Course:
Mathematics for IT Professionals

Lecture 4

Predicate Logic

By
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Outline

The topics to be treated in this lecture are:

- Predicates
- Preconditions and Postconditions
- Quantification
  - Universal Quantifiers
  - Existential Quantifiers
- Restricted Domains
- Logical Equivalence
- Translation of Sentences to Logical Expression

Lecture Learning Outcomes

At the end of the session, you will be able to

- explain predicate logic
- understand preconditions and postconditions of predicates
- determine the difference between existential and universal quantifiers
- use counterexamples to determine validity of statements
- translate English sentence to logical expression
- have knowledge on applications of predicate logic

Introduction

- Propositional logic is not adequate enough to express the meaning of all kinds of statements
  - E.g. Suppose we know that,

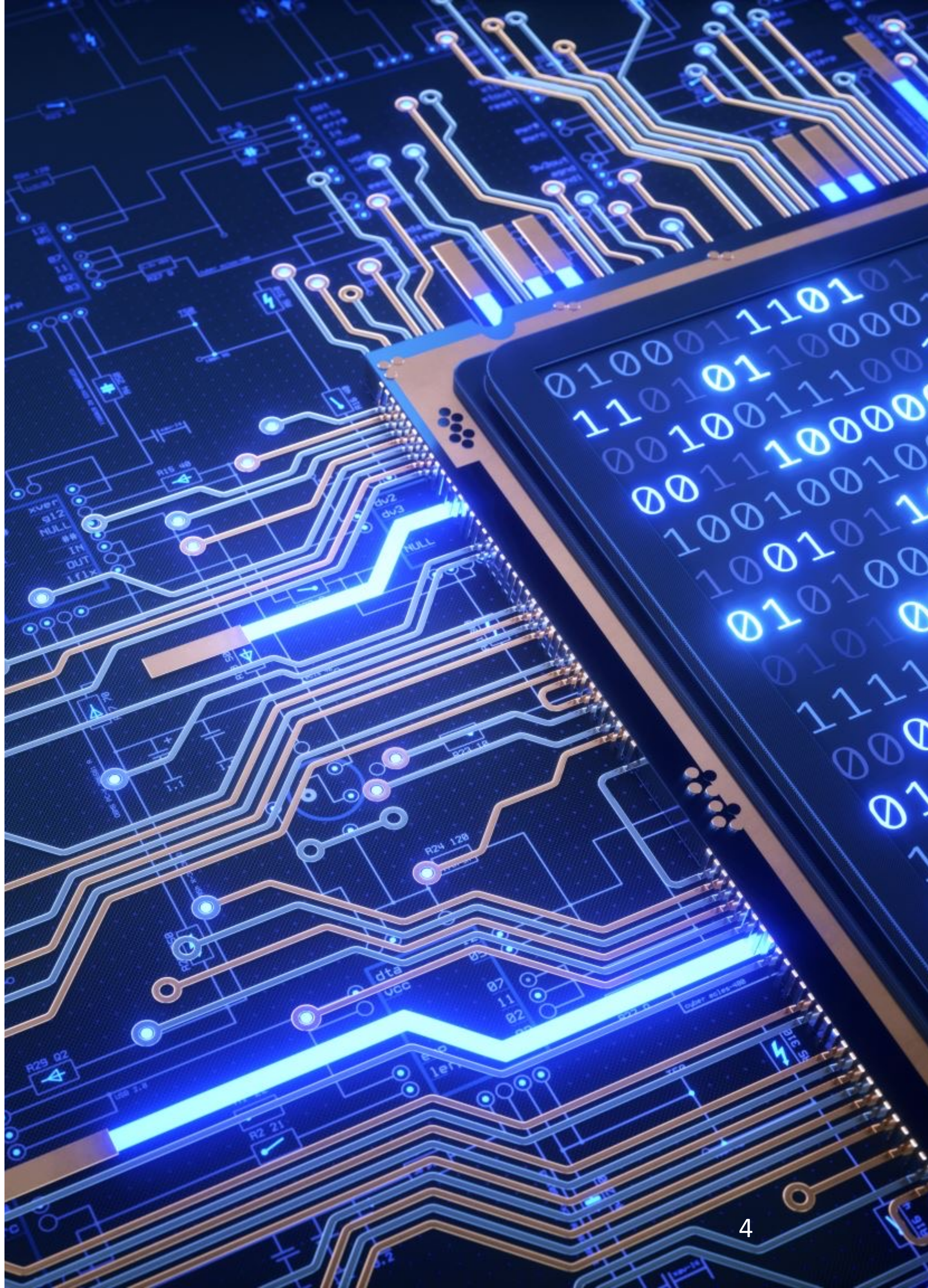
    “Every student of University of Ghana is punctual”

    But, based on propositional logic, we can conclude that

    “Kwame is punctual”
  - E.g. Suppose we know that,

    “The Projectors are not functioning properly”

    “The projector in the Lecture Room 1 is not working properly”
- Predicate logic is helpful to express a wide range of statements



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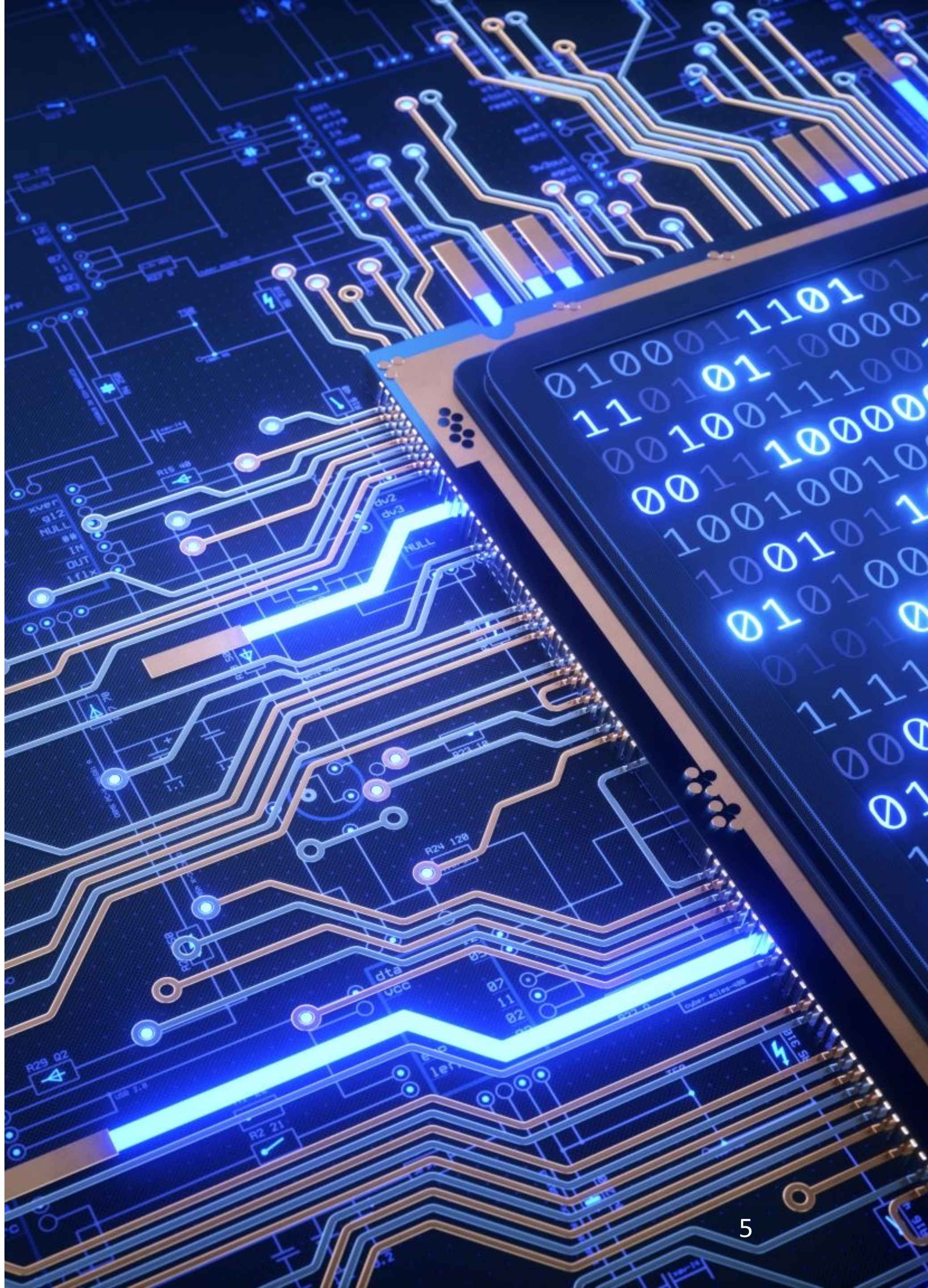
What is Predicate Logic?

- Predicate logic is an extension of propositional logic that permits concisely reasoning about whole classes of entities.

E.g., “x>1”, “x+y=10”, ”x+y=z”

- Such statements are neither true or false when the values of the variables are not specified.



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What is Predicate Logic?

- A ***predicate*** is part of a statement which asserts something about the subject
  - E.g. “x is greater than 3”,
    Let, P denote the predicate “is greater than 3”
    Then the above statement can be written as $P(x)$, where x is the subject that varies. Given a value to x, $P(x)$ becomes a proposition.

- The statement $P(x)$ can also be said as, the value of the **propositional function** P at x.

- E.g.:
  “$x = y + 3$” → $Q(x, y)$
  “$x + y = z$” → $A(x, y, z)$

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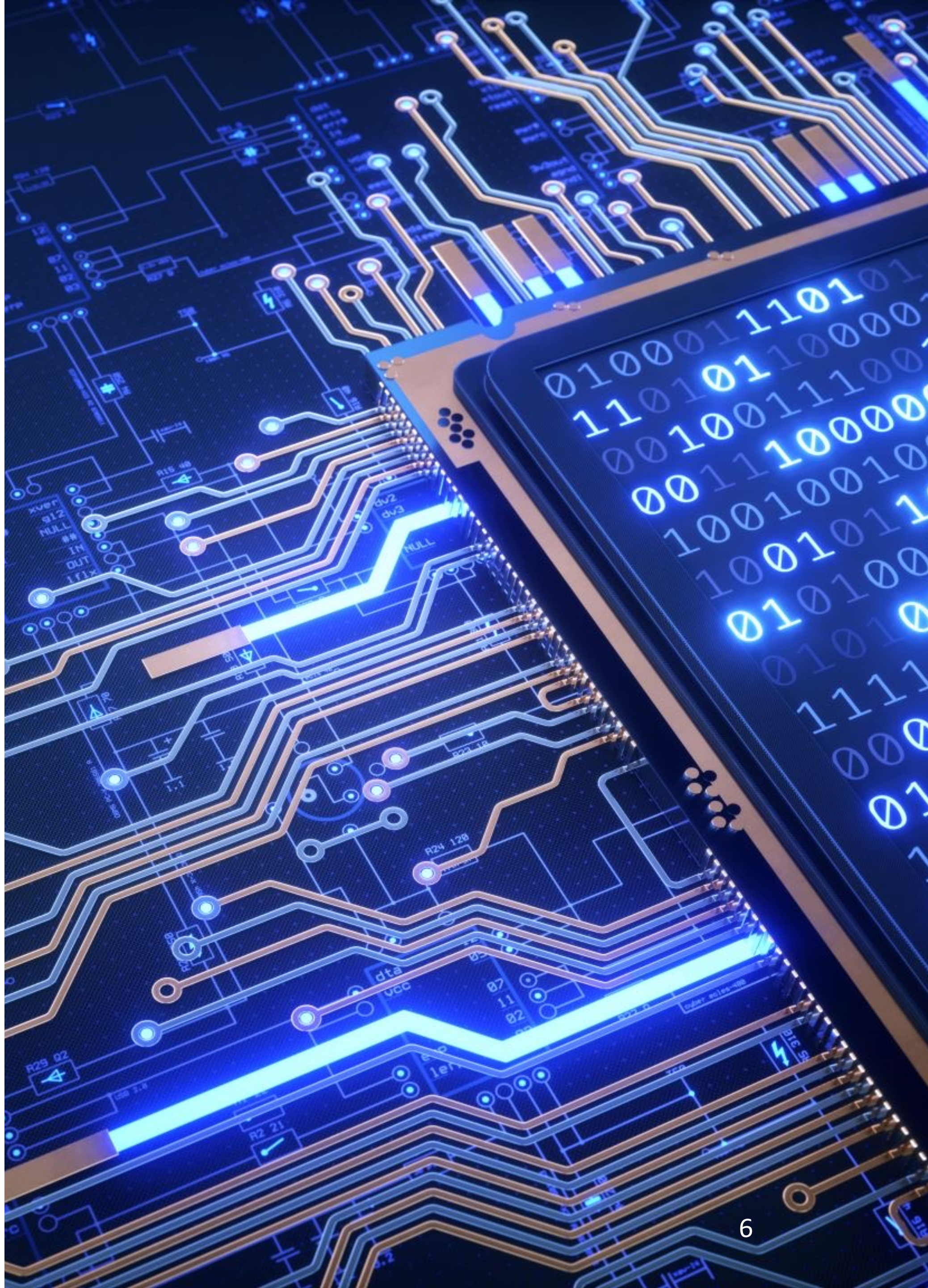


Subjects and Predicates

- The proposition

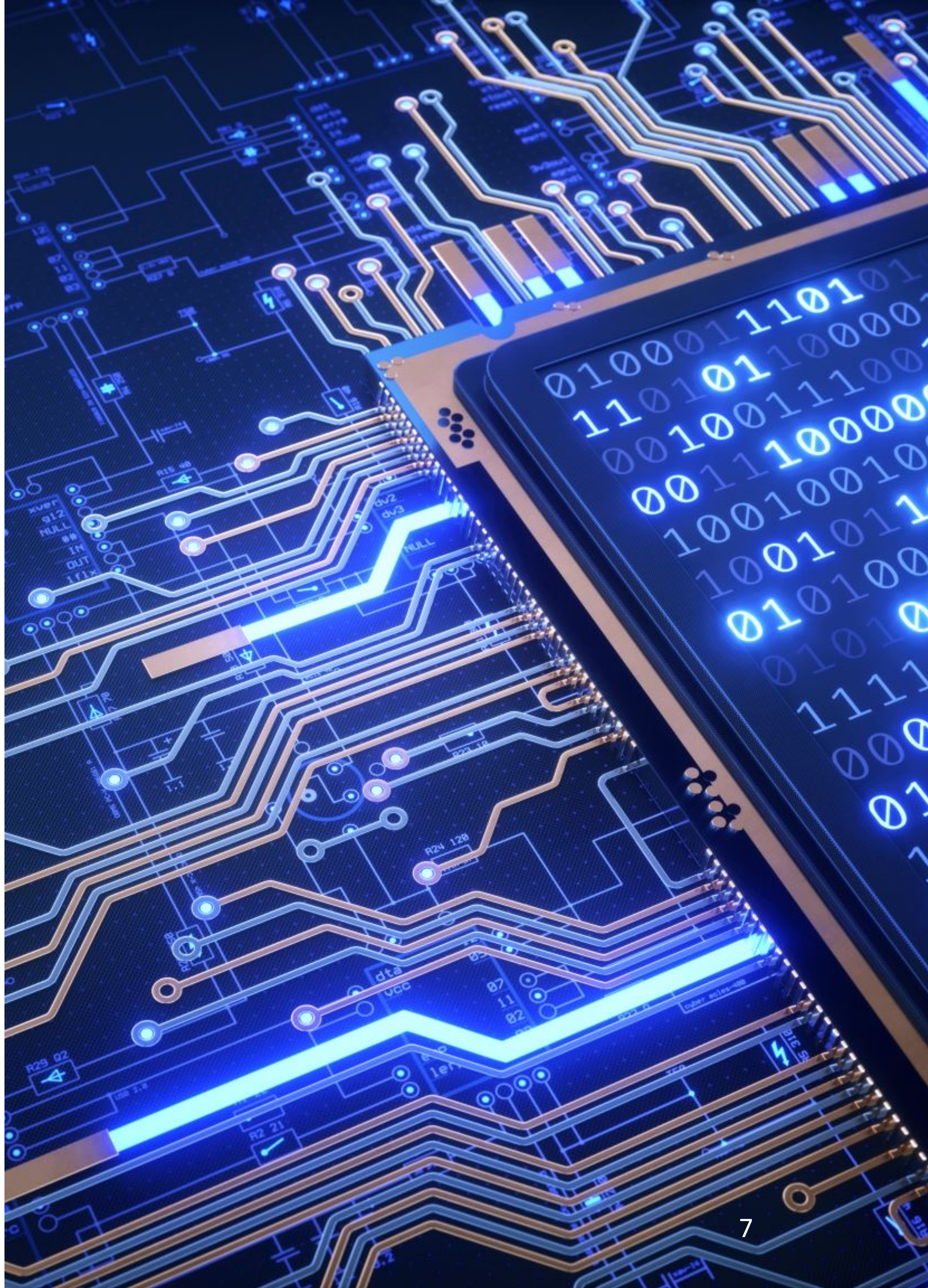
  “The cat is sleeping”

  has two parts:
  - “the cat” denotes the subject - the object or entity that the sentence is about.
  - “is sleeping” denotes the predicate- a property that the subject can have.

- **Let’s try this**
- Determine the subject and predicate of the sentence

  “Computer *x* is under attack by an intruder”

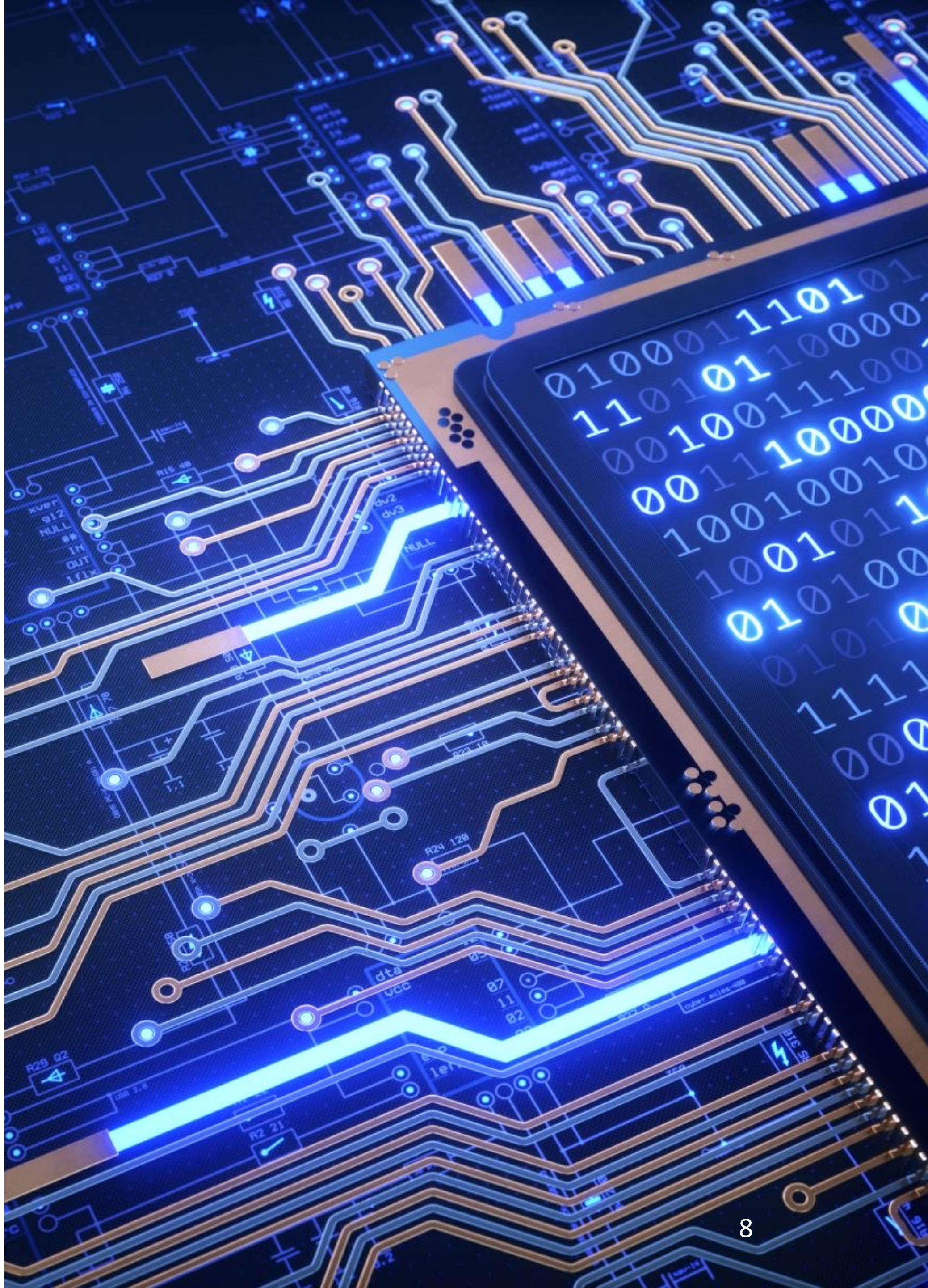
Answer: “Computer x” → subject and “is under attack by an intruder” → predicate



Examples of Predicates

- *P*(*x*) denotes "*x* > 3." State the truth values of P(x = 4) and P(x = 2)?
  - *Answer*: P(x = 4) is TRUE and P(x = 2) is FALSE

- *A*(*x*) denotes "Computer *x* is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are the truth values of A(CS*1*), *A*(CS2), and *A*(MATH1)?
  - *Answer*: *A*(CS1) is FALSE and *A*(CS2), *A*(MATH1) are TRUE

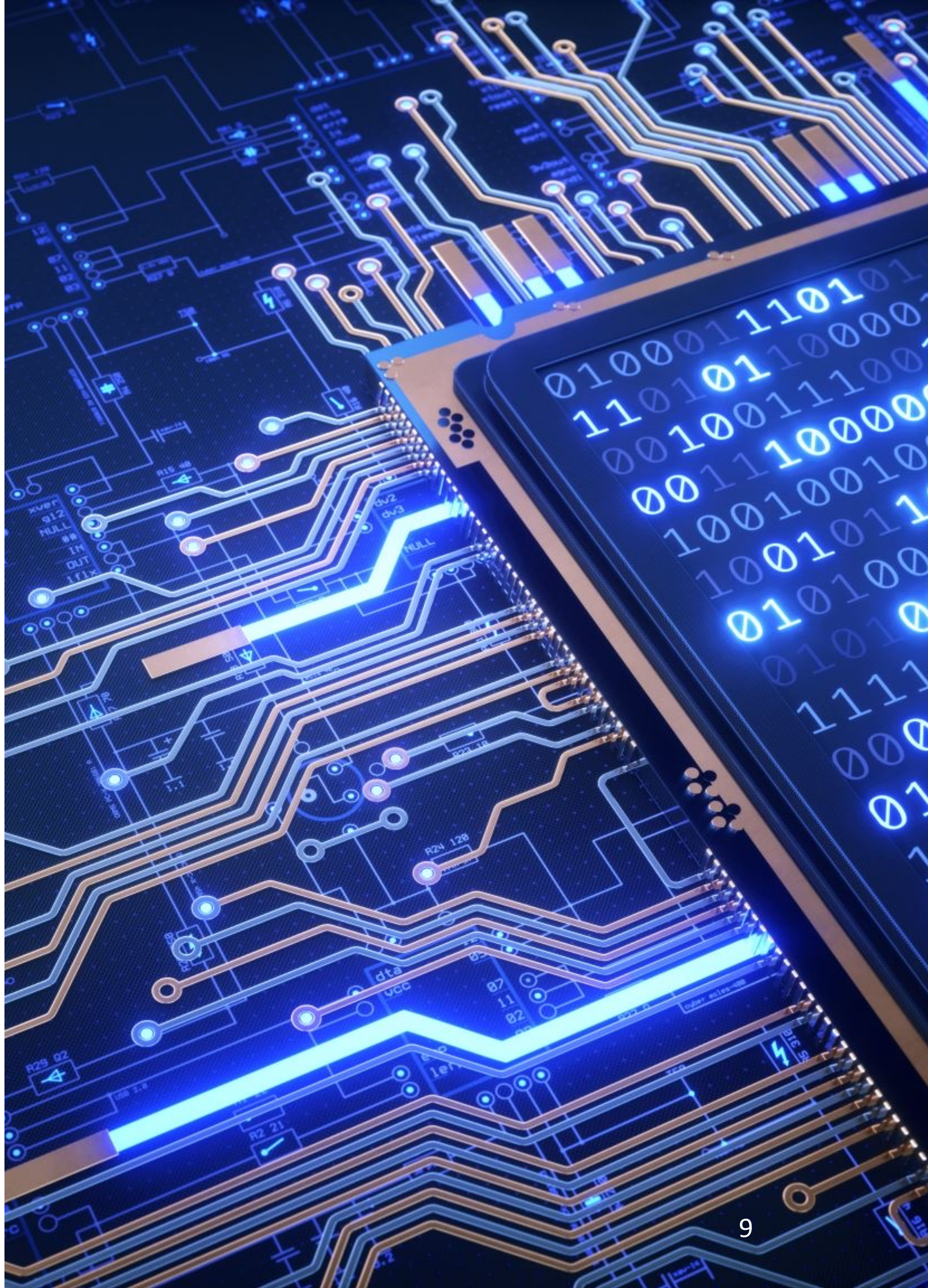


Examples of Predicates

- Let $Q(x, y)$ denotes the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?
  - *Answer*: $Q(1, 2)$ is FALSE and $Q(3, 0)$ is TRUE

- *Predicates in Computer programs*:

  e.g. **if** $x > 0$ **then** $x := x + 1$
  Here, $P(x)$ is "$x > 0$". Upon taking a value for x , the truth value of it is obtained and the decision of whether to execute $x := x + 1$ is made.



Precondition and Postcondition

- Computer program statements that describe the valid input are known as *preconditions*, and the statements that describe the valid output are known as *postconditions*

- *Preconditions* and *postconditions* can be represented as predicates to verify the correctness of the program

e.g. Consider the program statements,

temp := x   x := y   y := temp

Here, the predicate for **precondition** is,

$P(x, y)$ is "$x = a$ and $y = b$" and

for **postcondition**

$Q(x, y)$ is "$x = b$ and $y = a$", for some values a, b

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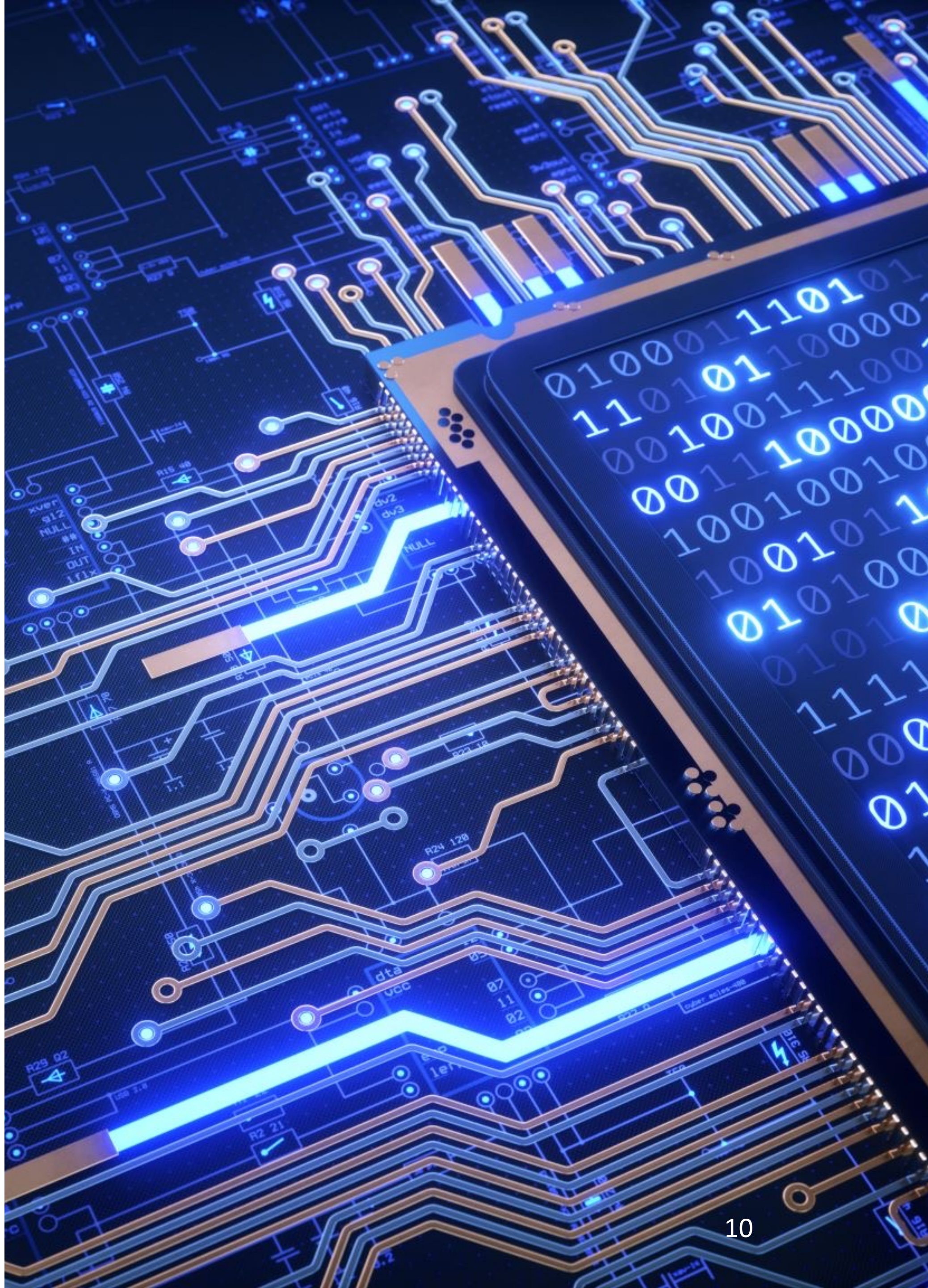

Quantification

- *Quantifiers* are the words, which are used to express the extent to which a predicate is true over a range of elements.
  - e. g. For all, some, a few, many, none
- **Quantification** is a way of creating a proposition from a propositional function (predicate) using the quantifiers.

- Two important types of quantification:
  - Universal quantification
  - Existential quantification

- The area of logic that deals with predicates and quantifiers is called the **predicate calculus** or **predicate logic**

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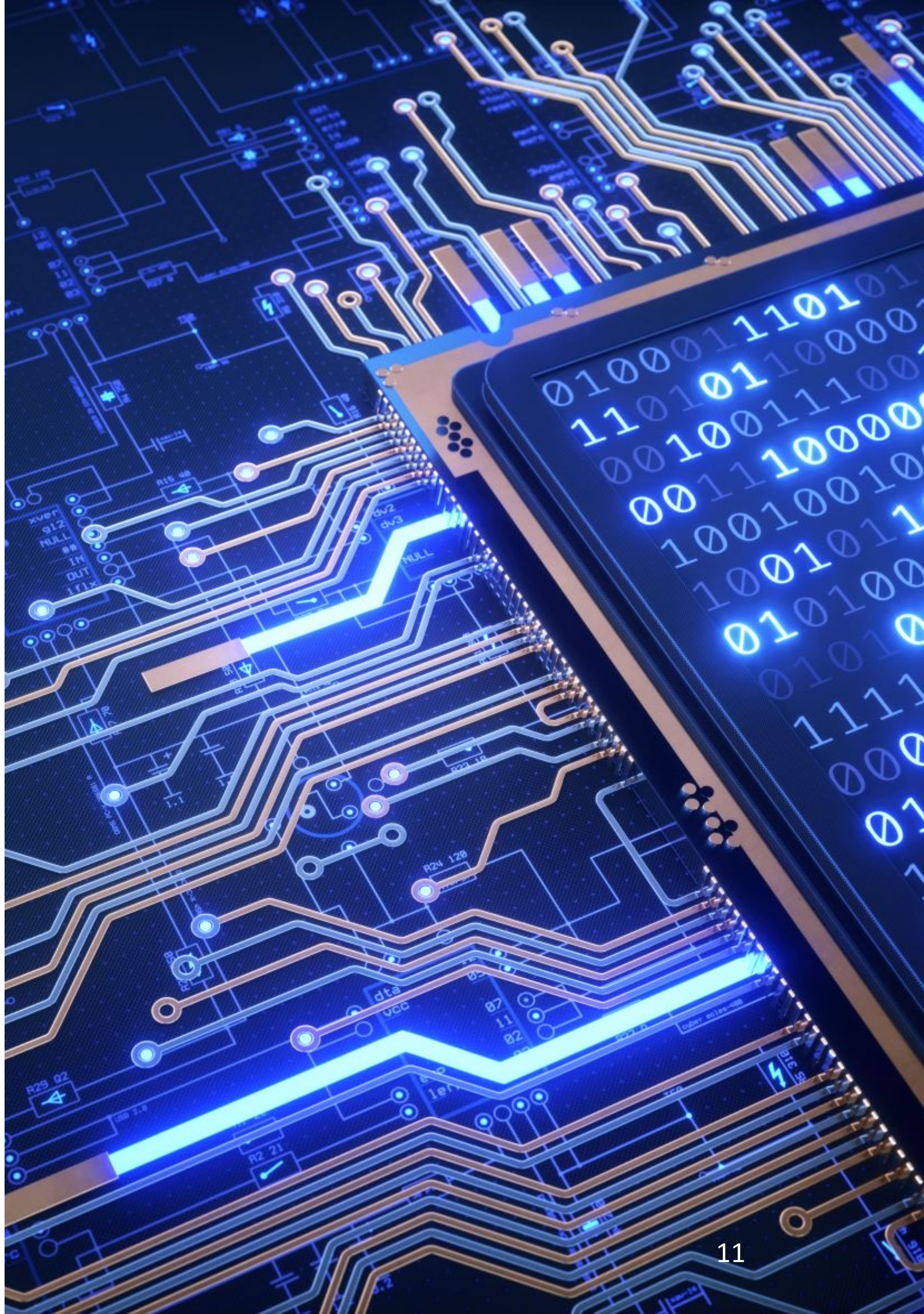

Universal Quantification

- ***Domain*** (or *Domain of discourse* or *universe of discourse*):
  - The set of possible values that a variable can assume
- The ***universal quantification*** of $P(x)$ is the statement

“$P(x)$ for all values of x in the domain.”

This is denoted by using the notation $\forall x\, P(x)$, where $\forall$ is the **universal** quantifier, which means “for all” or “for every”.

- An element in the domain for which $P(x)$ is FALSE is called a **counterexample** of $\forall x\, P(x)$
- The domain must always be specified whenever $\forall$ is used
- The meaning of $\forall x\, P(x)$ changes when the domain changes

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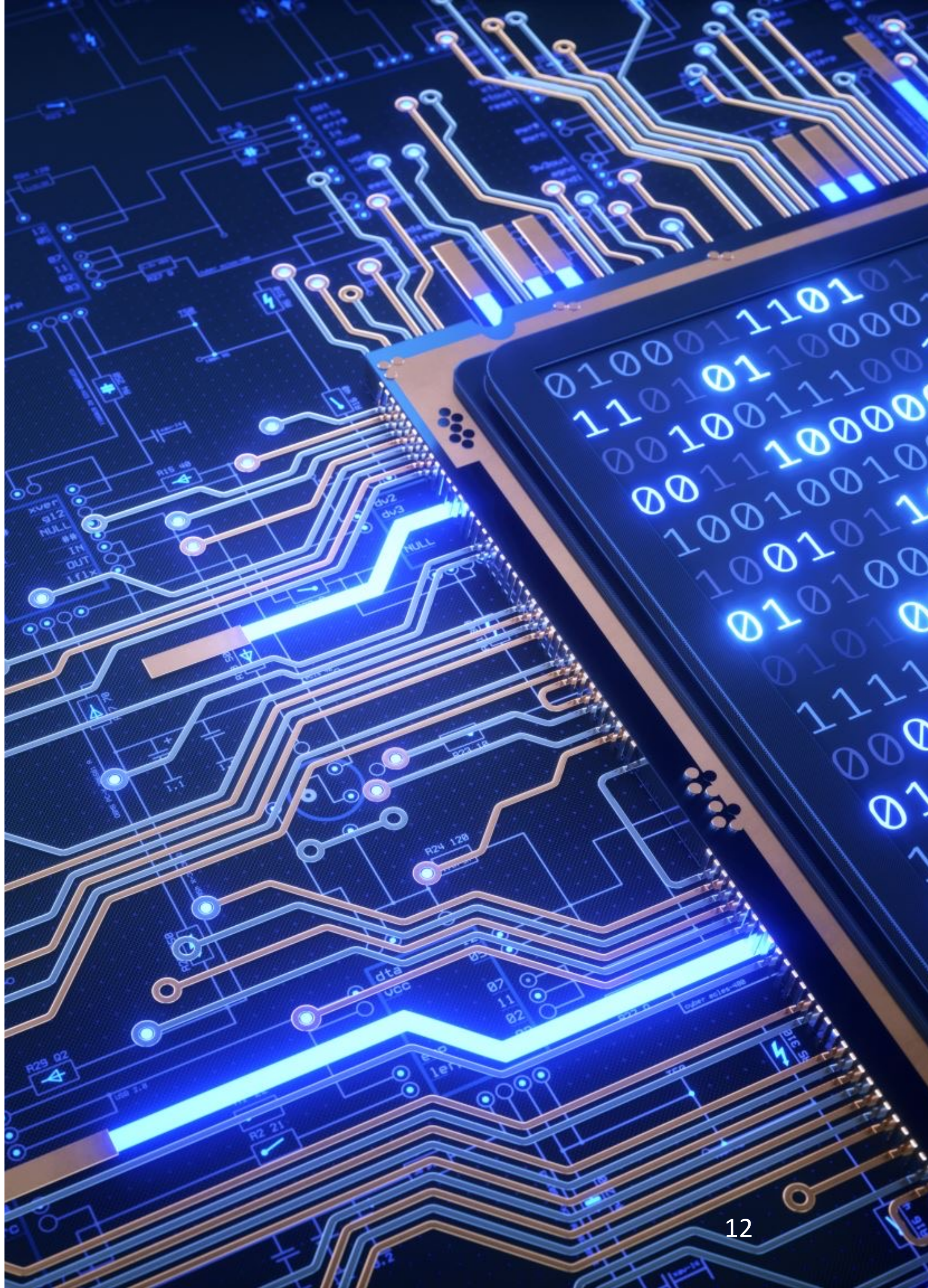
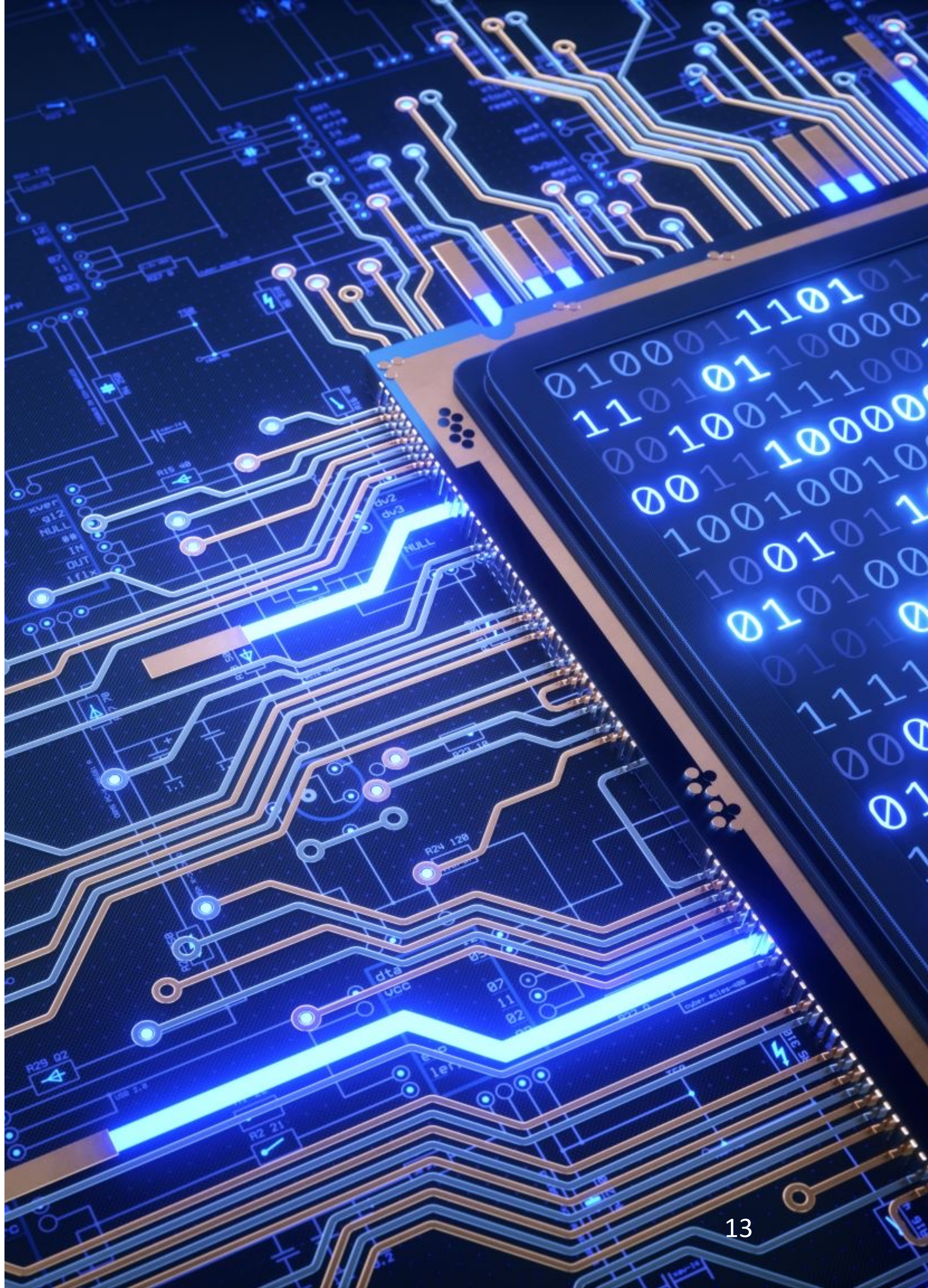


Worked Examples

- Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x\, P(x)$ where the domain consists of all real numbers?
  - *Answer* : TRUE

- Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?
  - *Answer* : FALSE ($x = 3$ is a counterexample)

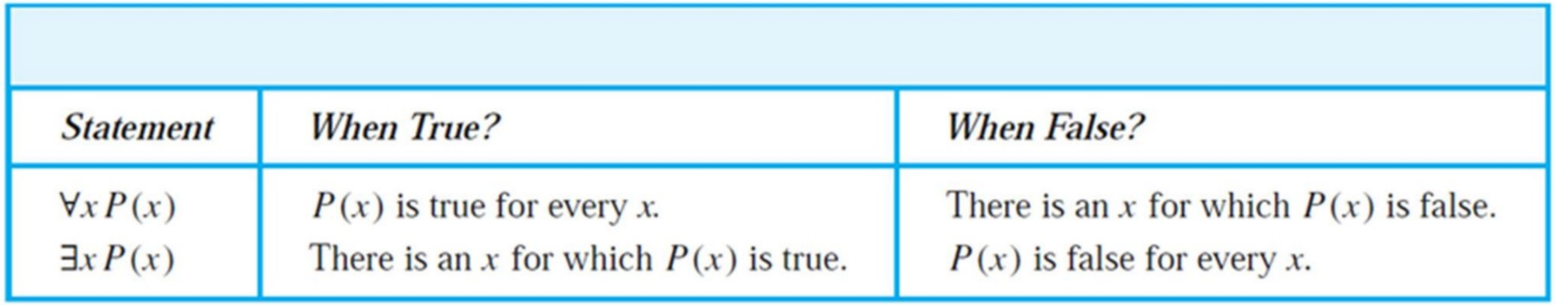
- Suppose that $P(x)$ is "$x > 0$". What is the truth value of the quantification $\forall x\, P(x)$ for the domain of integers? Why?
  - *Answer* : FALSE ($x = 0$ is a counterexample)



Existential Quantification

- The ***existential quantification*** of $P(x)$ is the proposition

  "There exists an element x in the domain such that $P(x)$"
- This is denoted by using the notation $\exists x\, P(x)$, where $\exists$ is the *existential quantifier,* which means "there exists" or "there is"
- Similar to $\forall$, a domain must always be specified when $\exists$ is used and the meaning of $\exists x\, P(x)$ changes when the domain changes



Statement	*When True?*	*When False?*
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

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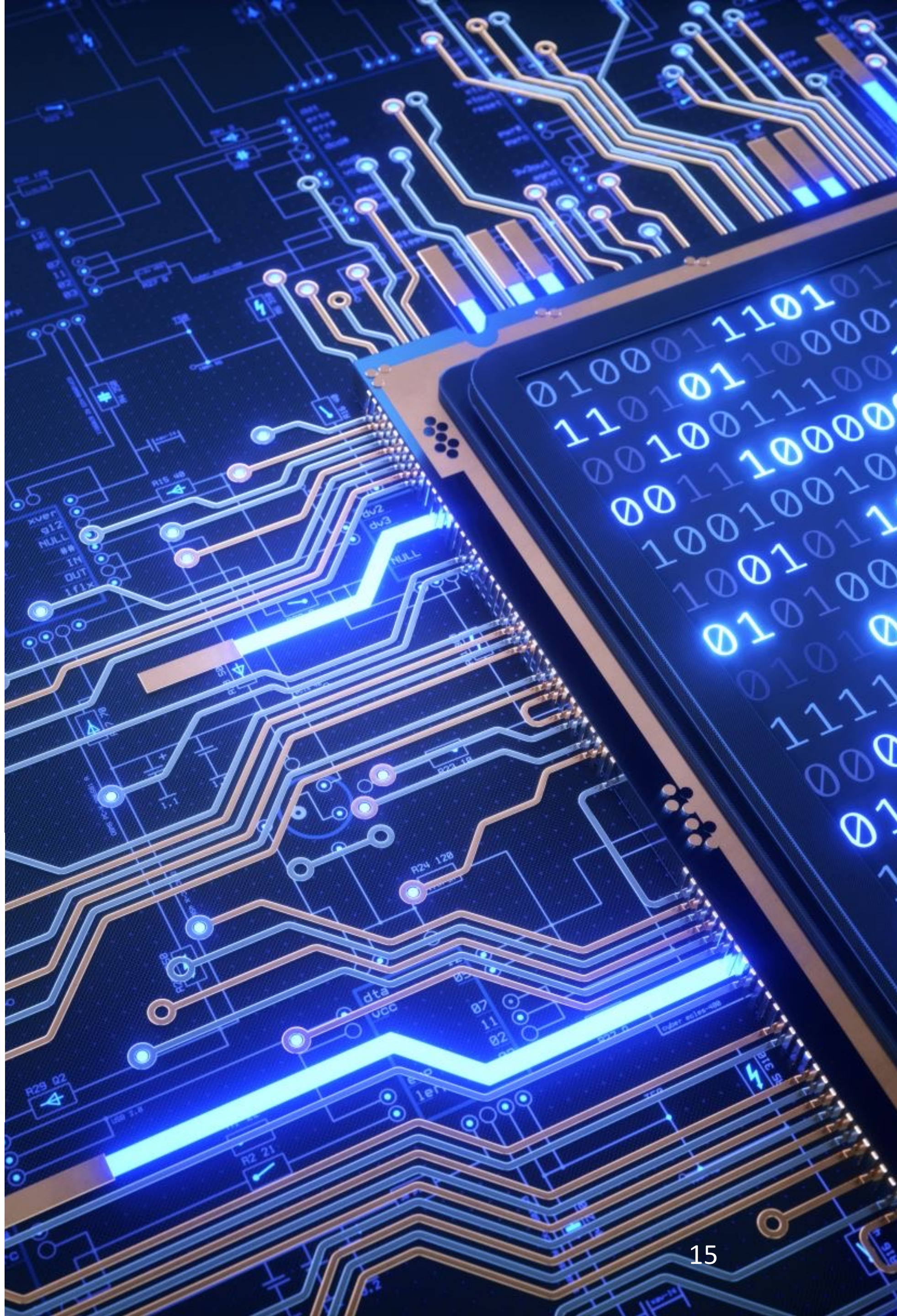


Existential Quantification: Examples

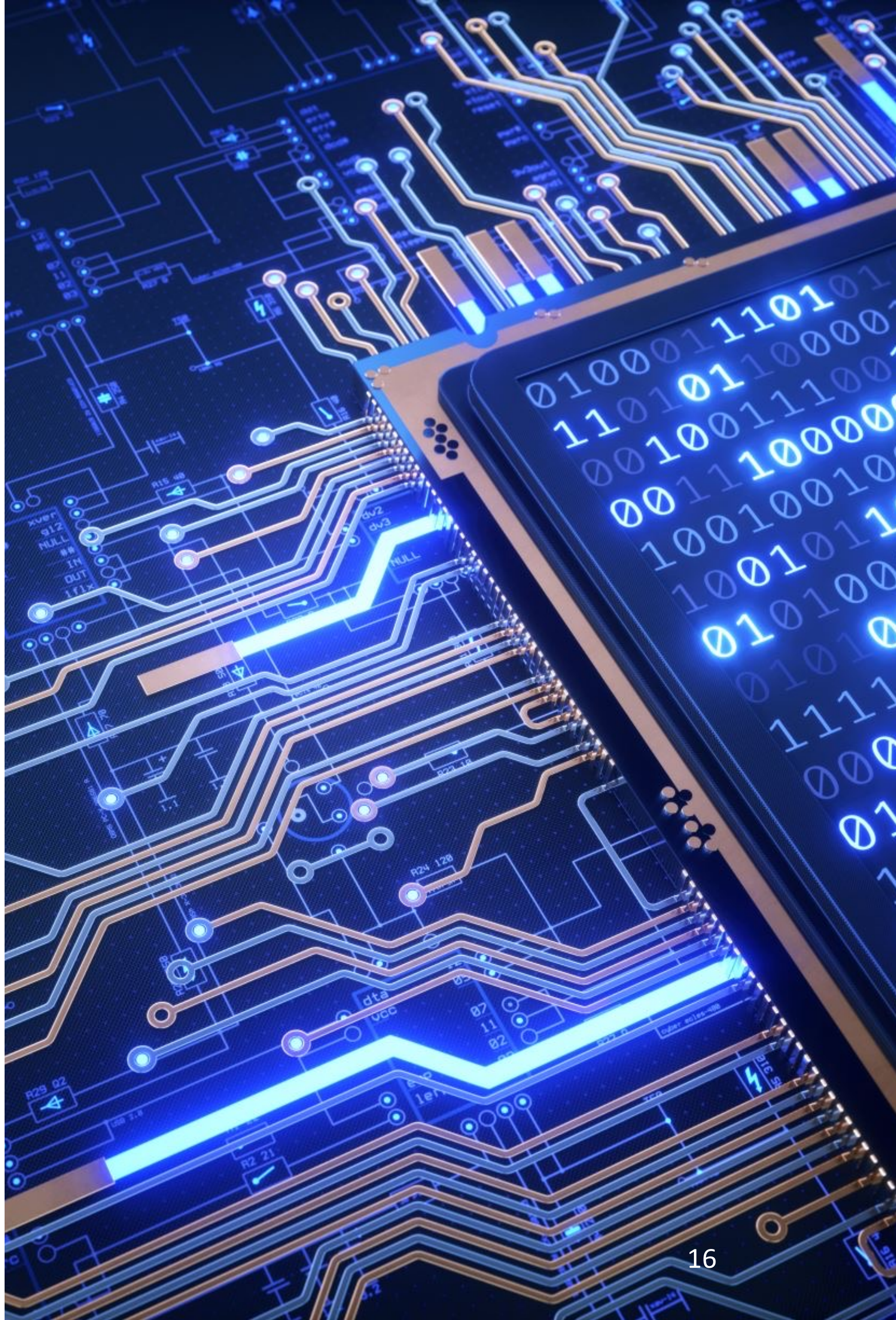
- Let $P(x)$ denote the statement "$x > 3$" What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?
  - *Answer* : TRUE

- Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?
  - *Answer* : FALSE

- **Note:** When the domain is empty,

$\forall x\, P(x)$ is always TRUE and

$\exists x\, P(x)$ is always FALSE



Exercise

- Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
  - a) $\forall x\,(x^2 \geq x)$
  - b) $\forall x(x > 0 \lor x < 0)$
  - c) $\forall x\,(x = 1)$

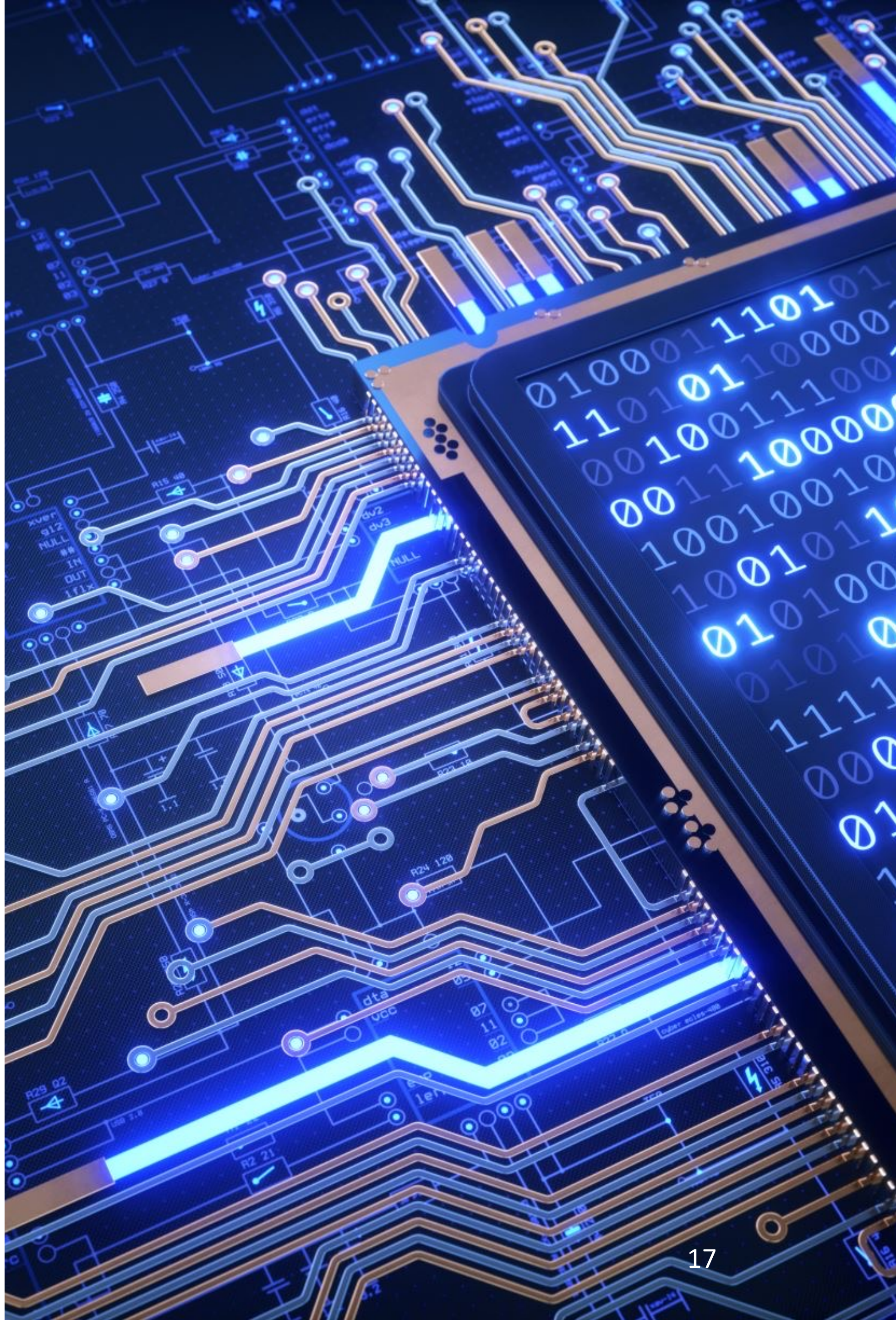
Answer:

Domain: All integers

a) $\forall x\,(x^2 \geq x)$ is TRUE. So, no counterexample exists

b) $\forall x\,(x > 0 \lor x < 0)$ is FALSE. The integer 0 is a counterexample

c) $\forall x\,(x = 1)$ is FALSE. Except the integer 1, all other integers are counterexamples



Quantifiers: Restricted Domains

- The restriction on the domain can be shown by including a condition on the variable written after the quantifier

- Examples: Let the domain be a set of real numbers

1) $\forall x < 0\ (x^2 > 0)$ denotes that, for all real numbers x with $x < 0$, $x^2 > 0$, which means, "the square of a negative real number is positive"

2) $\exists z > 0\ (z^2 = 2)$ denotes that, there exists a real number z with $z > 0$ such that $z^2 = 2$, which means, "there is a positive square root of 2"

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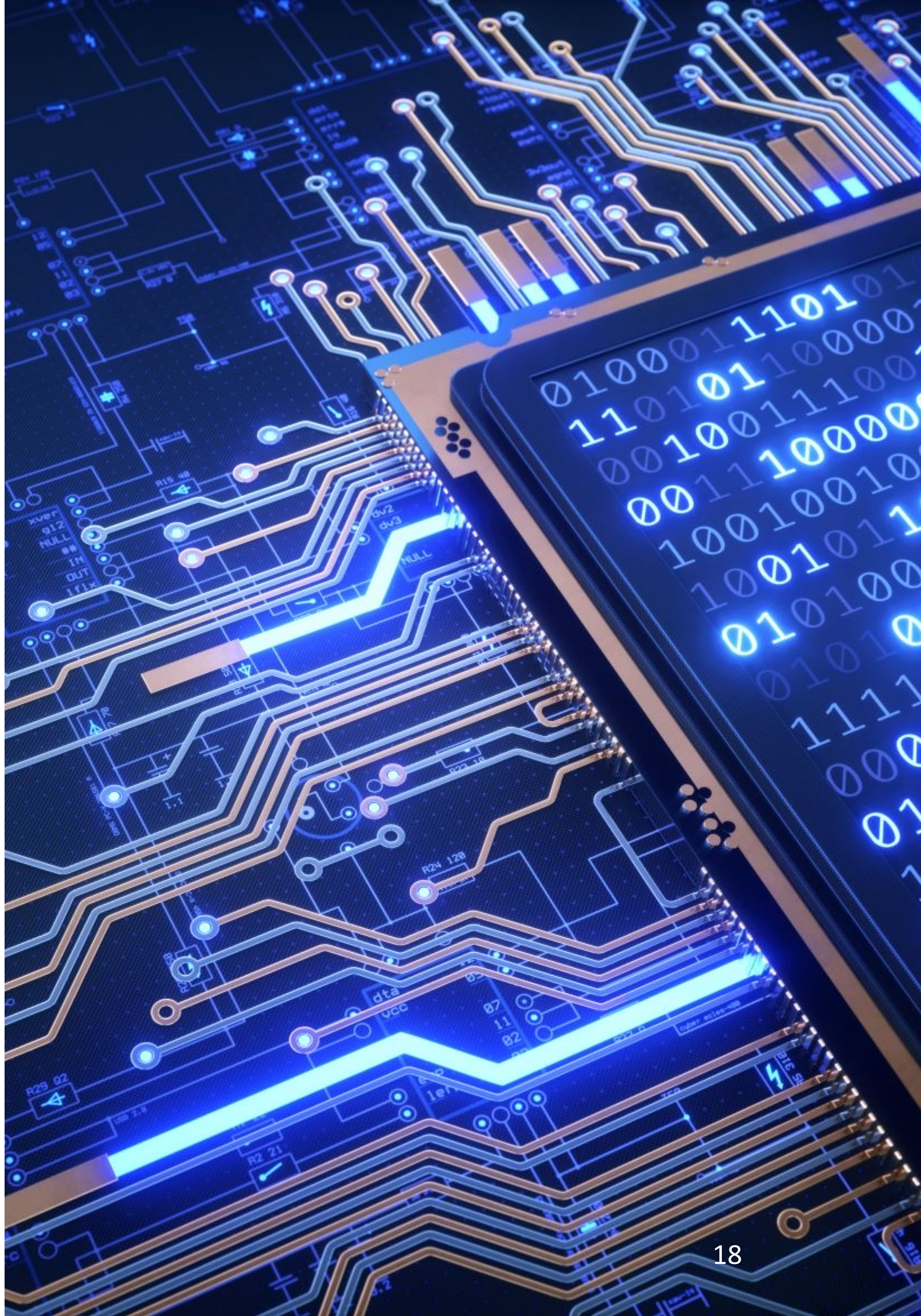

Quantifiers: Restricted Domains

- The *restriction of a universal quantification* is the same as the universal quantification of a conditional statement.
  - e.g. $\forall x < 0\ (x^2 > 0)$ is same as $\forall x\ (x < 0 \rightarrow x^2 > 0)$
- The *restriction of an existential quantification* is same as the existential quantification of a conjunction.

e.g. $\exists z > 0\ (z^2 = 2)$ is same as $\exists\ z\ (z > 0 \land z^2 = 2)$

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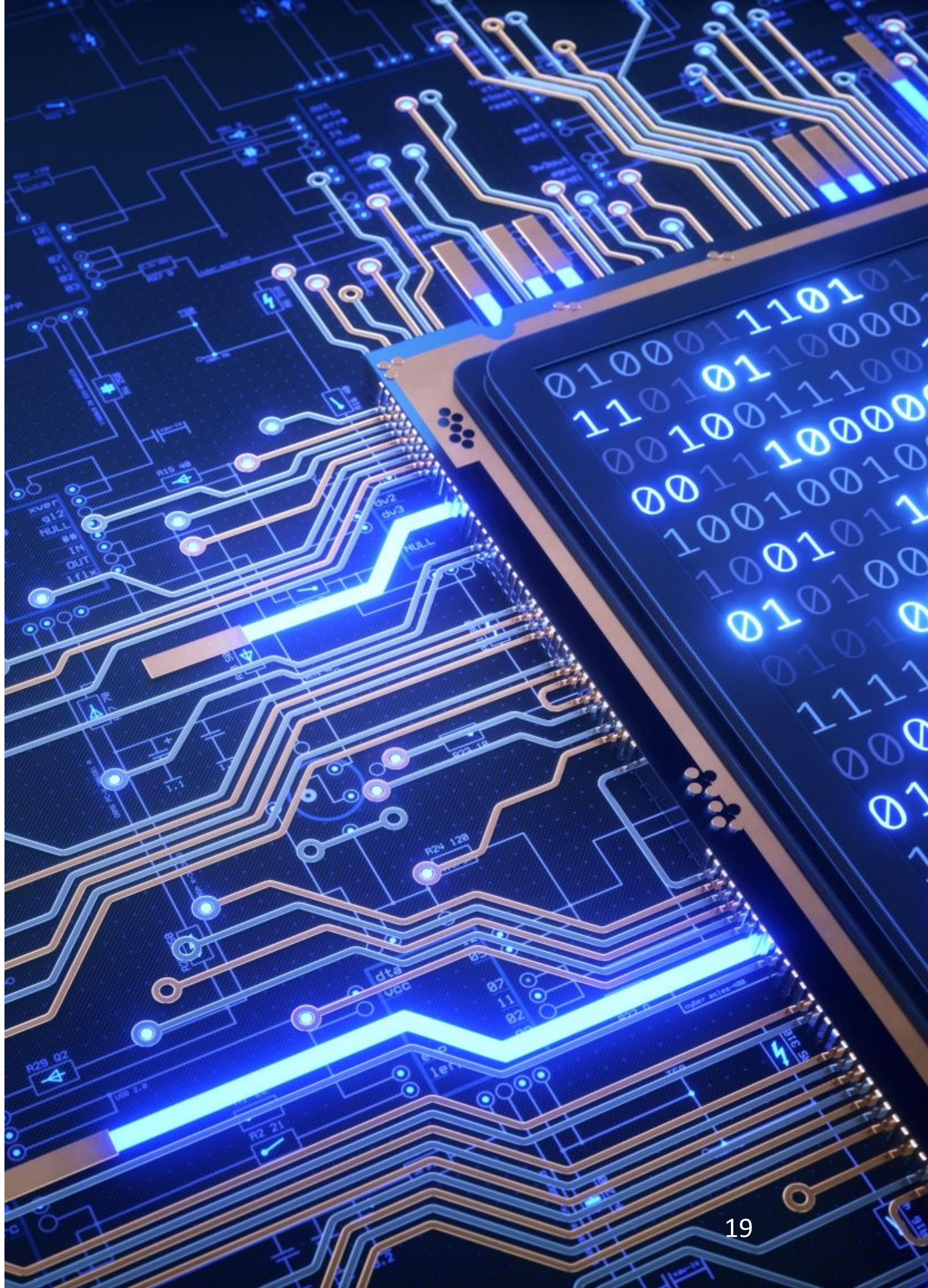


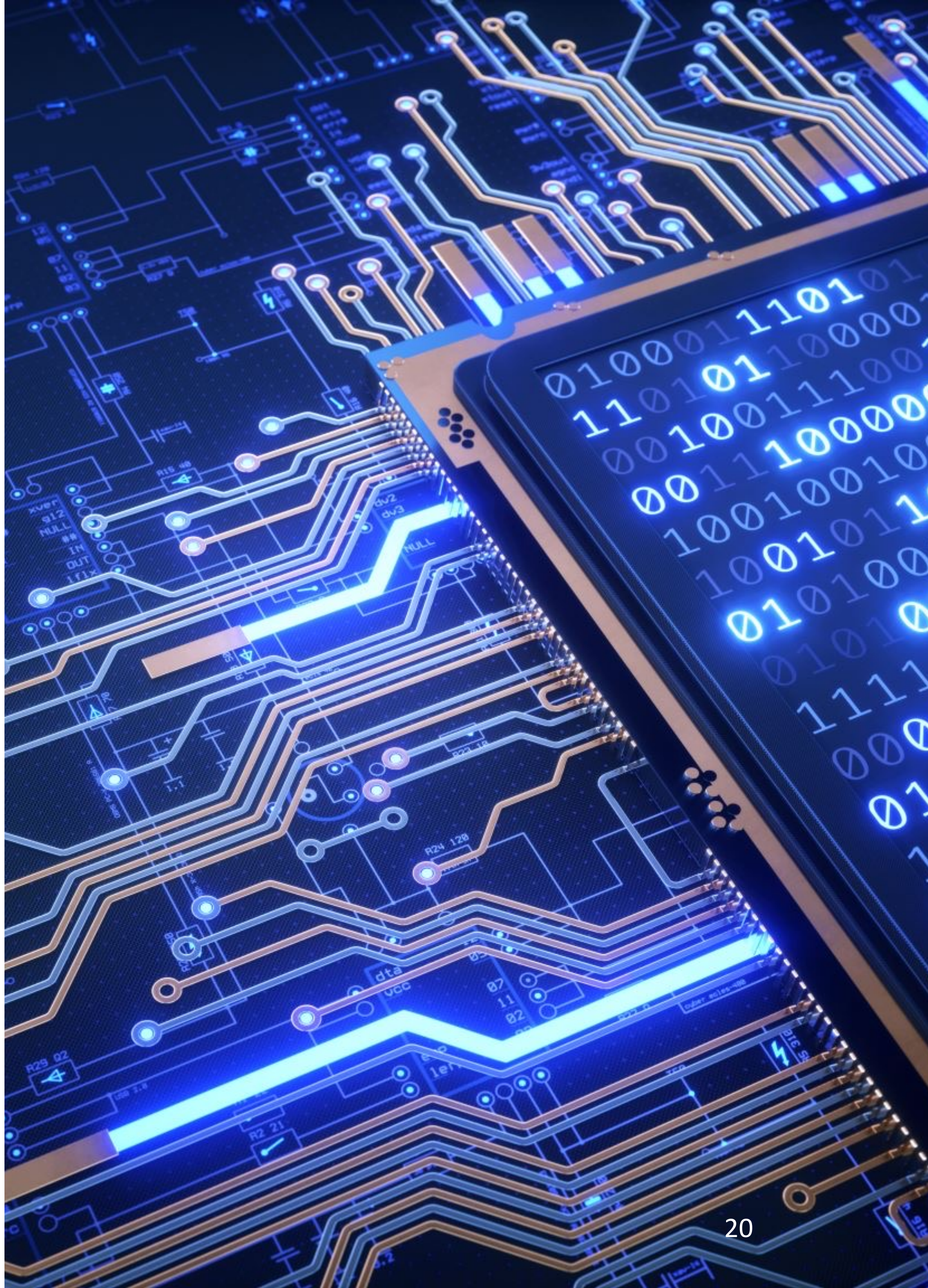
Quantifiers: Precedence, Scope of Variables

- The quantifiers ∀ and ∃ have higher precedence than all logical operators

  e.g. $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

  but not $\forall x\, (P(x) \vee Q(x))$
- The occurrence of a variable is said to **bound** when a quantifier uses it, otherwise it is said to be **free**

e.g. In $\exists x\, (x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free



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Quantifiers: Precedence, Scope of Variables

- The part of a logical expression to which a quantifier is applied is called the **scope** of that quantifier.

– In $\exists x\,(P(x) \land Q(x)) \lor \forall x\,R(x)$, all variables are bound.
The scope of the first quantifier, $\exists x$, is the expression $P(x) \land Q(x)$ and the scope of the second quantifier, $\forall x$, is the expression $R(x)$.

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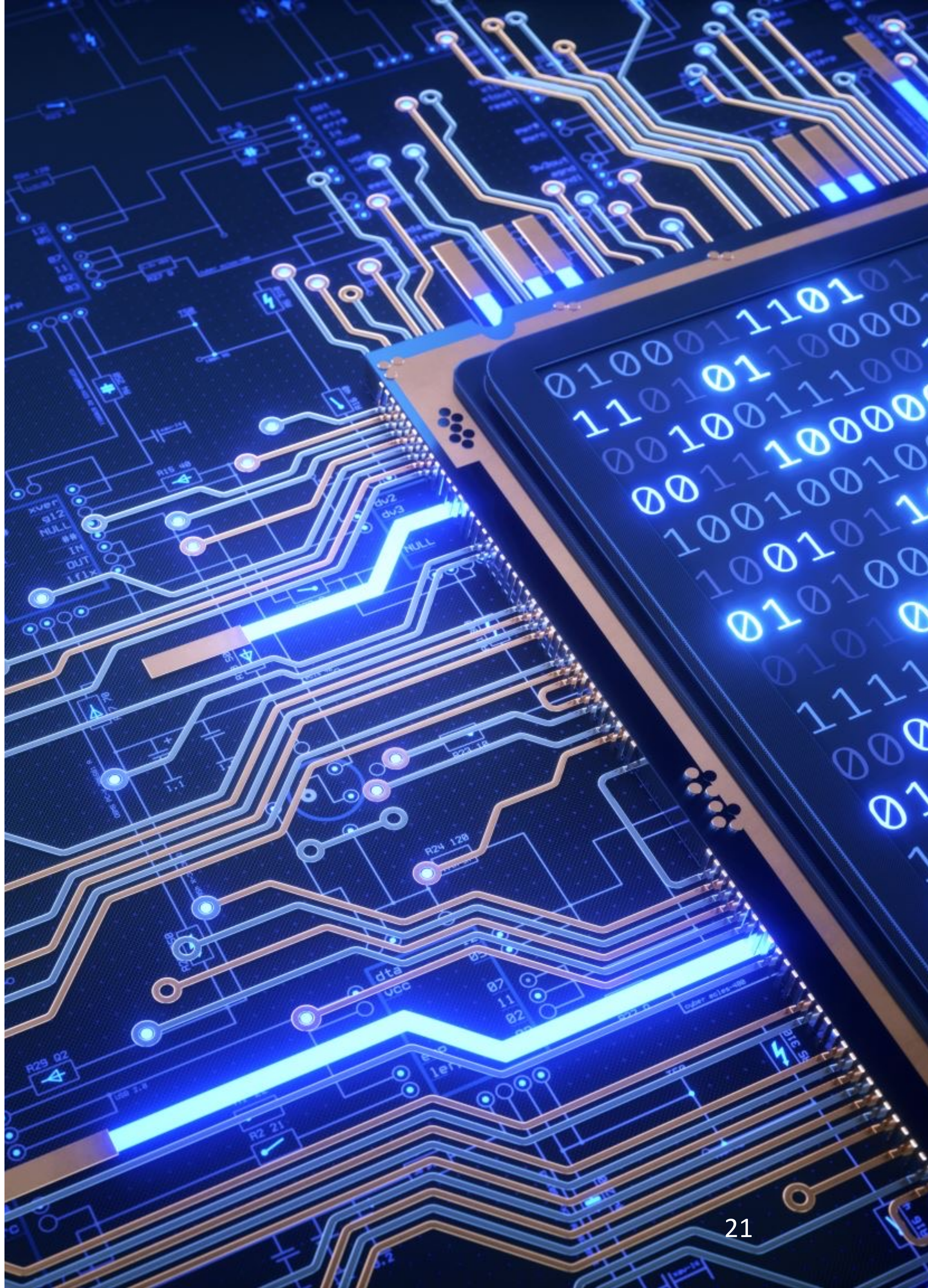

Quantifiers: Logical Equivalence

- Statements involving predicates and quantifiers are *logically equivalent* if and only if
  - they have the *same truth value* no matter which predicates are substituted into these statements
  - and which domain of discourse is used for the variables in these propositional functions

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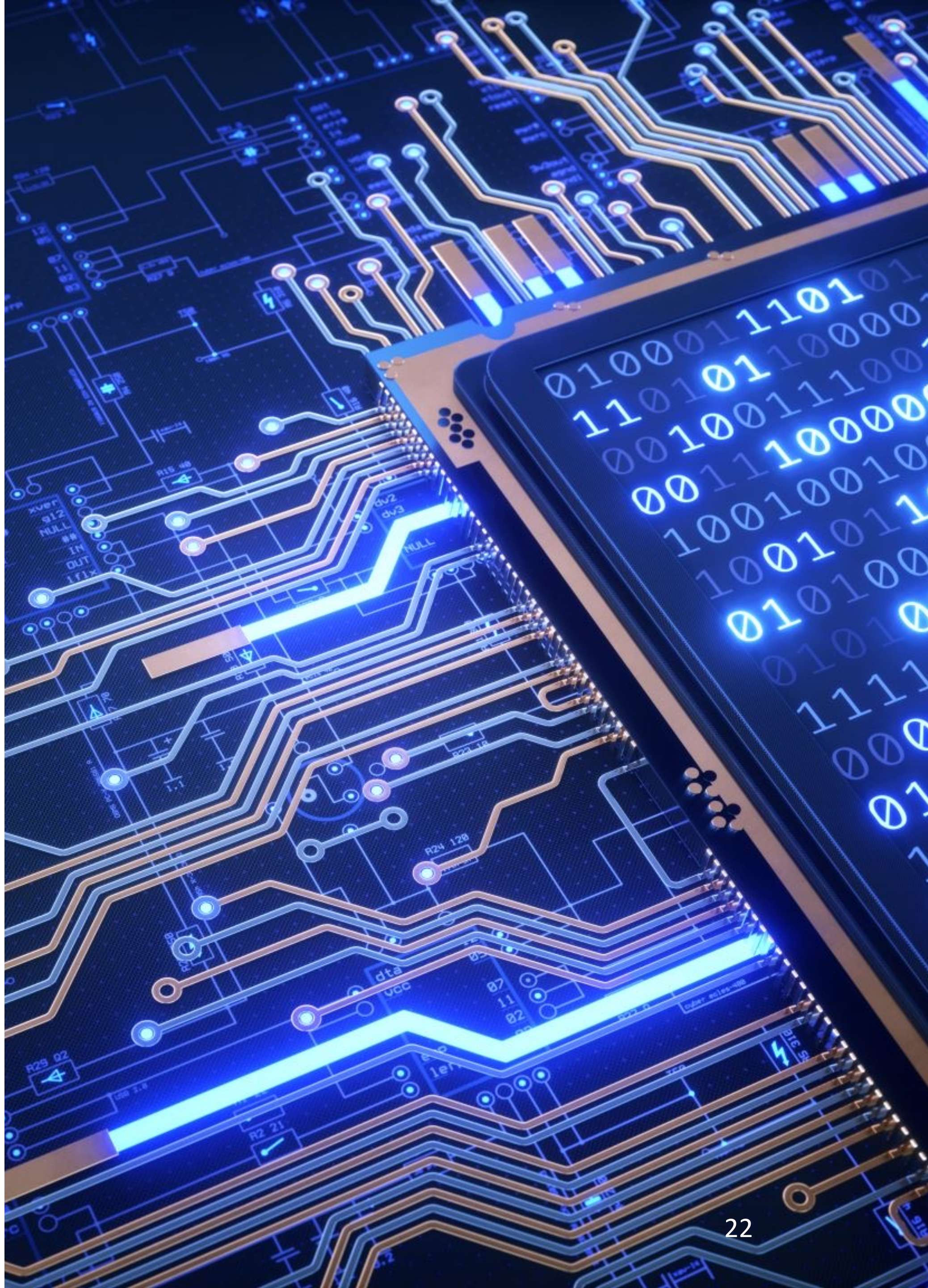

Quantifiers: Logical Equivalence

- The notation $S \equiv T$ is used to indicate that, two statements S and T involving predicates and quantifiers are logically equivalent.

- It may be noted that, we can distribute a universal quantifier over a conjunction and an existential quantifier over a disjunction, but not vice-versa.

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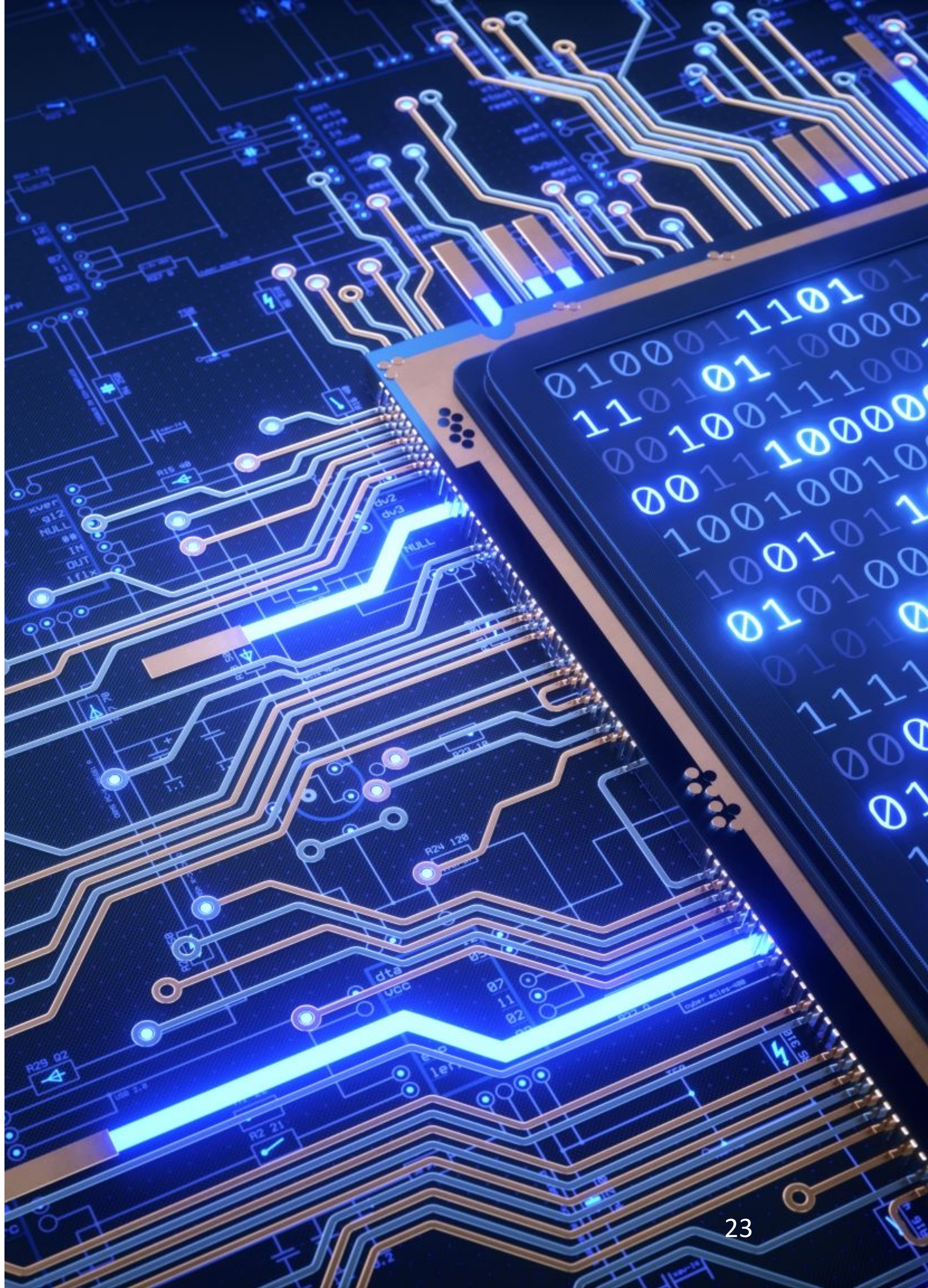


Exercise

- Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

Answer:

Let P(x) be any propositional function that is TRUE for some values in the domain and FALSE for some values.

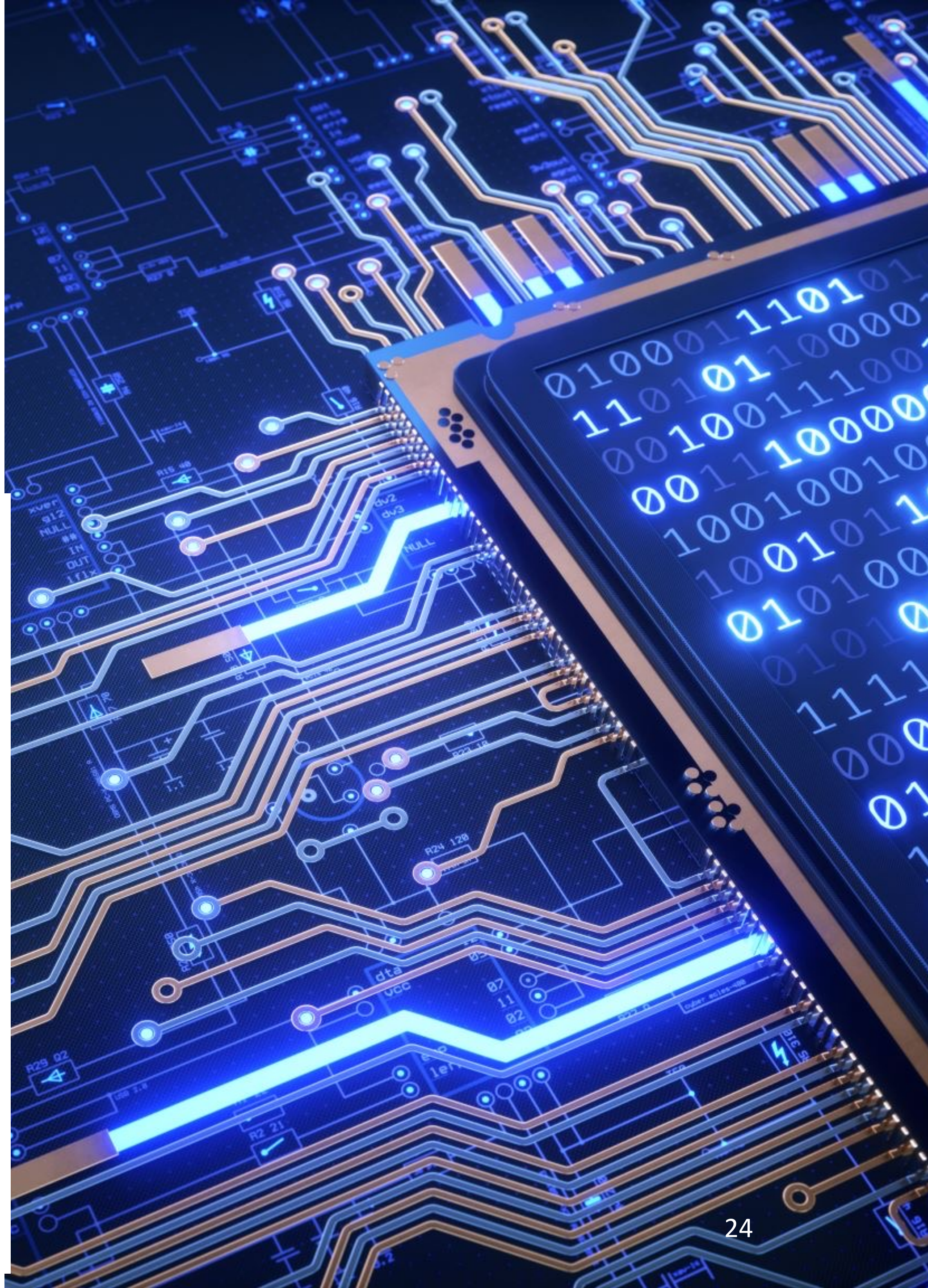
Also, let Q(x) be a propositional function that is FALSE for all the values. Then, $P(x) \rightarrow Q(x)$ will be sometimes FALSE and sometimes TRUE.

In effect, $\forall x(P(x) \rightarrow Q(x))$ is always FALSE,

Again, since P(x) is TRUE for some values and FALSE for some values in the domain, $\forall x P(x)$ is FALSE. Also, $\forall x\ Q(x)$ is TRUE.

In effect, $\forall x P(x) \rightarrow \forall x Q(x)$ is TRUE always.

The given statements $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are not equivalent.

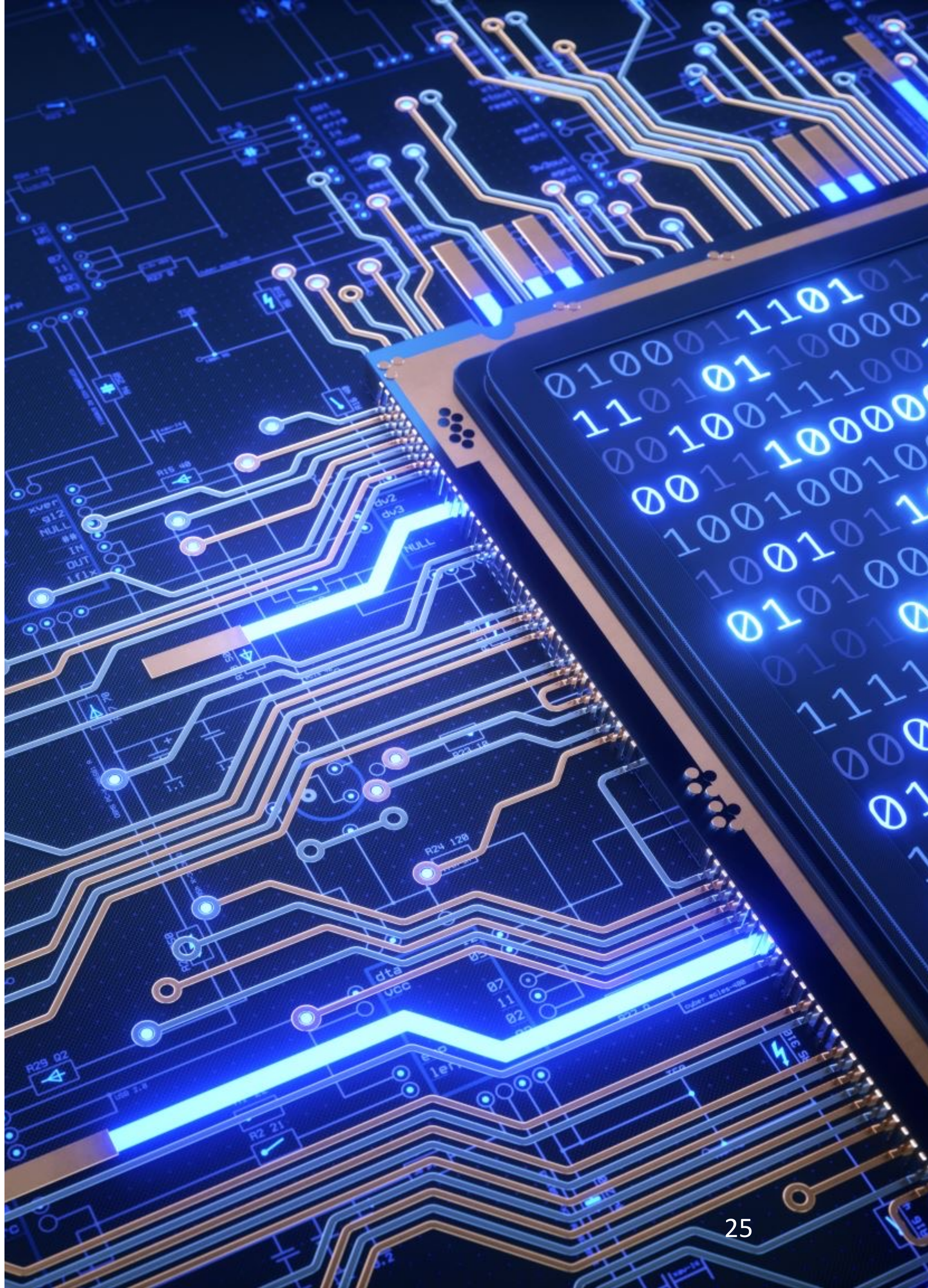


Exercise

- Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

Answer:

- Both statements are TRUE precisely when at least one of P(x) and Q(x) is TRUE for at least one value of x in the domain.
- And both statements are FALSE when both P(x) and Q(x) are FALSE.
- Hence, it is evident that, both the statements always have same truth value.
- Thus, they are equivalent.



Negation of Quantifier Expressions

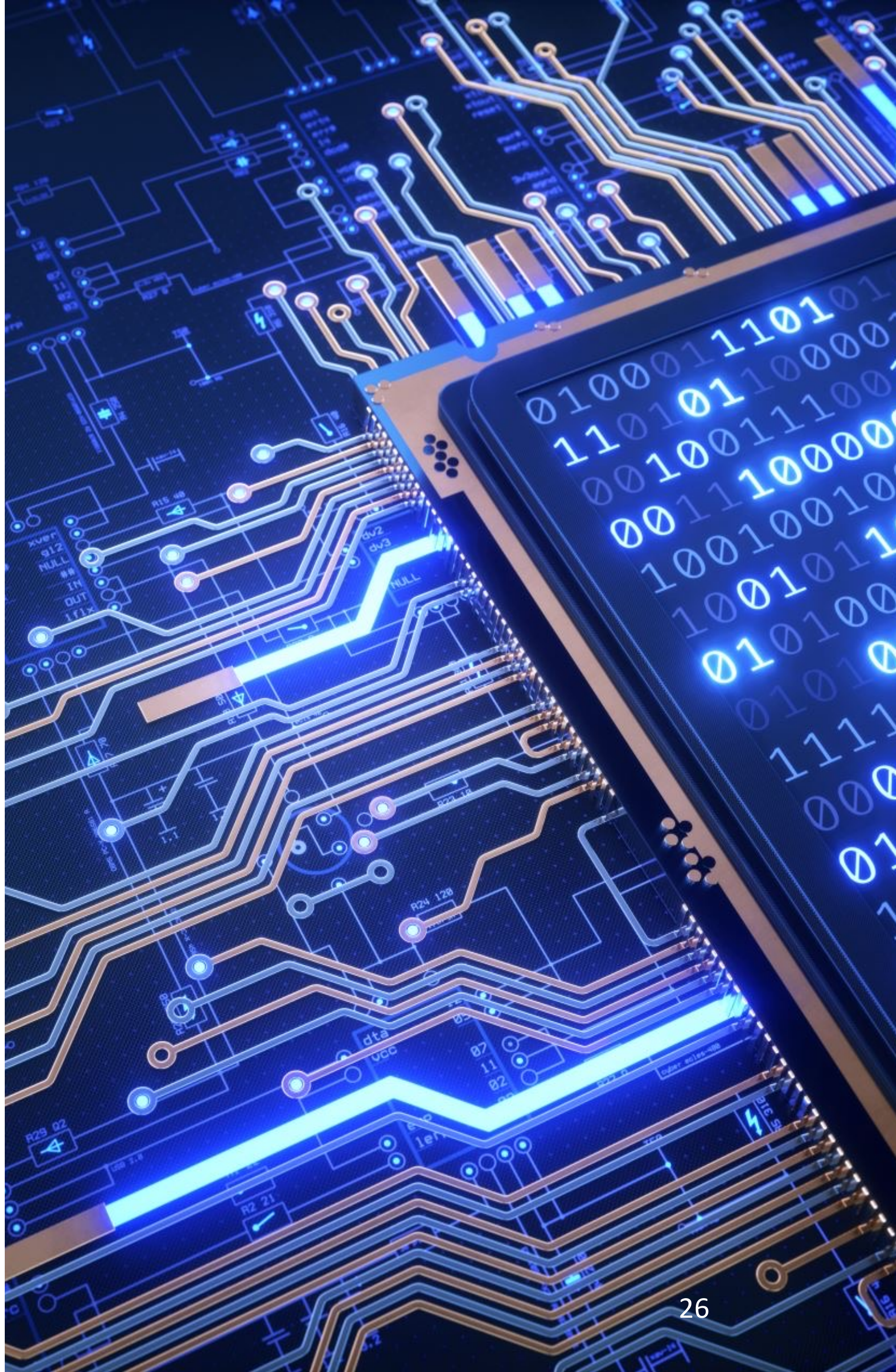
- The negation of quantified expressions $\forall x\, P(x)$ and $\exists x\, Q(x)$ are represented by $\neg\forall x\, P(x)$ and $\neg\exists x\, Q(x)$ respectively and the following equivalences hold.

$$\left.\begin{array}{l} \neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, Q(x) \equiv \forall x\, \neg Q(x) \end{array}\right\} \text{De Morgan's Laws for Quantifiers}$$

- Example:

1) “Every student in your class has taken a course in calculus” = $\forall x\, P(x)$ then
$\neg\forall x\, P(x)$ = “It is not the case that every student in your class has taken a course in calculus”
≡ “There is a student in your class who has not taken a course in calculus” = $\exists x\, \neg P(x)$

2) Workout the same way as above for
“There is a student in this class who has taken a course in calculus.”



Translation of English Sentence to Logical Expressions

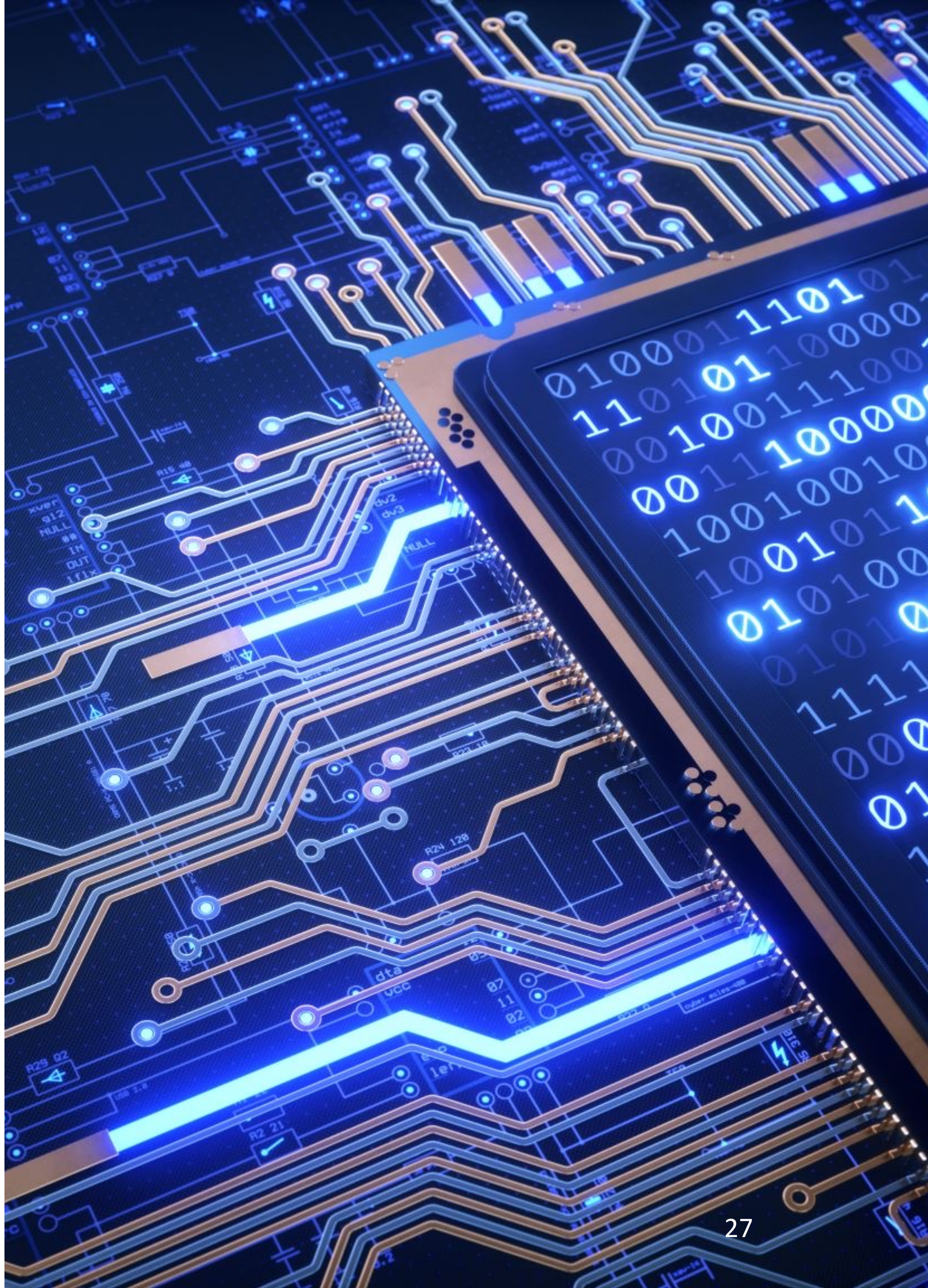
- Useful in the fields of Artificial intelligence and software engineering and logic programming

Example 1: Express the statement “Every student in this class has studied calculus” using predicates and quantifiers when the domain of discourse is ‘all persons’

Answer: $\forall x\,(S(x) \rightarrow C(x))$

Example 2: Express the statement “Every student in this class has visited either Canada or Mexico” using predicates and quantifiers.

Answer : $\forall x\,(S(x) \rightarrow (C(x) \vee M(x)))$

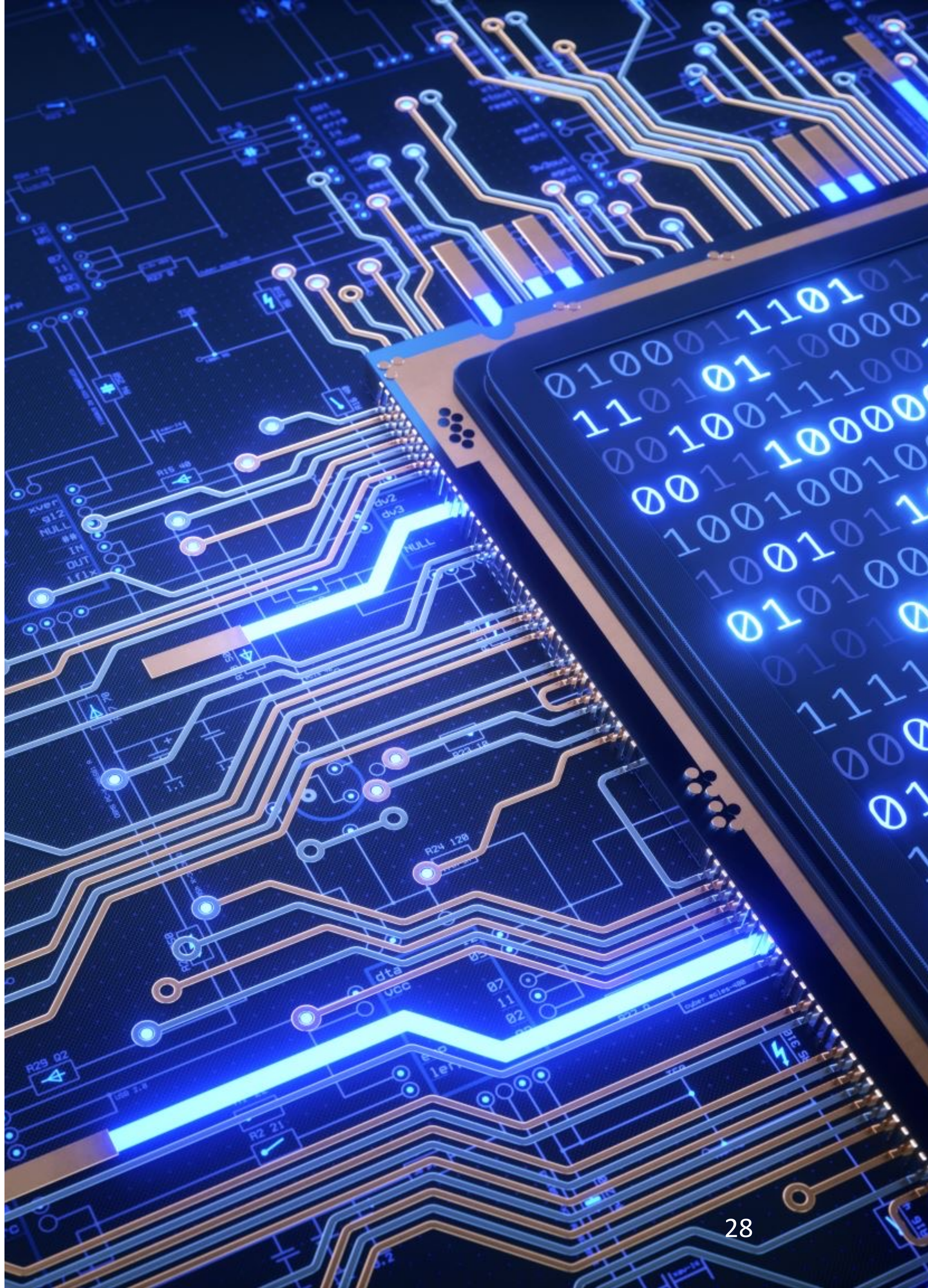


Translation of English Sentence to Logical Expressions

Example: Use predicates and quantifiers to express the system specification

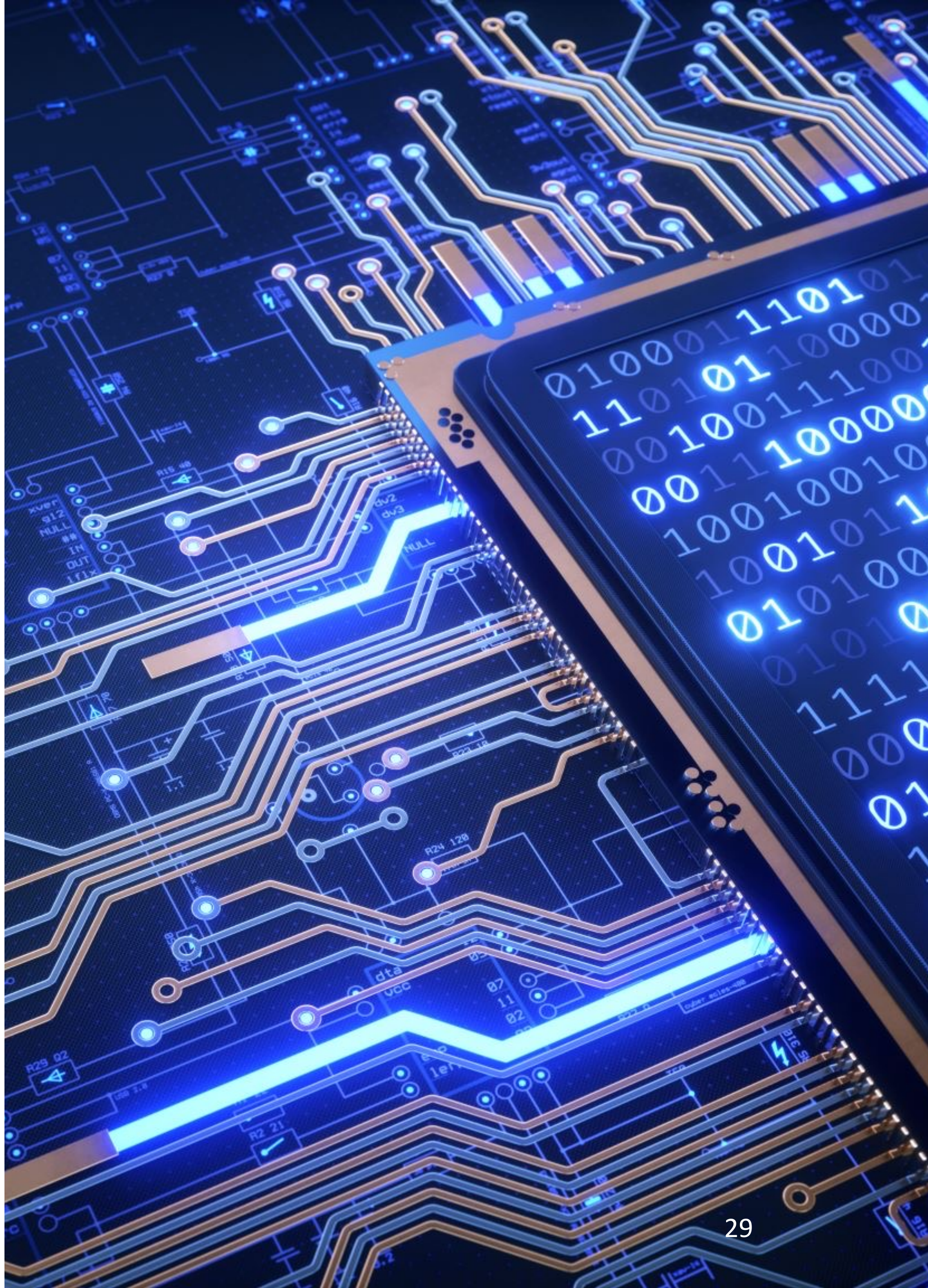
"Every mail message larger than one megabyte will be compressed"

Answer: $\forall m\,(S(m, 1) \rightarrow C(m))$



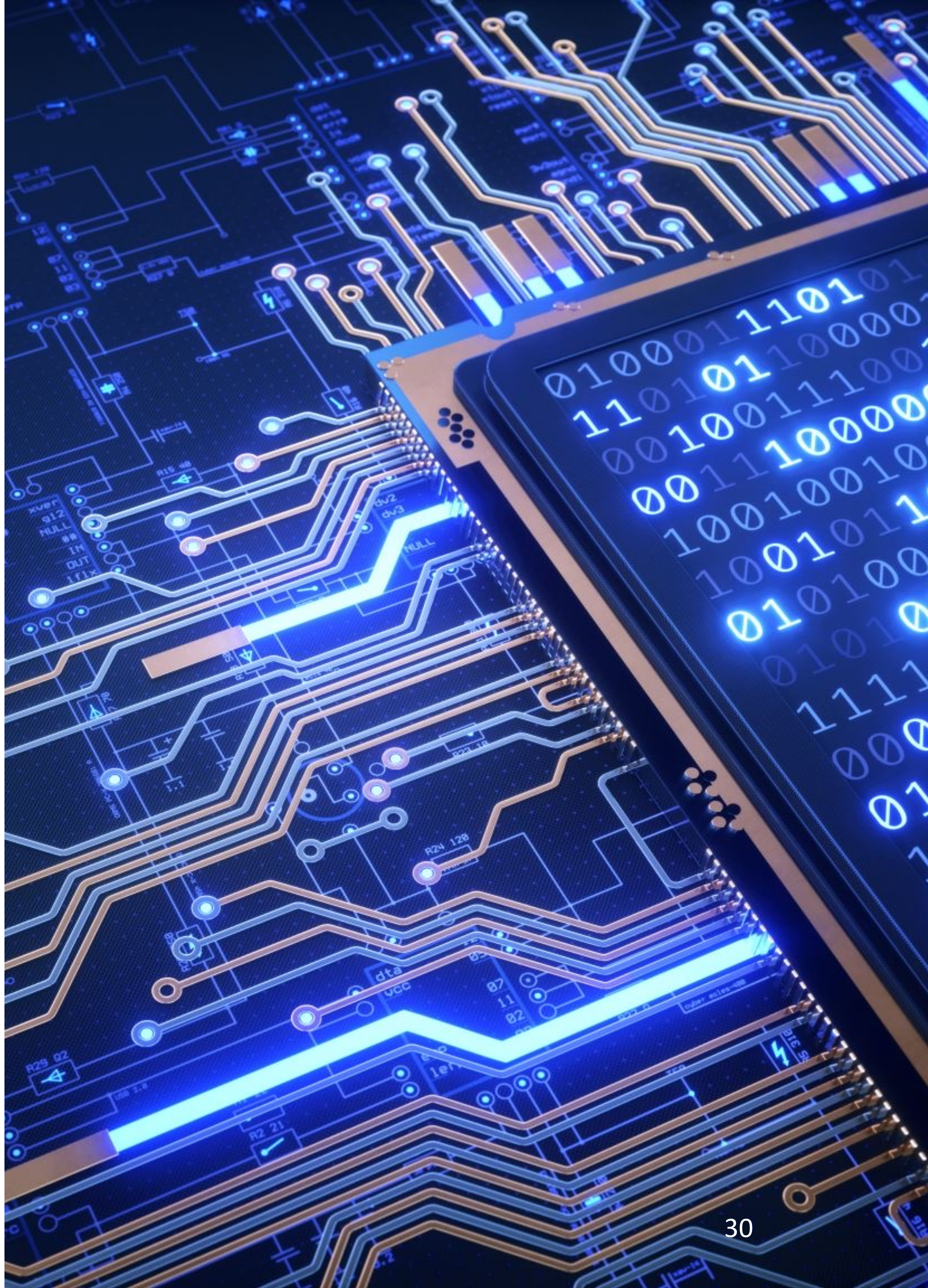
Applications of Predicate Logic

- It is the formal notation for writing perfectly clear, concise, and unambiguous mathematical definitions, axioms, and theorems for any branch of mathematics.
- Supported by some of the more sophisticated database query engines.
- Basis for automatic theorem provers and many other Artificial Intelligence systems.



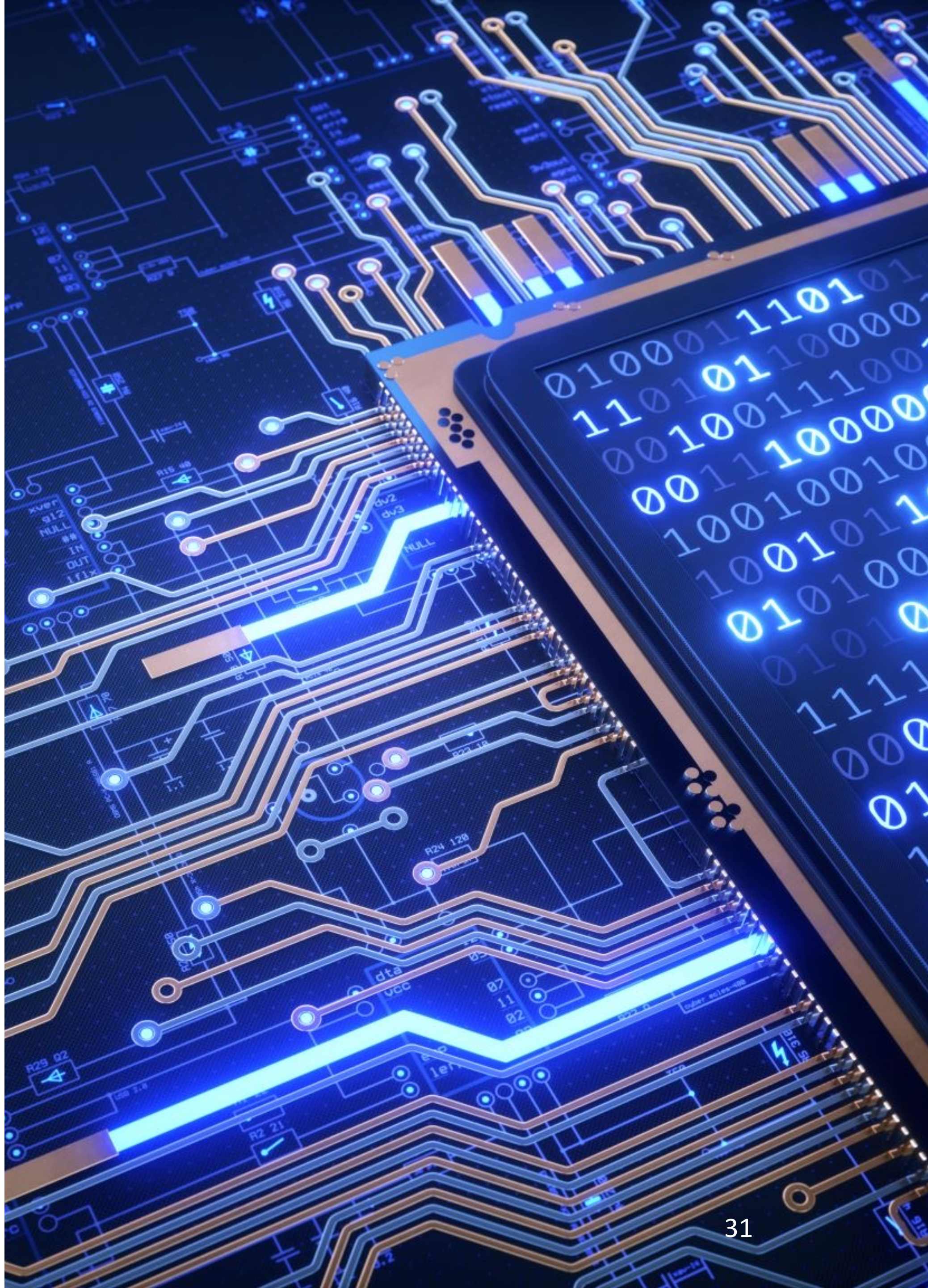
Applications of Predicate Logic

- Basis for query languages such as SQL in database systems
- Used in knowledge representation and reasoning in AI systems
- Formalizing mathematical theories and proofs
- Applied in natural language understanding and semantics in NLP systems
    - to represent the meaning of sentences and extract semantic relationships between entities



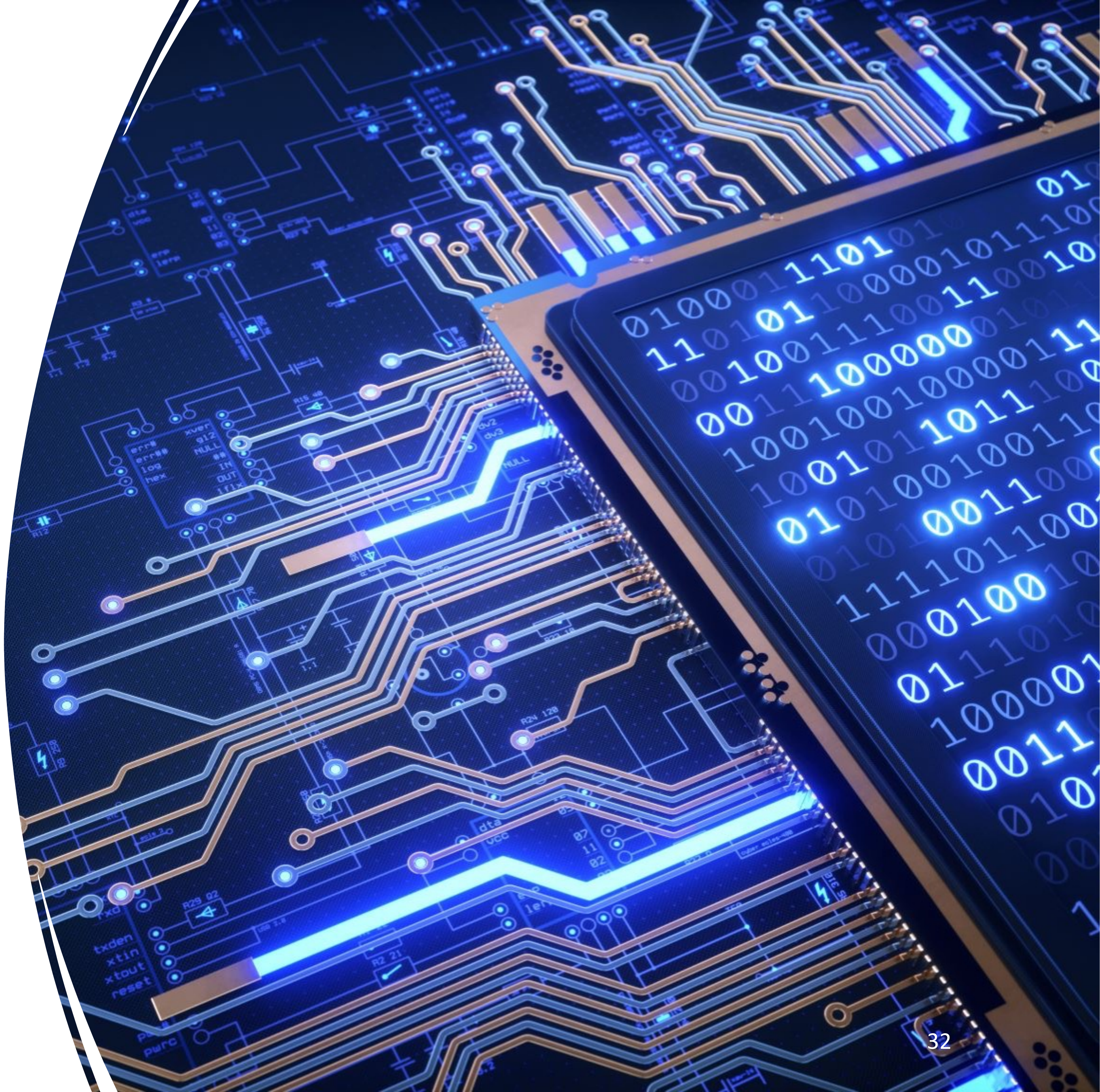
Applications of Predicate Logic

- Used in formal methods for verifying the correctness of software and hardware systems
- Used in digital circuit design and verification
- Used in software engineering for formal specification and verification of software systems
- Used for representing knowledge and semantic relationships on the web



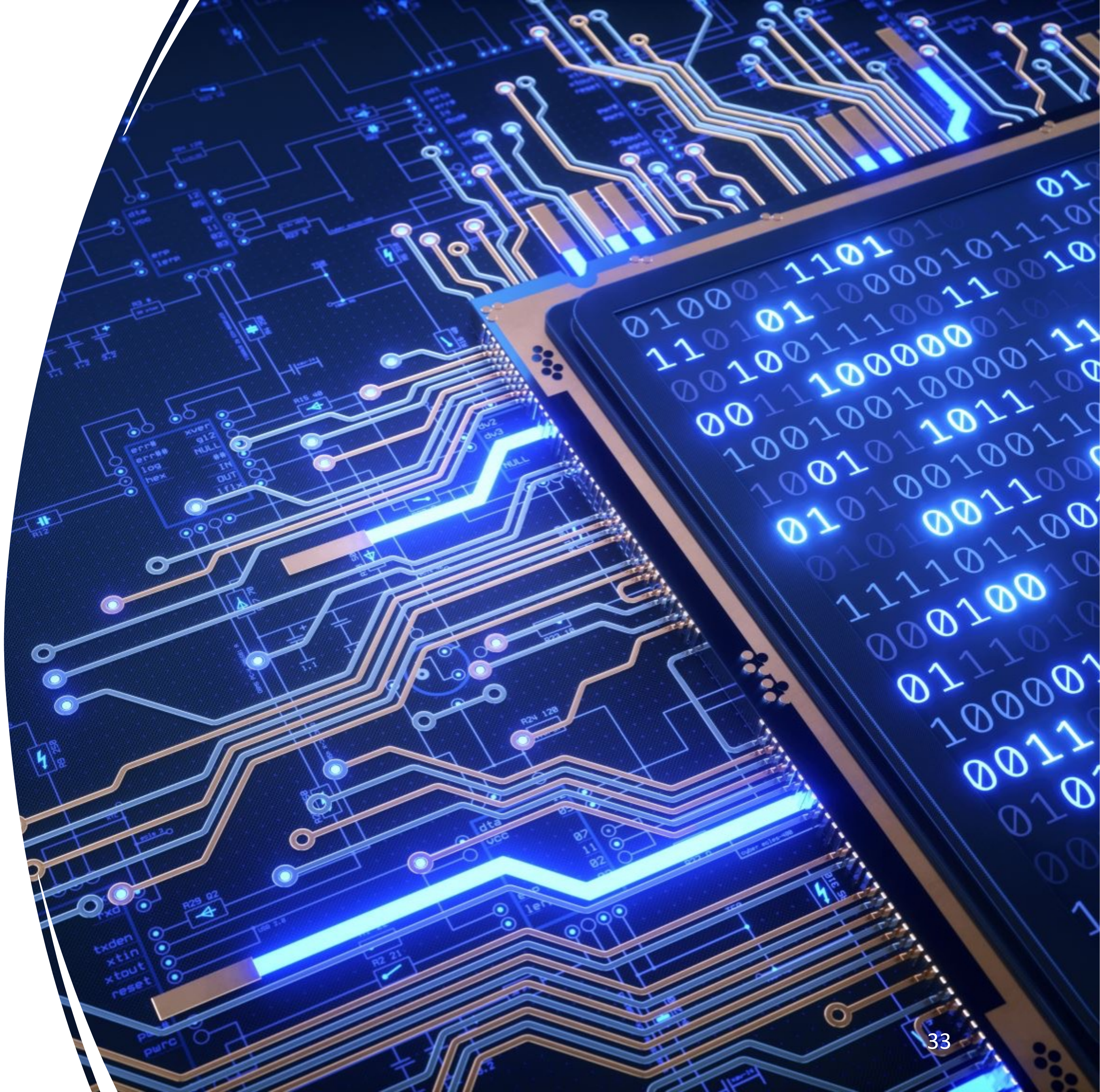
Summary

- Subjects and Predicates
- Preconditions and Postconditions
- Quantifications
    - Existential quantifiers ∃
    - Universal quantifiers ∀
- Translation of English sentences to logical expressions



Reference

Rosen, K. H. (2012). *Discrete mathematics and its applications (7th Edition)*. McGraw-Hill.
Chapter 1



See you next time!

Thank you!