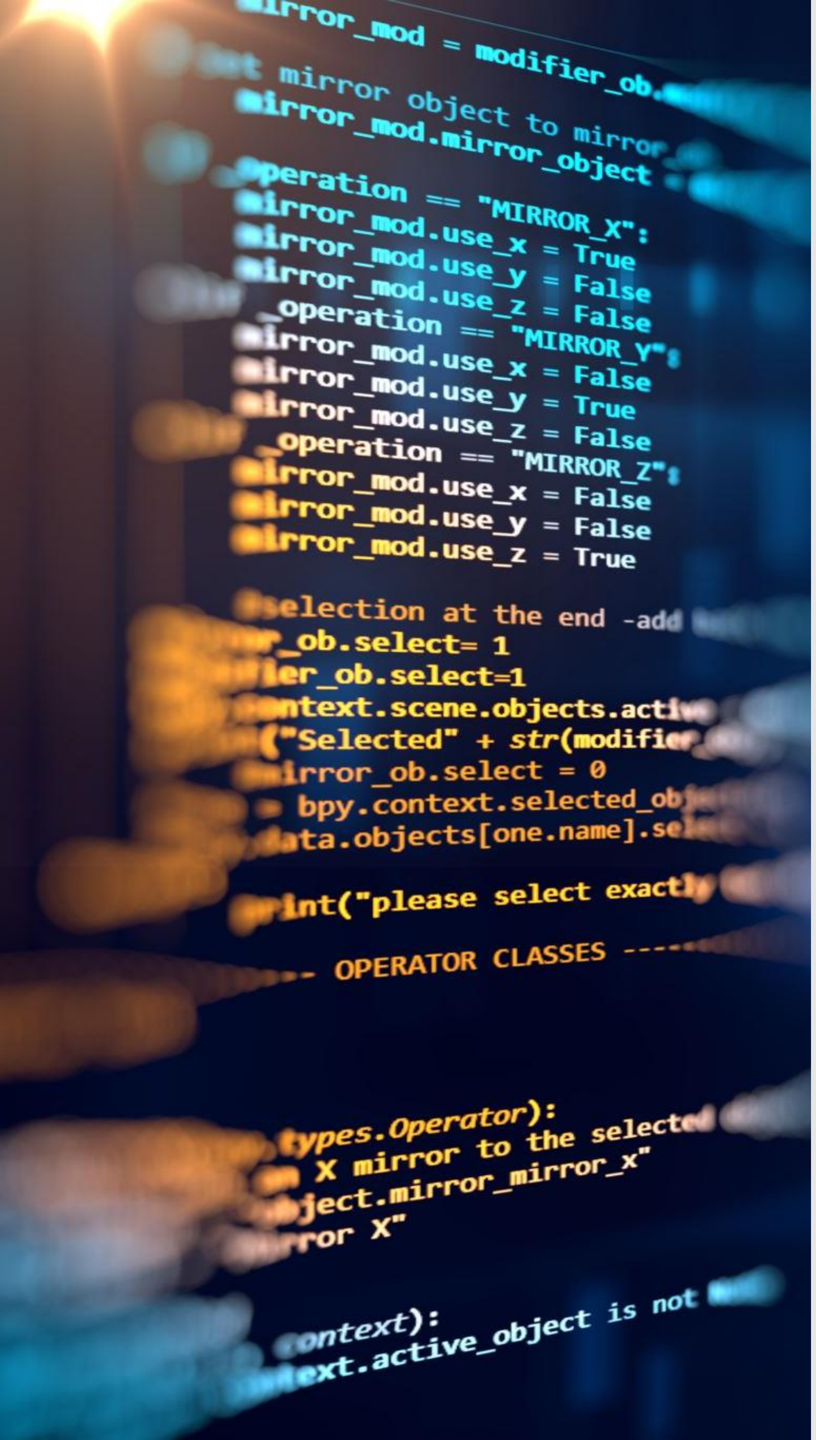
Steps

1. Initialize a tree with a single vertex, chosen arbitrarily.
2. Choose any edge with the smallest weight.
3. Proceed sequentially by adding edges of minimum weight incident a vertex in the tree, that does not result to a circuit
4. Stop when $n - 1$ edges have been added.

Example 1 (**Prim-Jarnik's algorithm**)

Find a minimum spanning tree of the weighted graph below





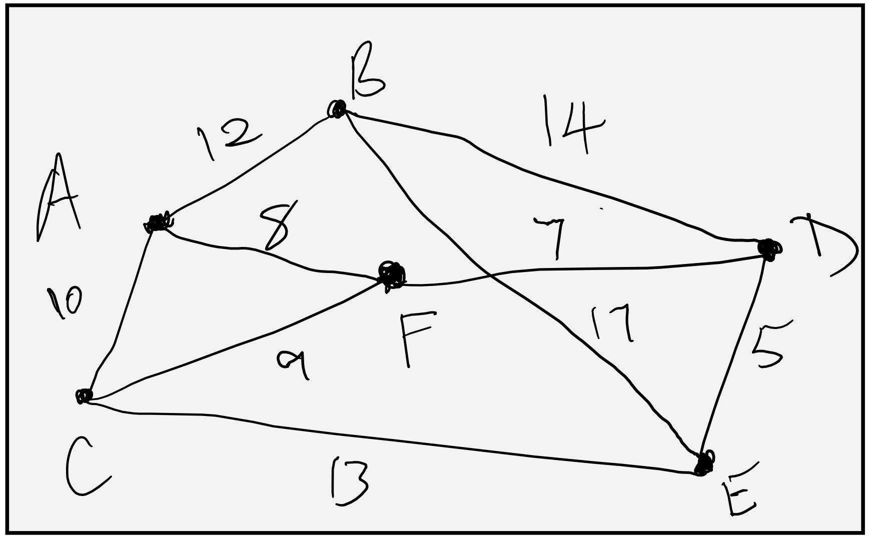
Kruskal’s algorithm

Steps

1. Choose an edge with minimum weight.
2. Successively add edges with minimum weight, that do not form a circuit, with any of the edge already taken.
3. Stop when $\mathrm{n} - 1$ edges have been added.

Example 2: Kruskal's Algorithm

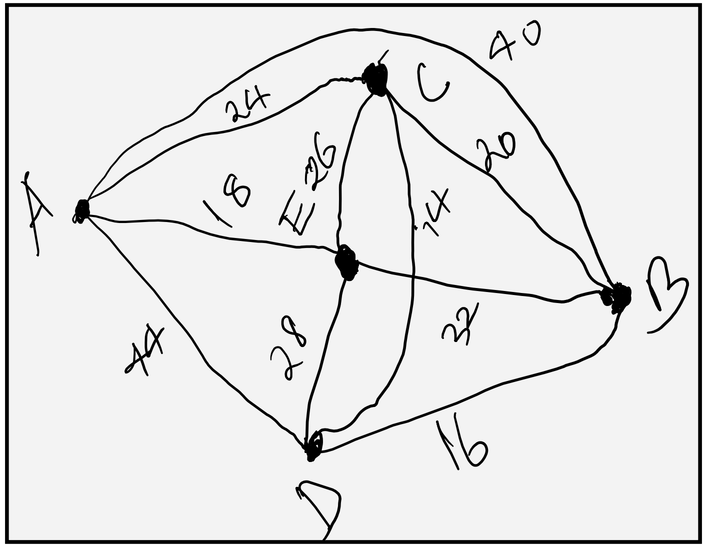
Find a minimum spanning tree of the weighted graph G below.



Edges	DE	DF	FA	FC	AC	AB	CE	BD	Total
Weight	5	7	8	9	10	12	13	14	41
Take	Yes	Yes	Yes	Yes	No	Yes	No	No	

Example 2

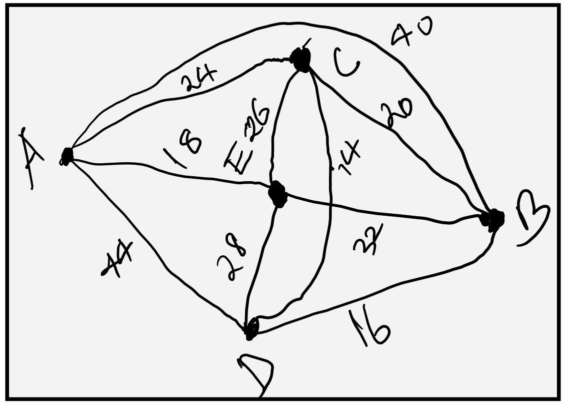
Five towns are connected by roads. The road network can be represented with the connected graph below (4-regular graph) with weights that represent the distance between the nodes or towns. Determine the minimum tree connecting all the towns using Kruskal's algorithm.



Edges									
Weight									

Example 3

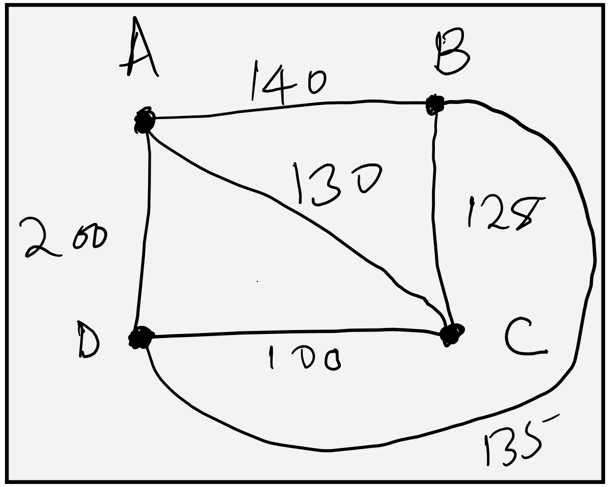
Five towns are connected by roads. The road network can be represented with the connected graph below (4-regular graph) with weights that represent the distance between the nodes or towns. Determine the minimum tree connecting all the towns using Prim's algorithm.



Edge	CD	DB	CA	AE	Total
Weight	14	16	24	18	72 km

Application: Local area networks

Consider four buildings A, B, C, and D. The buildings are to be linked with data cables. The length of the cables is as shown in the graph(3-regular graph) below. Determine the minimum length of cable that connects all the four buildings.



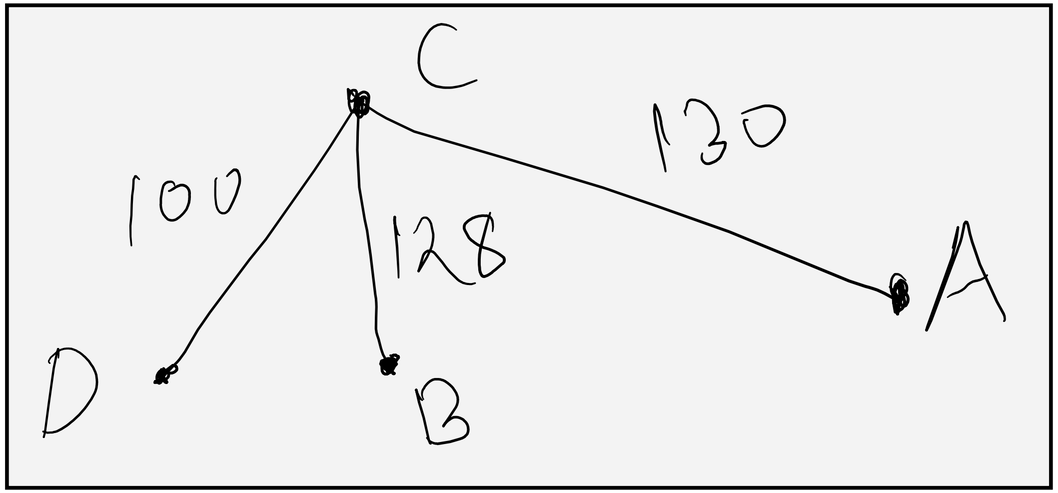
Edge						
Weight						

Contd...

It is cheaper and sufficient to install the 3 shortest cables; CD, BC, and AC.

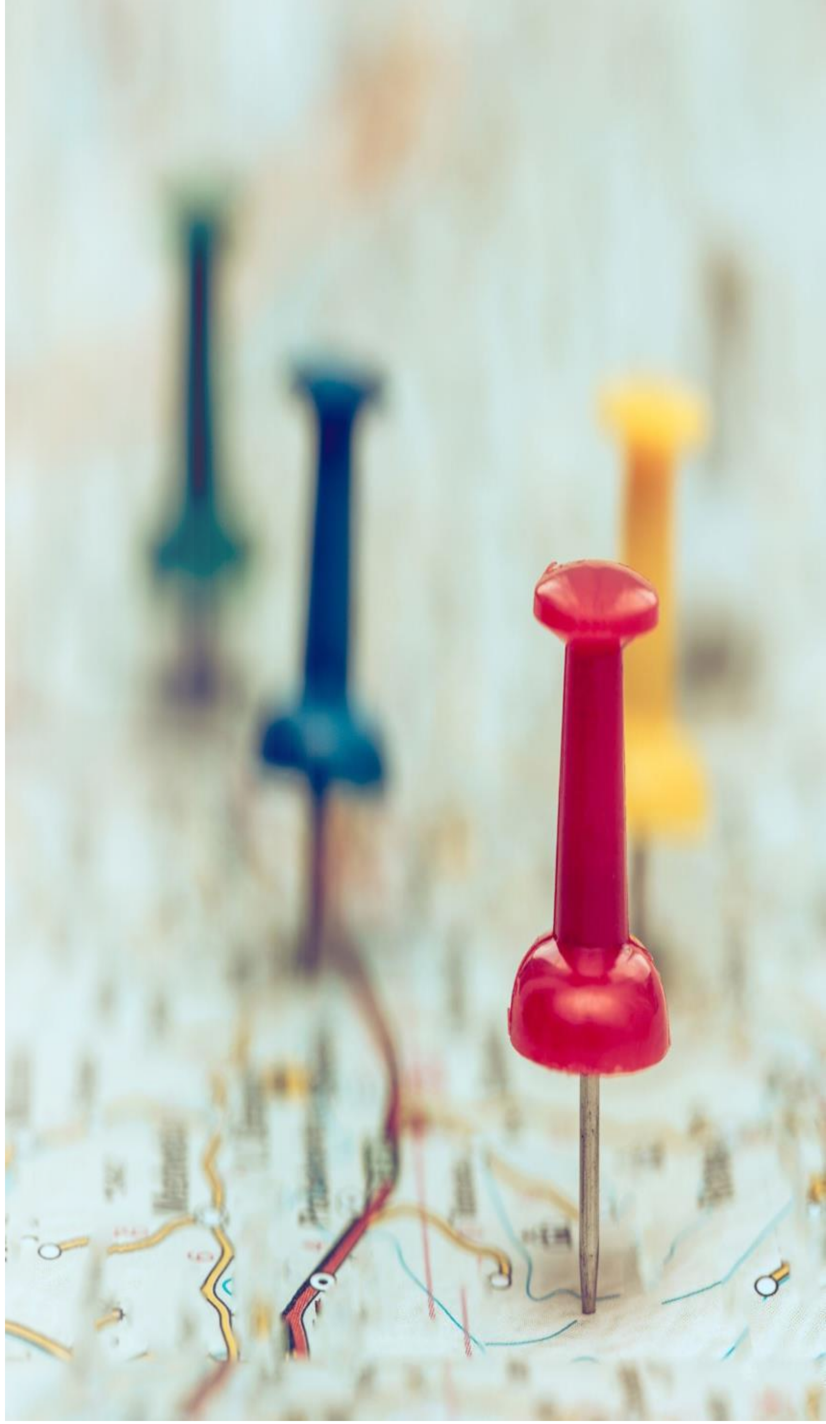
Any two buildings can easily connect with one another.

The three cables form a tree (a connected graph) and forms a minimal spanning tree, with C as the root, that provide the shortest distance.



Traveling-salesperson problem

- **Theorem 1:** A finite connected graph is said to be Eulerian if and only if each vertex has even degree.
- **Definition:** A Hamiltonian circuit in a graph G is a closed path that passes every vertex in the graph exactly once.
- **Theorem 2:** The complete graph K_n with $n \geq 3$ vertices, has $\mathrm{H} = \frac{(\mathrm{n}-1)!}{2}$ Hamiltonian circuits.



Traveling-salesperson problem

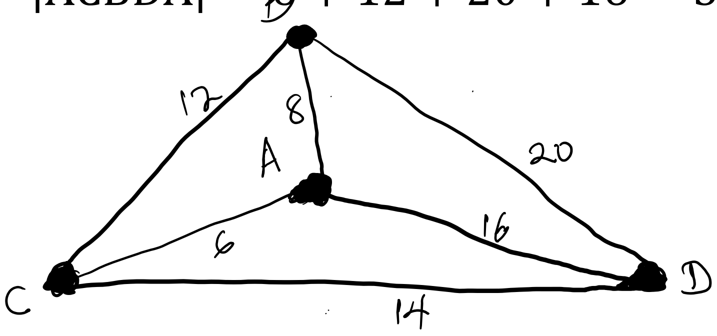
Let G be a complete weighted graph, the traveling-salesperson problem refers to finding a Hamiltonian circuit for G of minimum weight.

Example 1

Determine the Hamiltonian circuit of minimum weight for the complete weighted graph.

The graph G has 4 vertices, by the above theorem 2 it has $\mathrm{H} = \frac{(n-1)!}{2} = \frac{(4-1)!}{2} = \frac{3!}{2} = 3$ Hamiltonian circuits. Assume the circuits start at A then we have the following circuits;

- $|ABCDA| = 8 + 12 + 14 + 16 = 50$
- $|ACDBA| = 6 + 14 + 20 + 8 = 48$
- $|ACBDA| = 6 + 12 + 20 + 16 = 54$



ACDBA with weight 48 is the Hamiltonian circuit of minimum weight.

Nearest-neighbor Algorithm

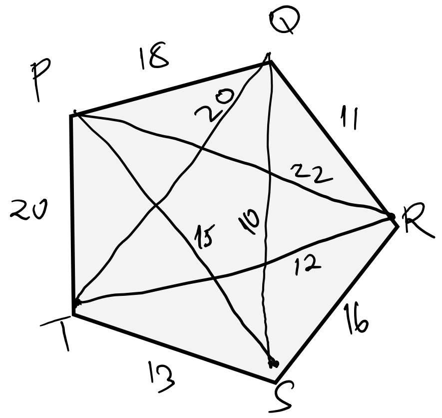
The NNA starts at a random vertex, then chooses the edge with the least weight to the next closest vertex (neighbor).

Proceed sequentially until a Hamiltonian circuit is completed.

Note the NNA may miss shorter routes due to its ‘greedy’ nature.

Example 1

- Consider the weighted graph below; The graph has $\frac{(5-1)!}{2}$=12 Hamiltonian circuits. We can apply NNA starting at (i) P (ii) Q (iii) R (iv) S (v) T to get the Hamiltonian circuits;
- $|PSQRTP| = 68$
- $|QSTRPQ| = 75$
- $|RQSTPR| = 76$
- $|SQRTPS| = 68$
- $|TRQSPT| = 68$



References

Lipschutz, S., & Lipson, M. (2007). *Discrete Mathematics*. McGraw-Hill.

Rosen, K. (2012). *Discrete mathematics and its application* (7th ed.). McGraw-Hill.

Wilson, R. J. (1998). *Graph Theory* (4th ed.). Addison Wesley Longman Ltd.