Course: Mathematics for IT Professionals

Lecture 5

Sets

By
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Outline

The topics to be treated in this lecture are:

- Forms of set representation
- Number set
- Cardinality of set
- Subset
- Power set
- Ordered tuple
- Cartesian product of sets
- Set operations
- Set Identities
- Computer representation of set elements
- Application of Sets

Lecture Learning Outcomes

At the end of the session, you will be able to

- understand the two forms of set representation
- differentiate between proper and improper subsets
- note how to write the elements in a power set
- understand the Venn diagram and its representation
- compute the Cartesian product of sets
- know how to represent the elements of a set in a computer

Introduction

- A **set** is a group or collection of objects
- A set is a fundamental discrete structure in which many important discrete structures such as the following are built
  - *Combinations*: unordered collections of objects used in counting
  - *Relations*: sets of ordered pairs that represent relationships between objects
  - *Graphs*: sets of vertices and edges that connect vertices
  - *Finite state machines*: used to model computing machines

- Examples:

  The set of all students in B.Sc. Information Technology

  The set of villages in Ghana

  The set of integers less than 100
  and so on…

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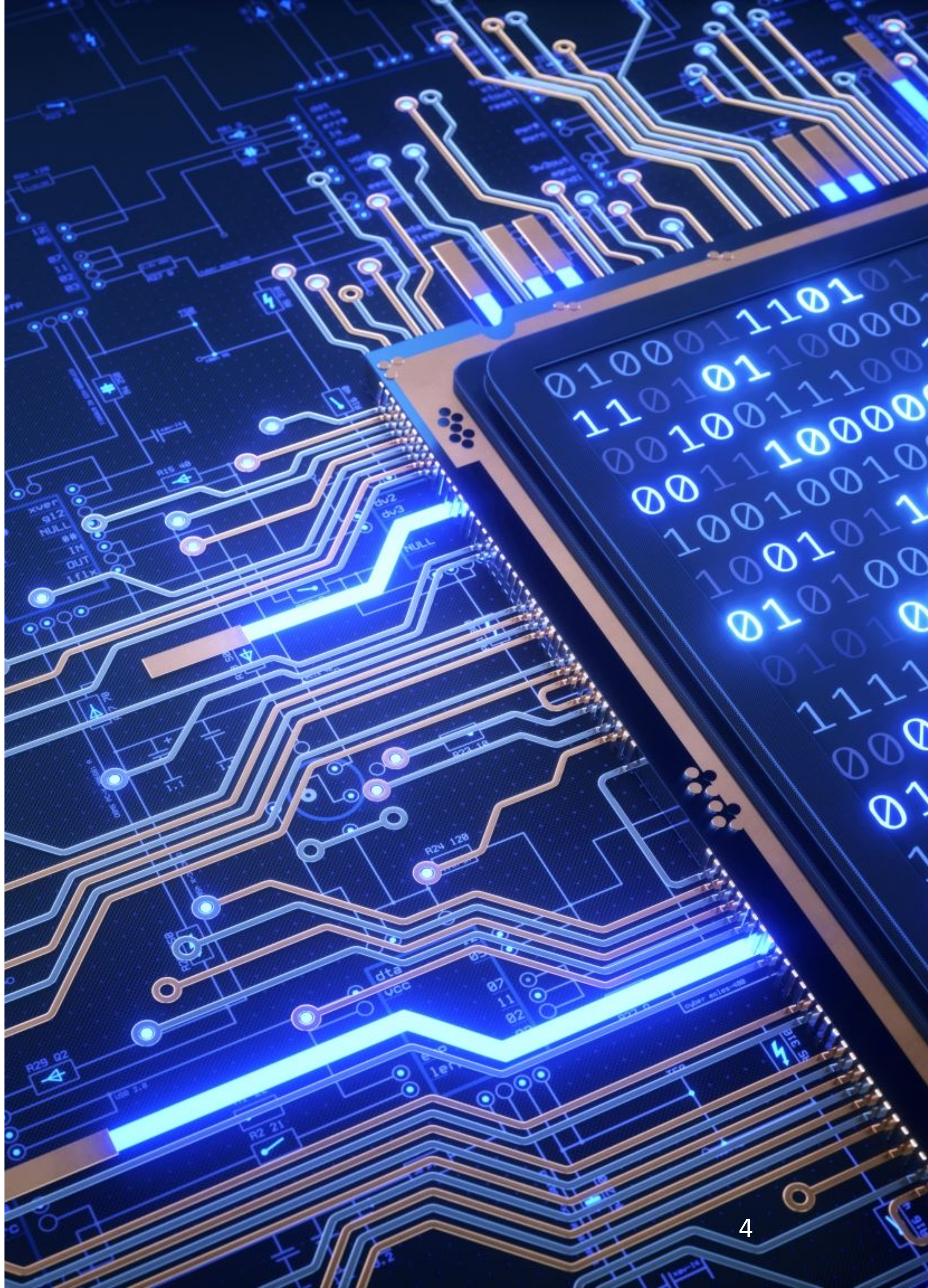

Definition and Notation

- A *set* is an unordered collection of objects
- The objects *contained* by a set are called *elements* or *members* of the set.
- The sets are denoted using the uppercase letters A, B, C, … etc.
- The elements of a set are denoted by the small case letters a, b, c,… etc.
- The notation $a \in A$ (read as 'a belongs to A') denotes that, a is an element of the set A.
- The notation $a \notin A$ (read as 'a does not belong to A') denotes that, a is not an element of the set A.

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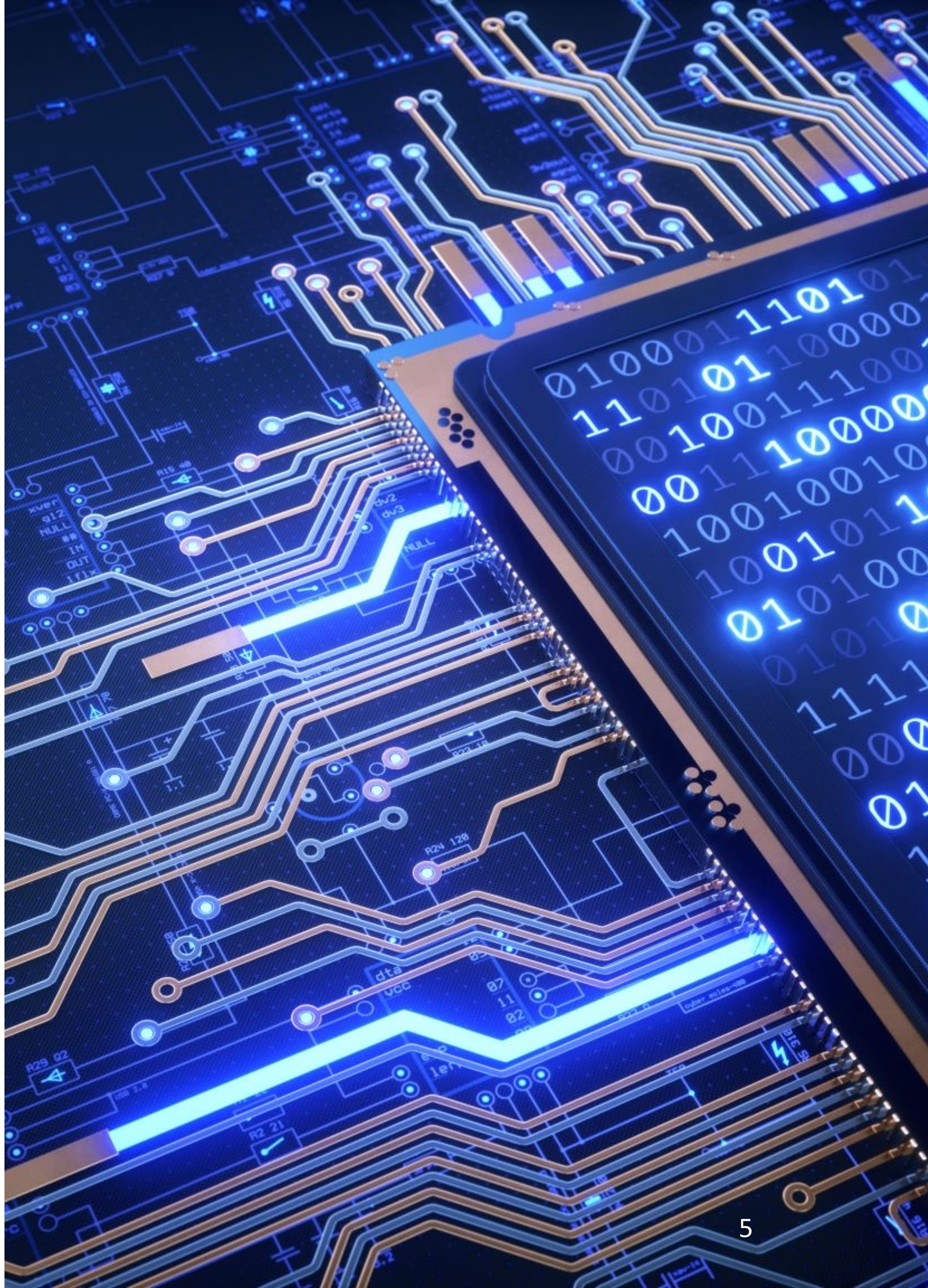
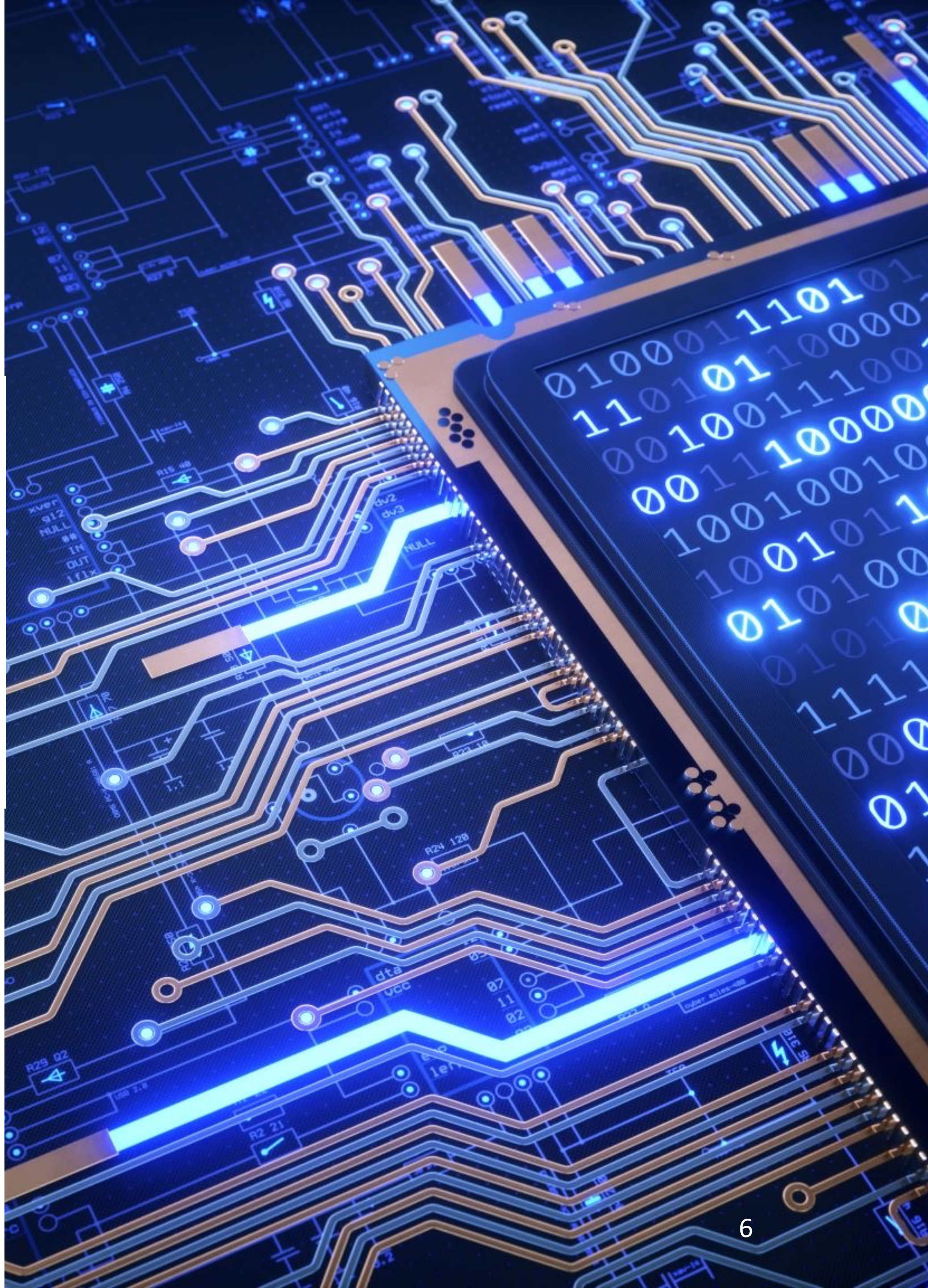


Representation of Sets

- A *set* can be represented in two ways:
  1) ***Roster Method***: All the elements of a set are listed between braces

     Examples:
     - The set of all vowels in English, $V = \{a, e, i, o, u\}$
     - The set of odd positive integers less than 10, $O = \{1, 3, 5, 7, 9\}$
     - The set of positive integers less than 100 $P = \{1, 2, 3, . . . , 99\}$



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Representation of Sets

- The other way of representing sets is:
  2) *Set builder Method*: State the properties to be satisfied by the elements, to be the members of the set

  Examples:
  - The set V of all vowels in English can be represented as

    $V = \{ x \mid x$ is a vowel in English alphabet$\}$
  - The set O of all odd positive integers less than 10 is represented as

    $O = \{ x \mid x$ is an odd positive integer less than 10$\}$

    or

    $O = \{x \in \mathbb{Z}^+ \mid x$ is odd and $x < 10\}$
  - The set $\mathbf{Q}^+$ of all positive rational numbers can be written as

    $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \frac{p}{q}$, for some positive integers p and $q\}$

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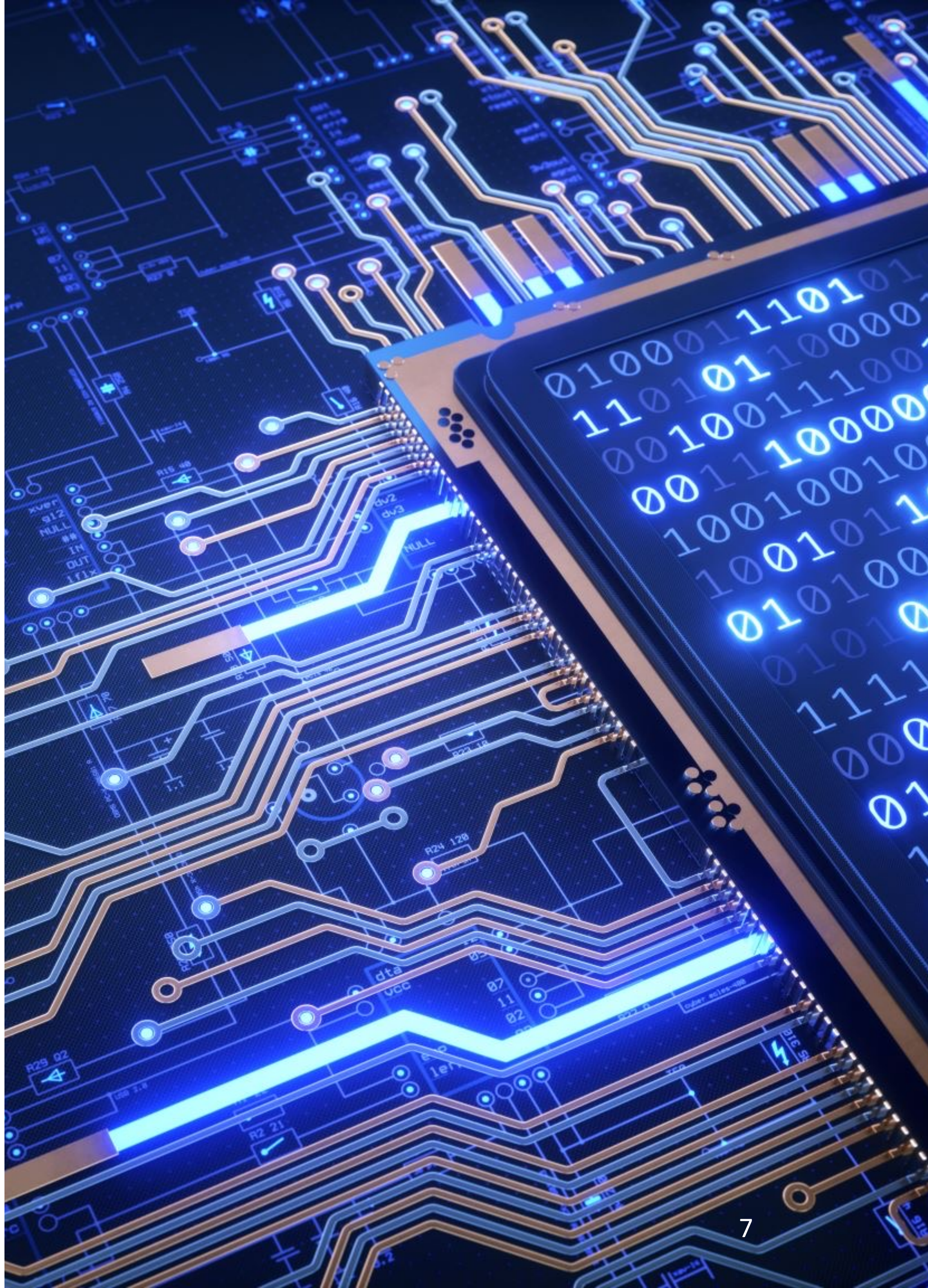
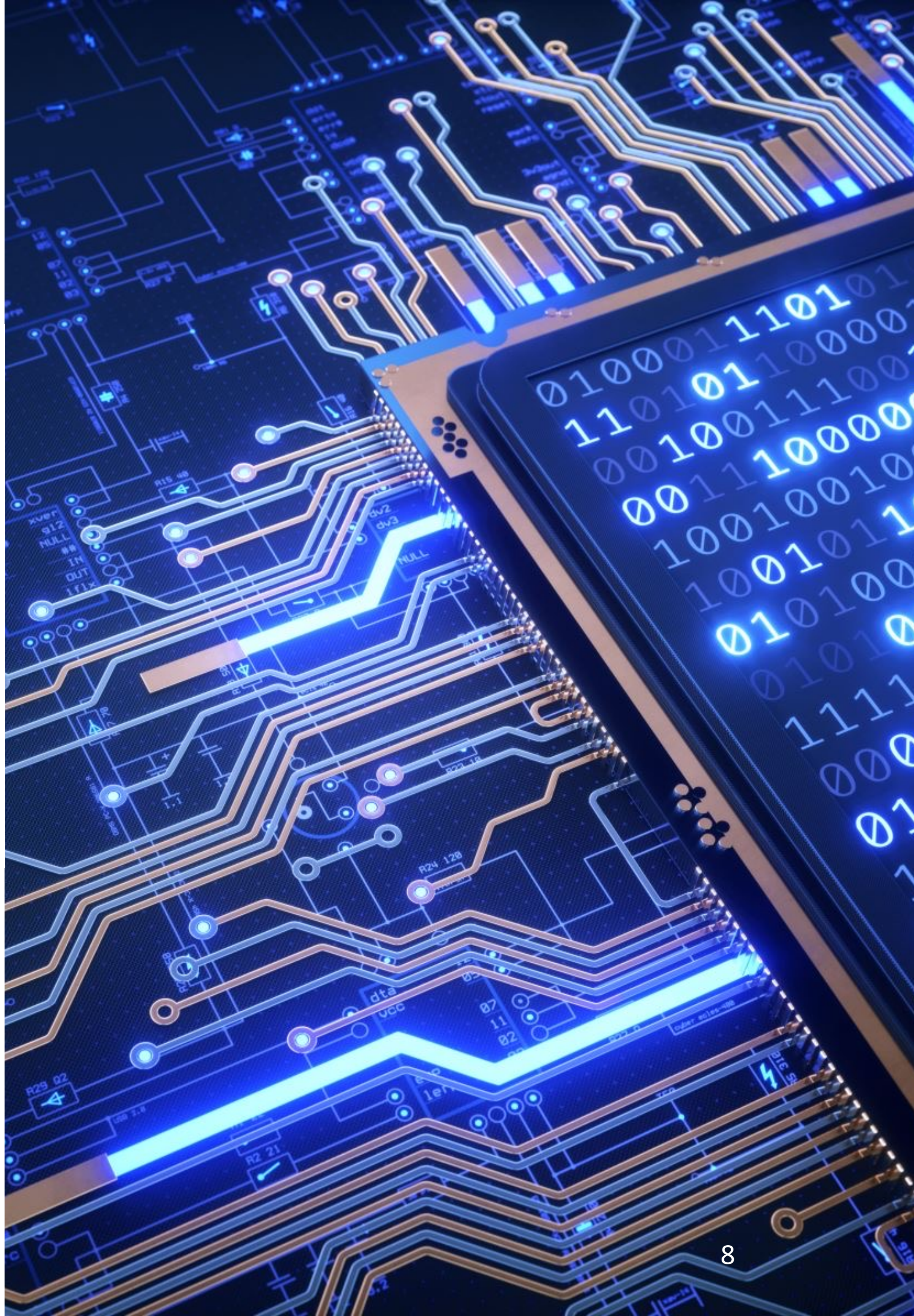


Number Sets

- The following sets, each denoted using a boldface letter, play an important role in discrete mathematics:

  N = {0, 1, 2, 3, . . .}, the set of **natural numbers** (0 may not be)
  Z = {. . . ,−2,−1, 0, 1, 2, . . .}, the set of **integers**
  $\mathbf{Z}^{+}$ = {1, 2, 3, . . .}, the set of **positive integers**
  Q = {*p*/*q* | *p* ∈ **Z**, *q* ∈ **Z**, and *q* ≠ 0}, the set of **rational numbers**
  R, the set of **real numbers**
  $\mathbf{R}_{+}$ the set of **positive real numbers**
  C, the set of **complex numbers**

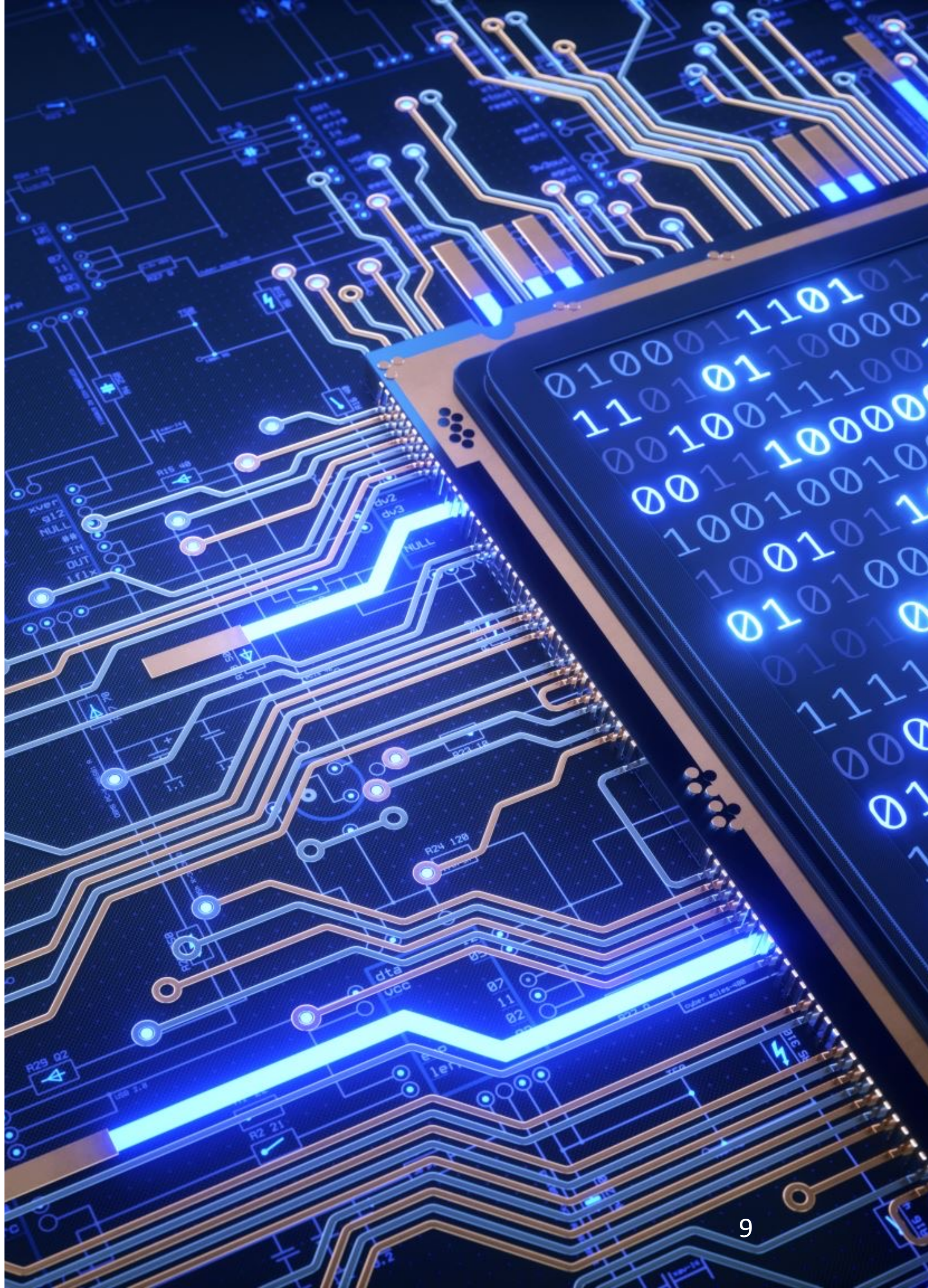


Natural Numbers

We define the Natural Numbers to be:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$

Note that the Naturals are "closed" under addition and multiplication.

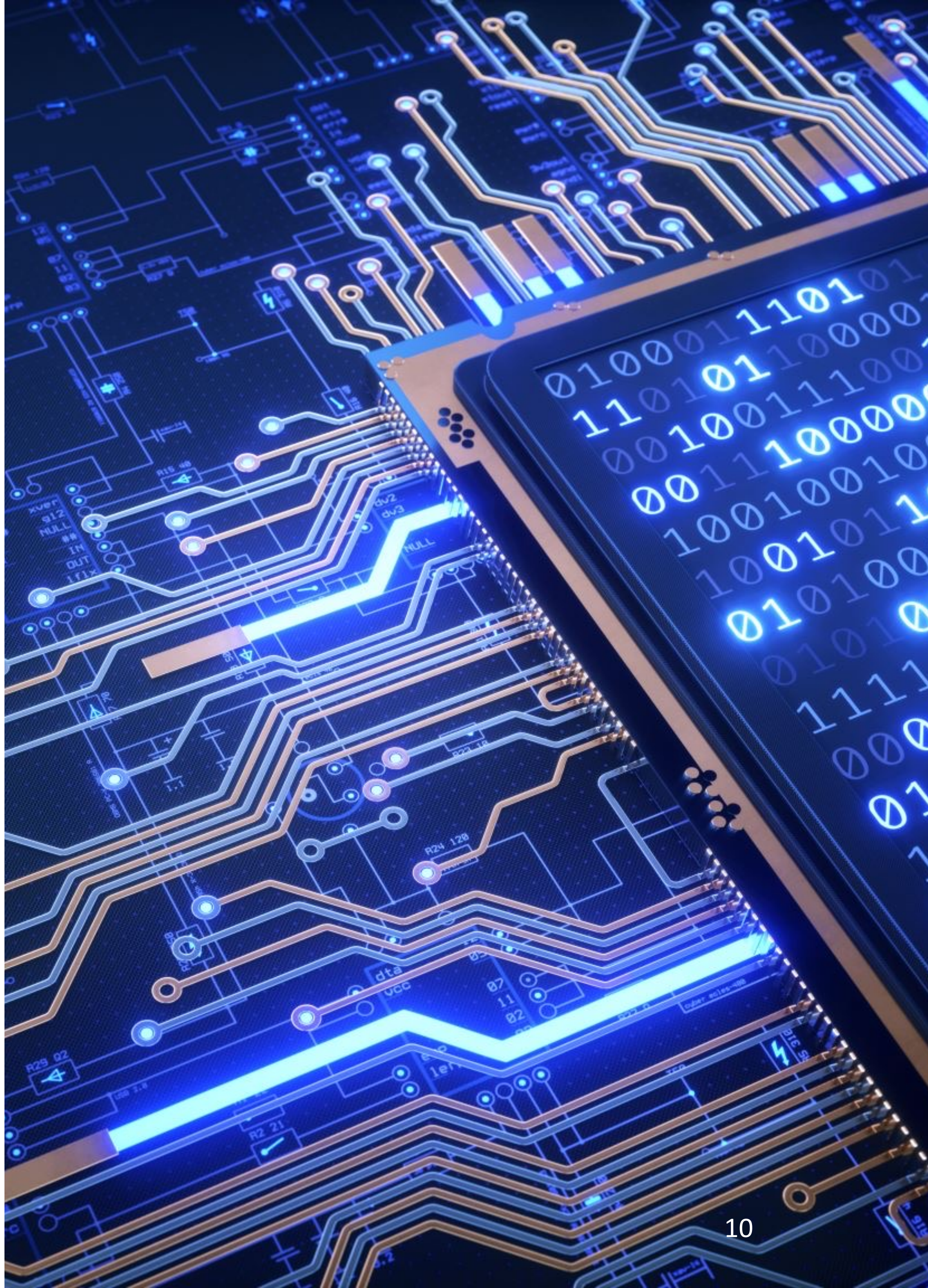


Integers

We define the Integers to be:

Z = {…, -2, -1, 0, 1, 2, 3, …}

Note that **Z** is “closed” under addition, subtraction, and multiplication.

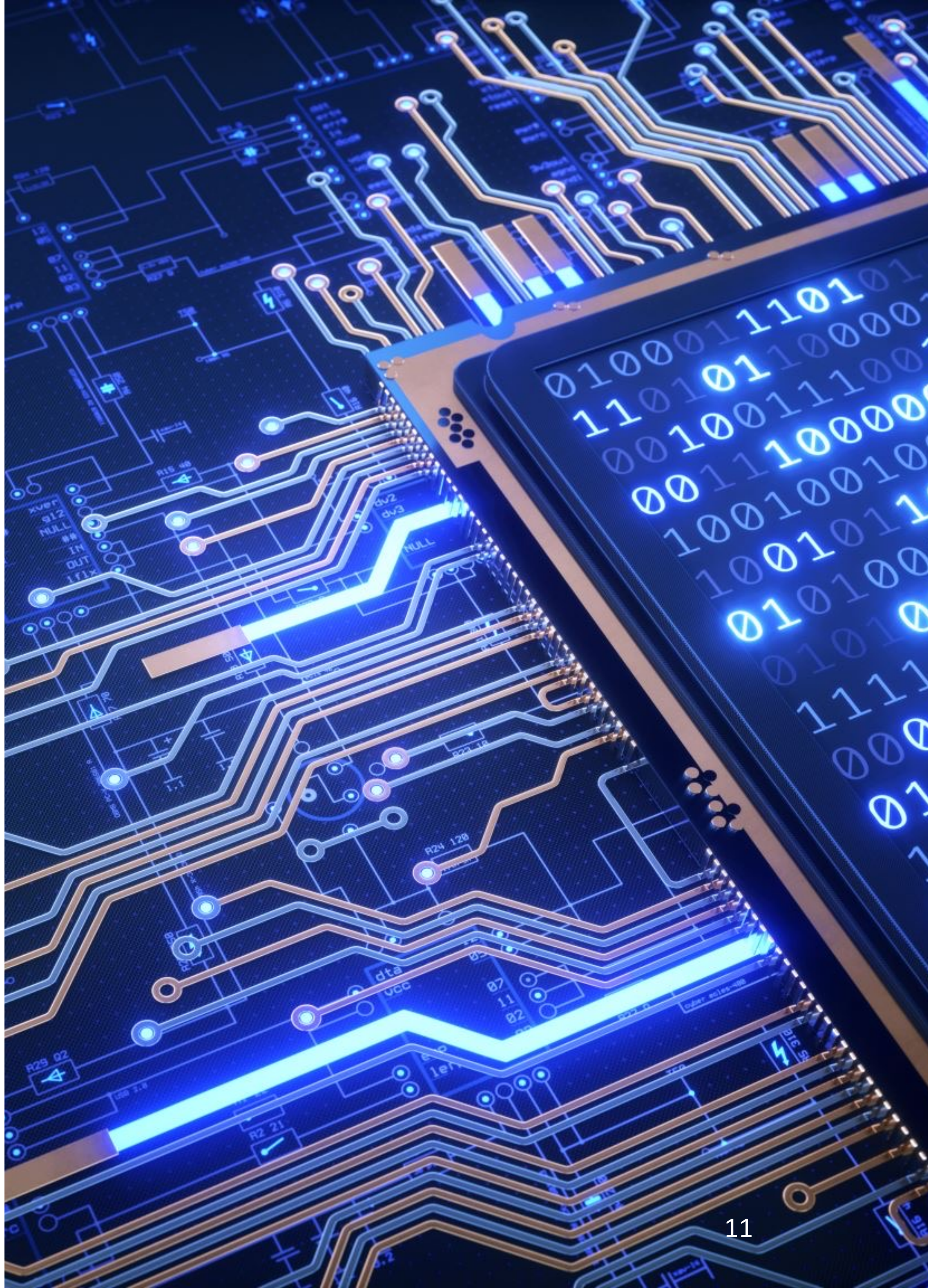


Rational Numbers

We define a Rational number to be:

$\mathbf{Q} = \{p/q \mid p, q \in \mathbf{Z} \text{ and } q \neq 0\}$

Note that **Q** is “closed” under addition, subtraction, multiplication, and non-zero division.

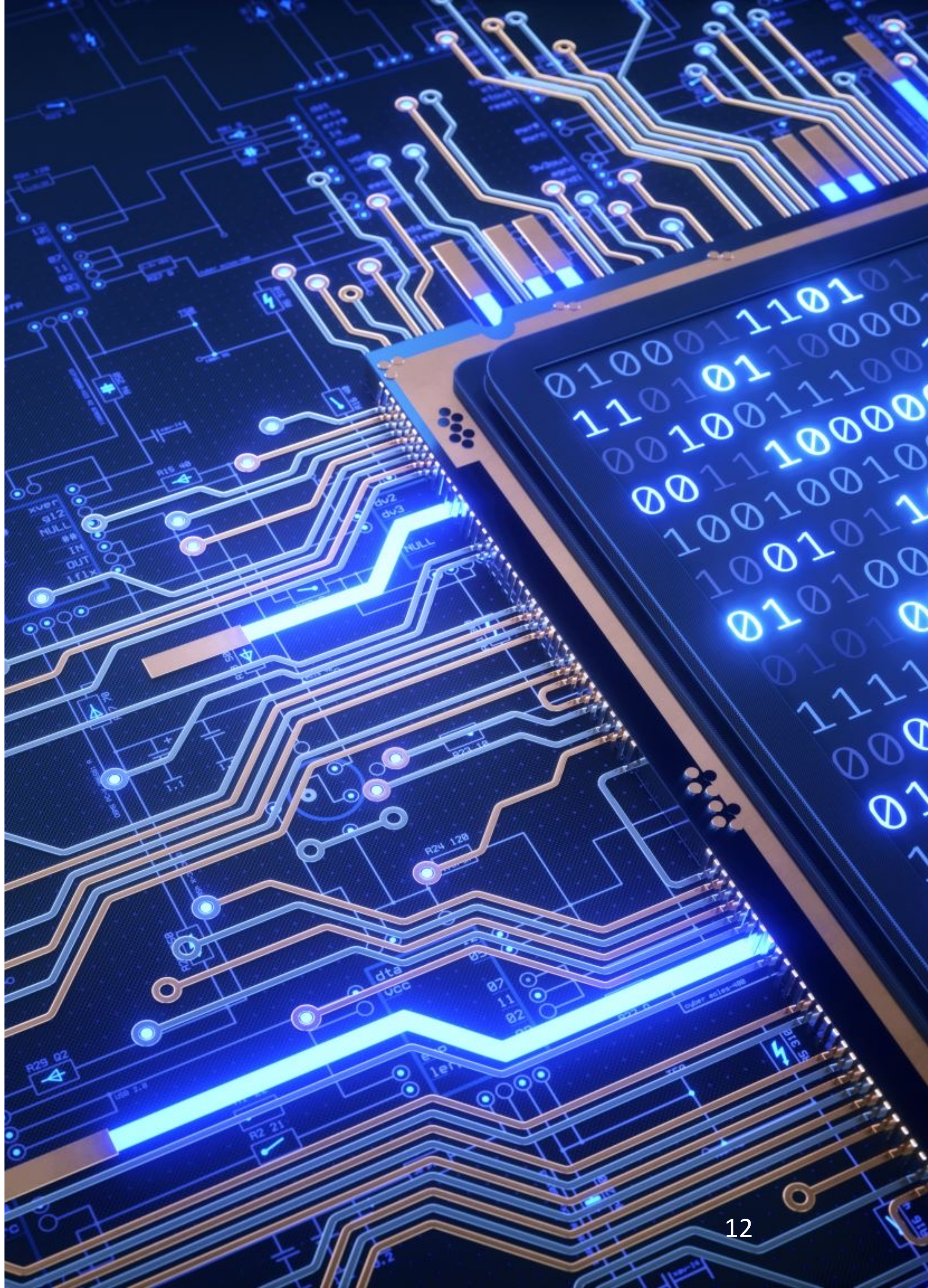


Irrational Numbers

I = {all infinite, non-terminating or non-repeating decimals}
Obviously, irrational numbers are impossible to write down exactly.

Thus, cannot be expressed as a ratio between two numbers and it cannot be written as a simple fraction

We use symbols to represent special values such as ***e*** and $\sqrt{2}$
The Irrationals are closed under + or ×
Irrational numbers cannot be written in a fraction.



Real Numbers

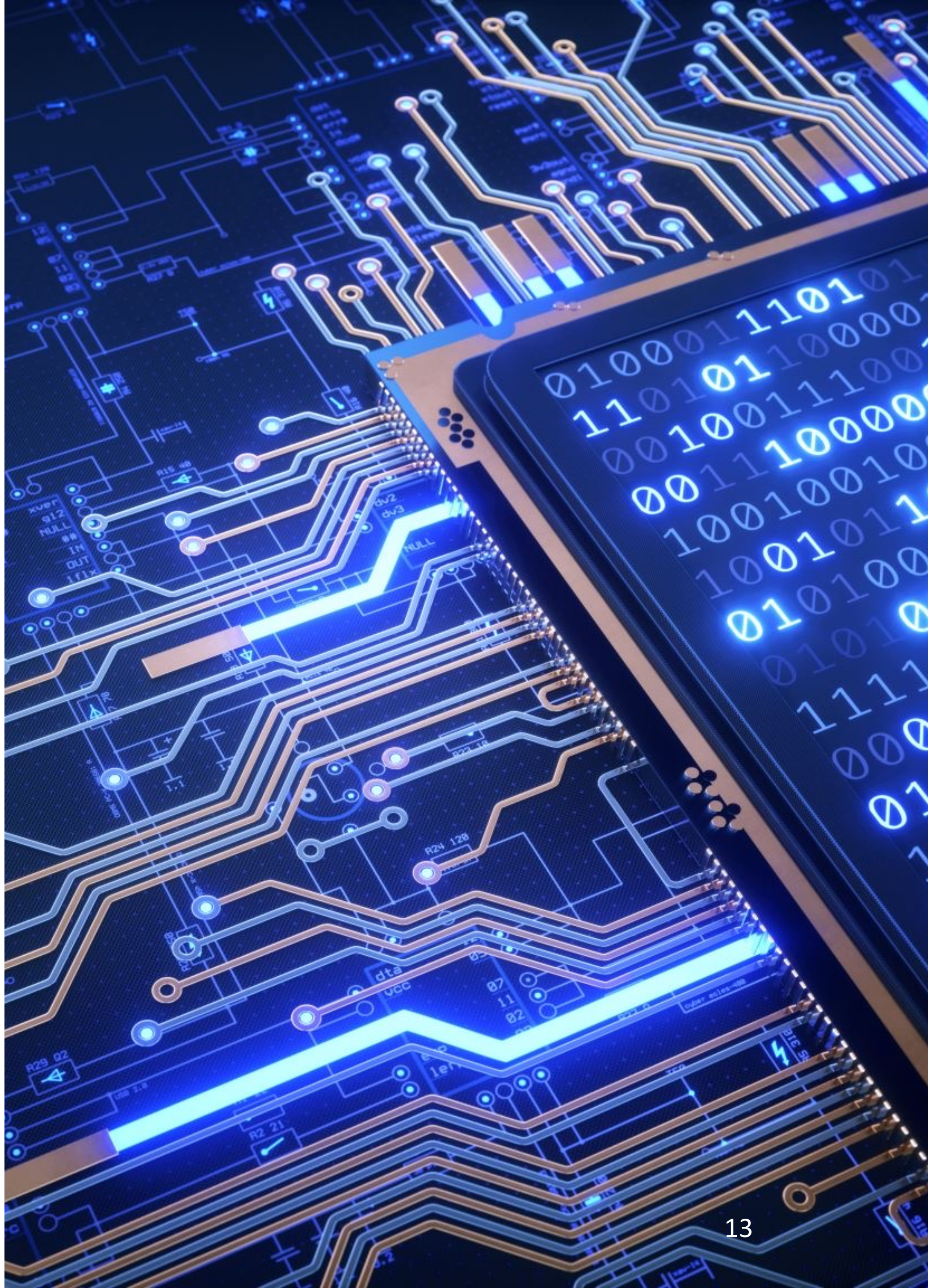
R = {all decimal expansions}

The Real Numbers are created by adjoining the Rationals with the Irrationals.

The Reals are closed under *all* operations.

The Reals form a *continuum*: we use the Real Number Line to represent this.

Real numbers are not *imaginary numbers.*



Complex Numbers

A combination of a real and an imaginary number in the form a + bi where ‘a’ and ‘b’ are real numbers, and ‘*i*' is a solution of the equation $x^2 = -1$.

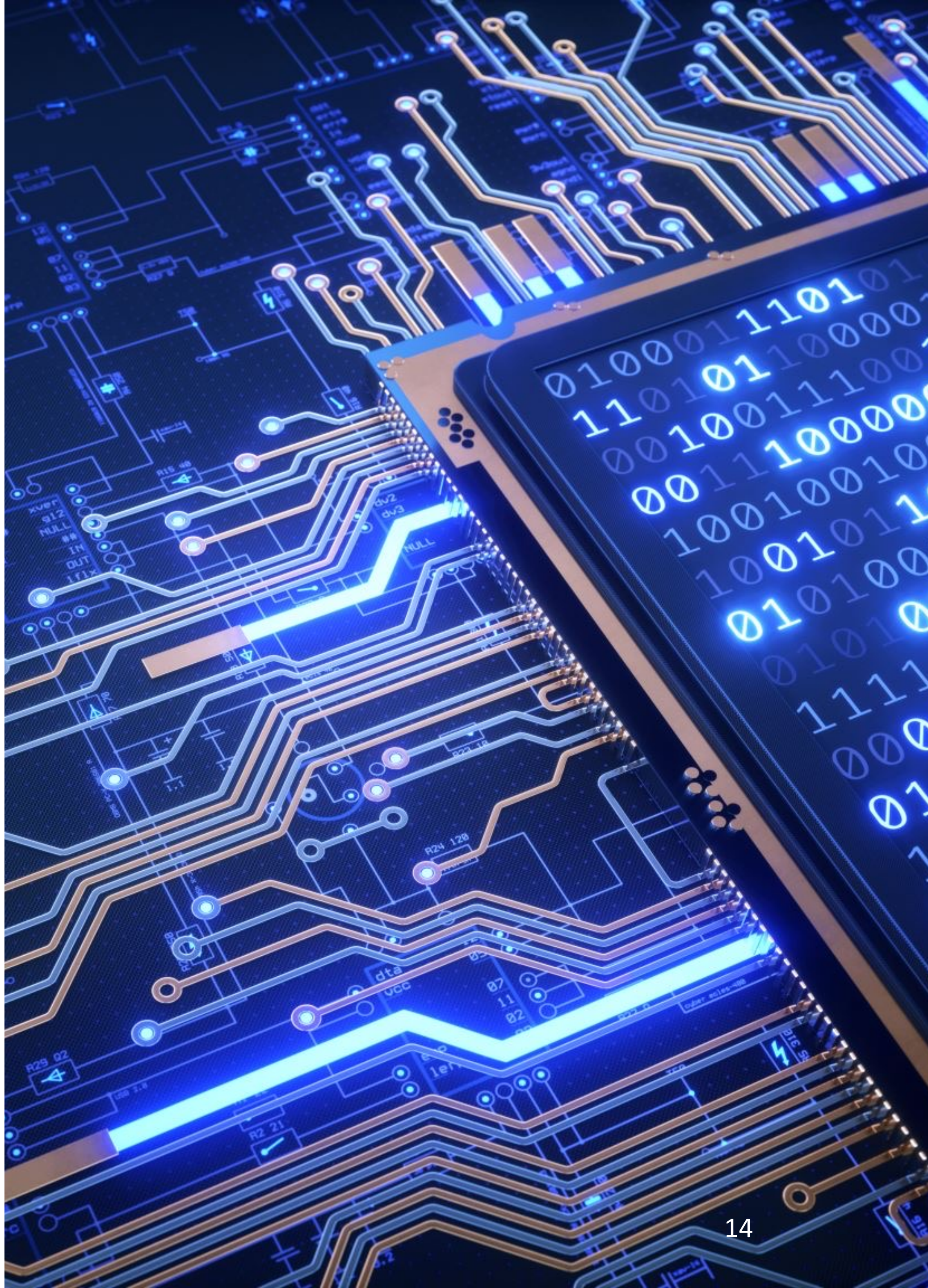
The Reals fall short when solving simple polynomial equations like $x^2 + 1 = 0$.

The Complex Numbers patch this hole.

$\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R} \text{ and } i = \sqrt{(-1)}\}$

Use the *Complex Plane* to represent these numbers.

The Complex Numbers are also a field.



Important Number Sets

- Notation for **intervals** of real numbers:

  Let *a* and *b* denote real numbers with $a < b$, the sets representing the real numbers from *a* to *b* can be written as follows.

  $[a, b] = \{x \mid a \leq x \leq b\}$ --- closed interval

  $[a, b) = \{x \mid a \leq x < b\}$ --- right-open interval

  $(a, b] = \{x \mid a < x \leq b\}$ --- left-open interval

  $(a, b) = \{x \mid a < x < b\}$ --- open interval

- *Note*: Sets can have other sets as members

  Examples : $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$, $\{a, e, i, \{0, 1\}\}$

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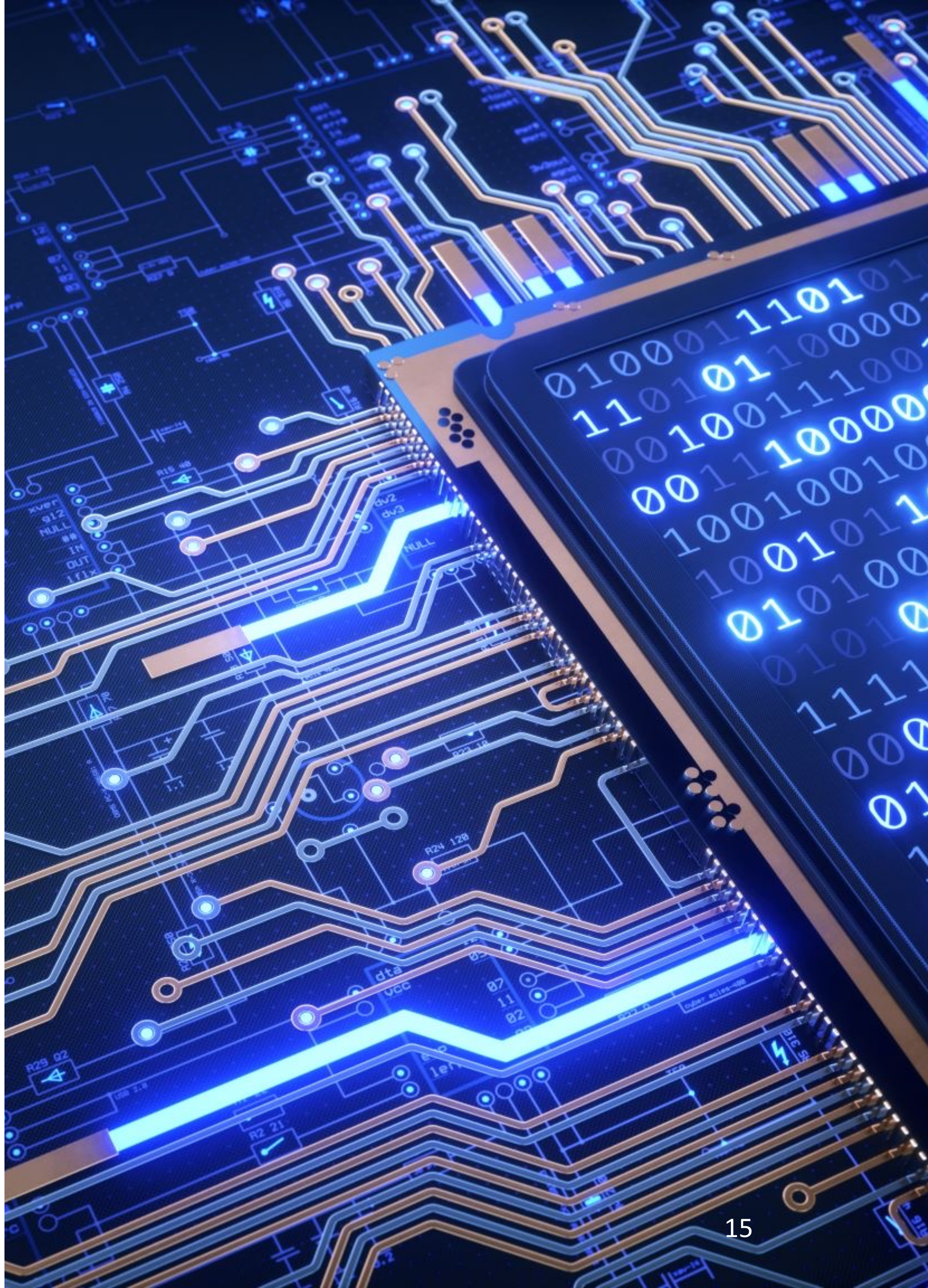


Equality of Sets

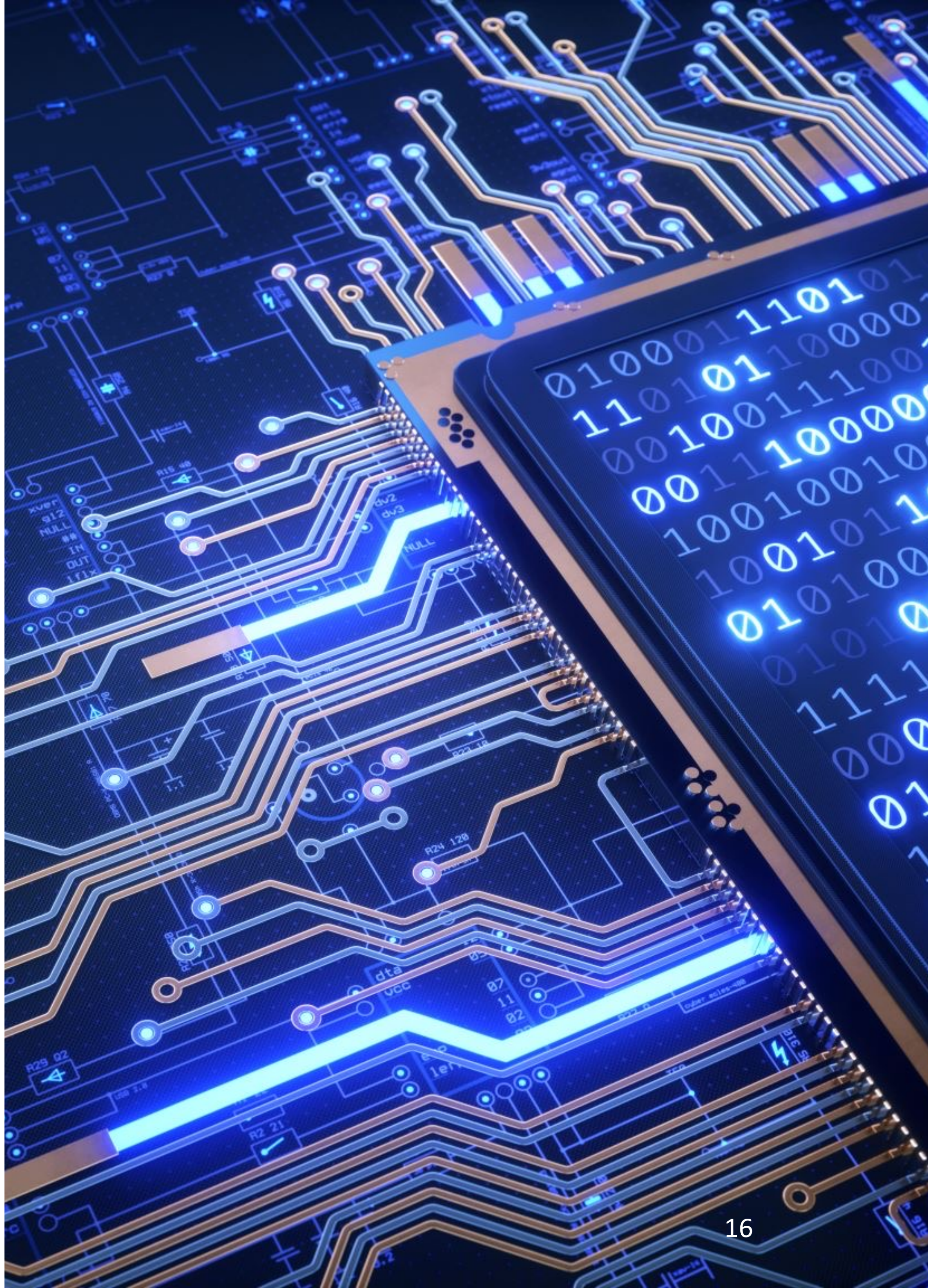
- Two sets are *equal* if and only if they have the same elements

  If A and B are sets, then A and B are equal if and only if
  $\forall x (x \in A \leftrightarrow x \in B)$
  The equality of two sets A, B is denoted as $A = B$

- Examples:

  {1, 3, 5} = {3, 5, 1}

  {1, 3, 5} = {1, 3, 3, 3, 5, 5, 5, 5}



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Empty Set

- An empty set is a set with *no elements*
- An empty set is also called as *null set*
- An empty set is denoted by either ∅ or { }
- Example:
  - The set of all positive integers that are greater than their squares

or

$$S = \{x \in \mathbf{Z}^+ \mid x > x^2\}$$

that means, $S = \{\ \} = \emptyset$

- *Note*: $\emptyset \neq \{\emptyset\}$

Empty set

A **singleton set** having **e**mpty set as member

Analogy:
A computer folder with exactly one empty folder inside

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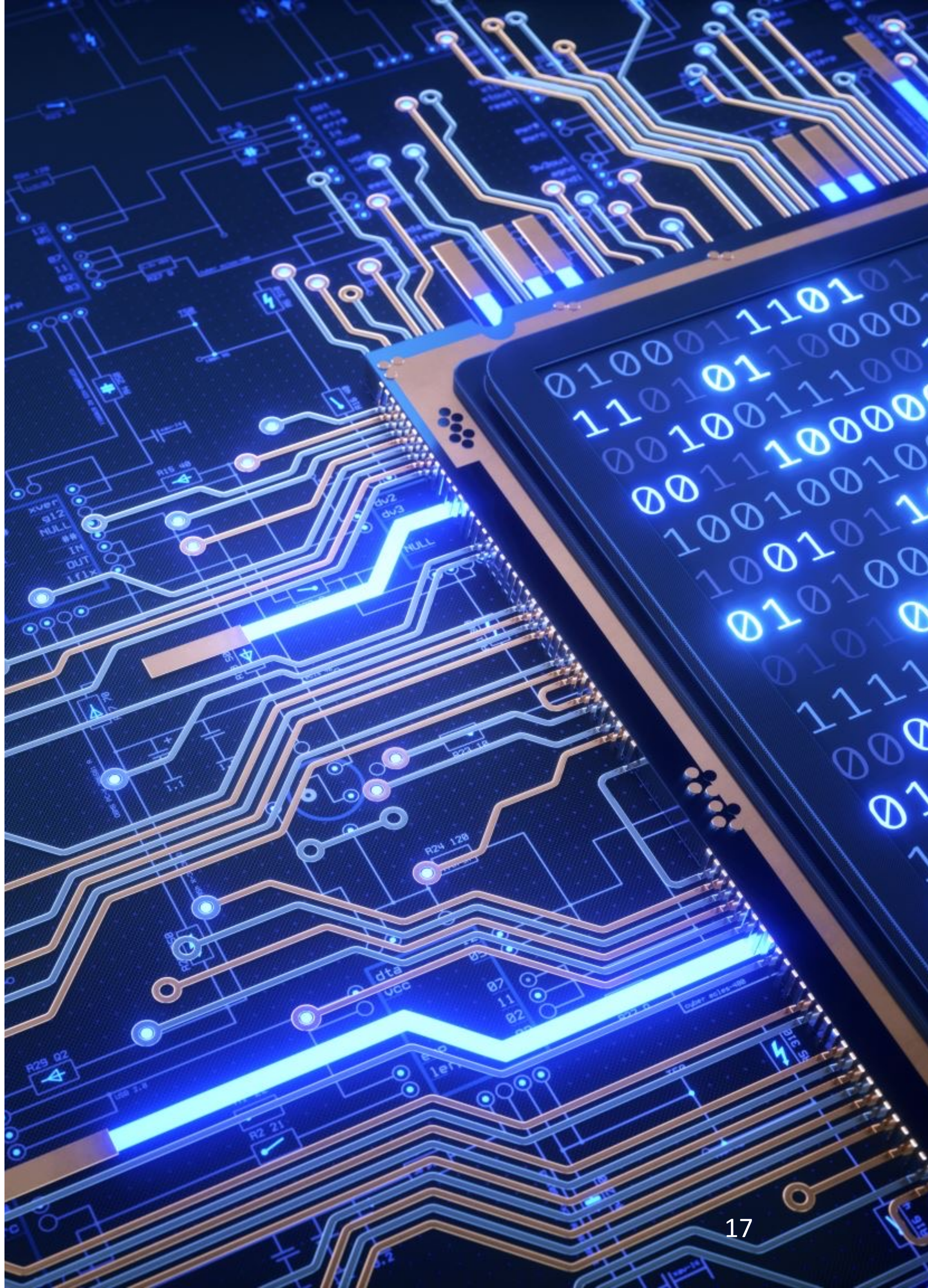
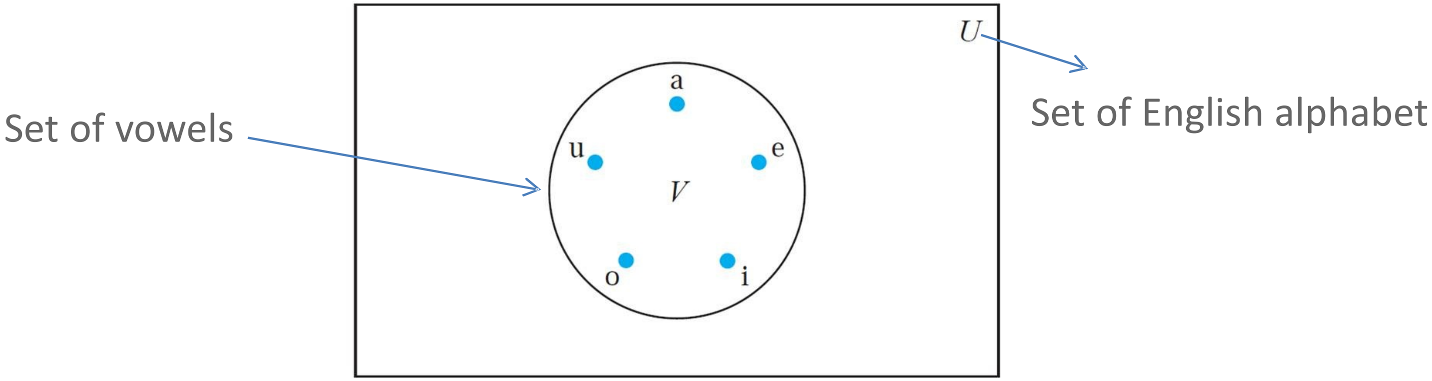


Venn Diagram

- Venn diagrams are used to represent the sets graphically and to indicate the relationship between two or more sets
- In Venn diagrams, the **universal set** *U*, which contains all the objects under consideration, *is represented by a rectangle*.
- *Circles* or other geometrical figures are used to represent sets



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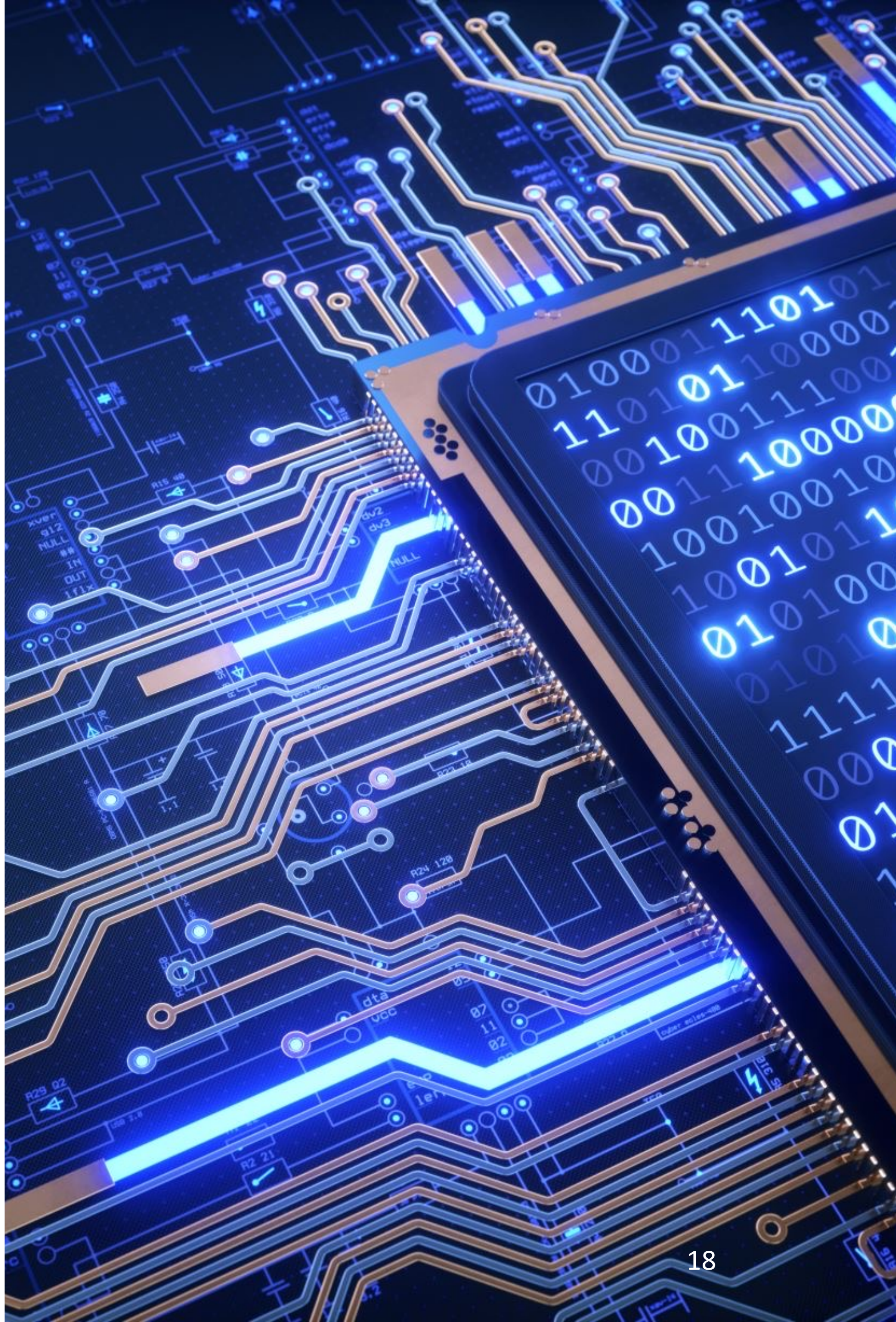
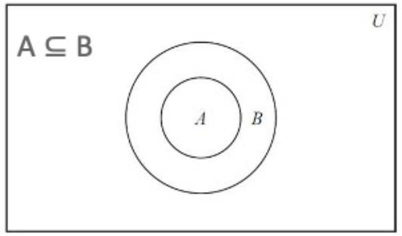


Subsets

- The set A is a subset of B if and only if every element of A is also an element of B. This is denoted by $A \subseteq B$
- $A \subseteq B$ if and only if the quantification $\forall x\,(x \in A \rightarrow x \in B)$ is true
- The notation $A \nsubseteq B$ means, *A is not a subset of B*
- Examples:
  - $\{1, 2, 3, 5, 7, 9\} \subseteq \{1, 2, 3, 5, 5, 7, 9\}$
  - $\{0, 1, 3, 5, 7, 9\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  - The set of rational numbers is a subset of the set of real numbers
- Every set is a subset of itself. That means, a set $A \subseteq A$
- *Empty set* is a subset of every set. That means, for a set A, $\emptyset \subseteq A$



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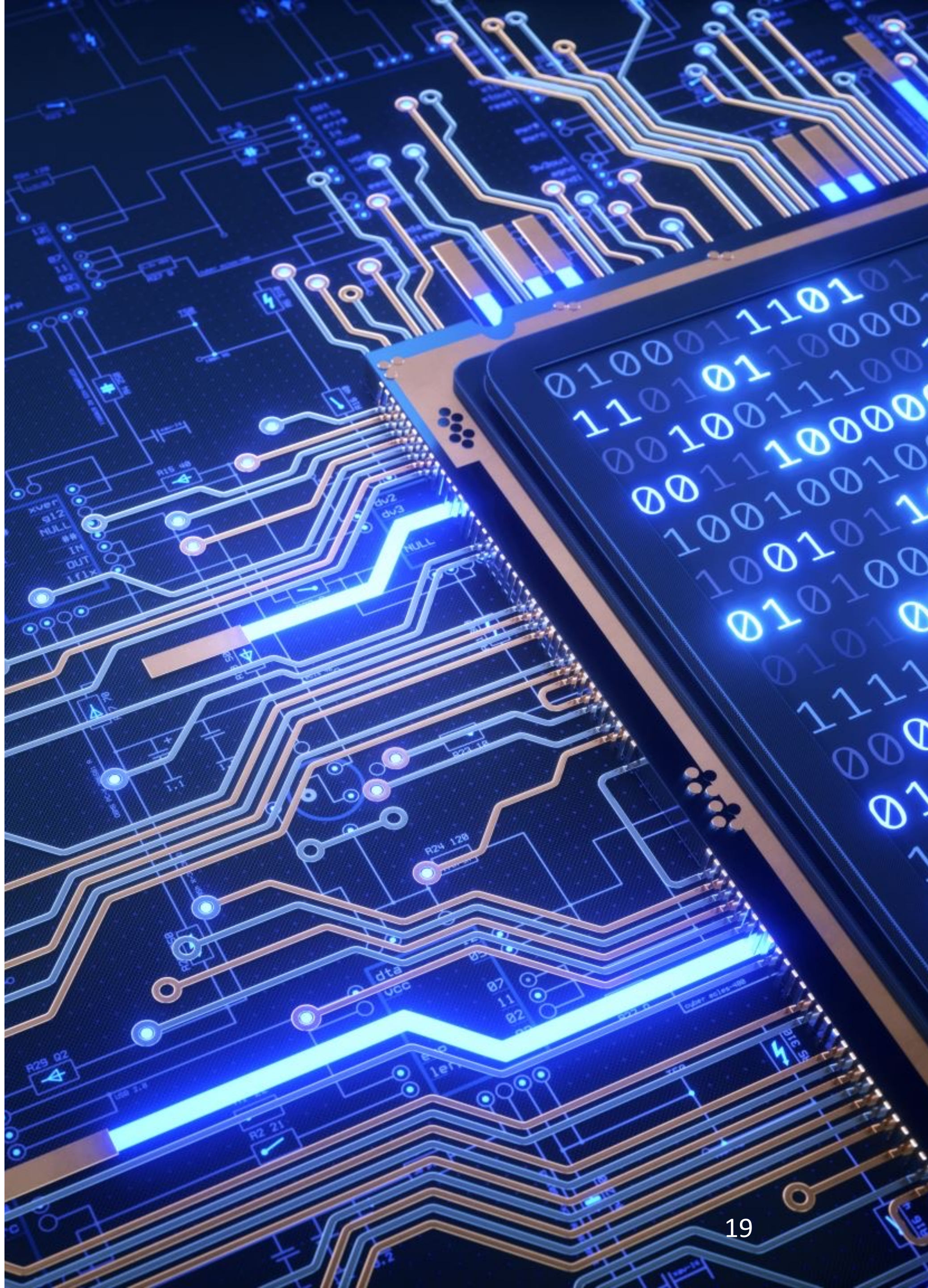

Proper Subsets

- $A \subset B$ denotes that, A is a subset of B but, $A \neq B$. In such case A is said to be a **proper subset** of B.
- $A \subset B$ if and only if $\forall x\, (x \in A \rightarrow x \in B) \land \exists x\, (x \in B \land x \notin A)$
- if A and B are sets such that $A \subseteq B$ and $B \subseteq A$, then $A = B$

or

$A = B$ if and only if $\forall x\, (x \in A \leftrightarrow x \in B)$

- Example:

  $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x$ is a subset of the set $\{a, b\}\}$ are equal sets

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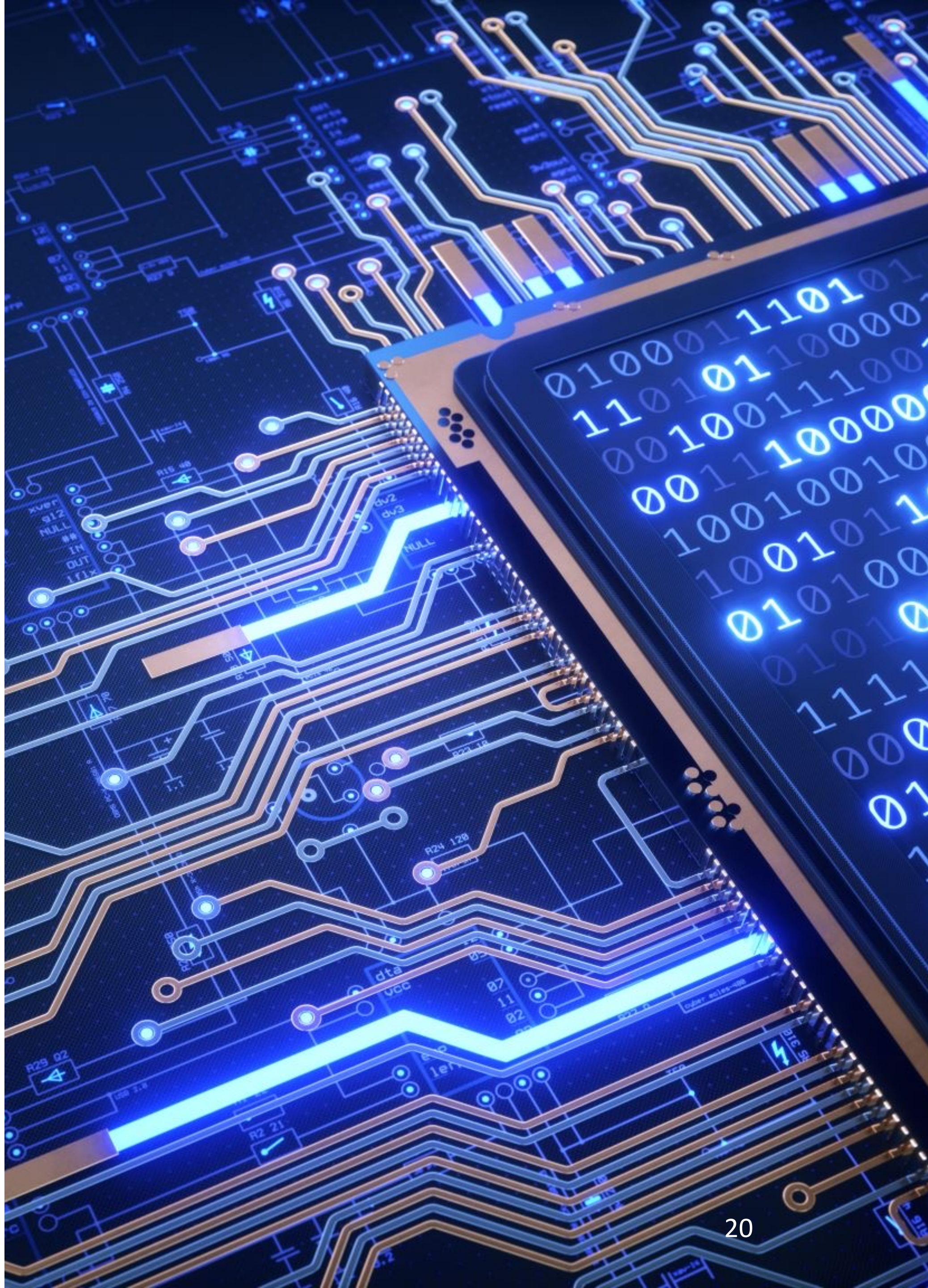

Cardinality of a Set

- The size of a set is simply *the number of **distinct** elements* in it.
- The size of a set is also called the ***cardinality***
- The *size* or *cardinality* of a set S is denoted by $|S|$
- If the number of elements in a set are finite and hence the cardinality of it can be determined, it is called as a ***finite set*** otherwise as an ***infinite set***
- Examples:
  - Let A be the set of odd positive integers less than 10. Then $|A| = 5$
  - Let S be the set of letters in the English alphabet. Then $|S| = 26$
  - $|\emptyset| = 0$
  - The set of positive integers is **infinite**

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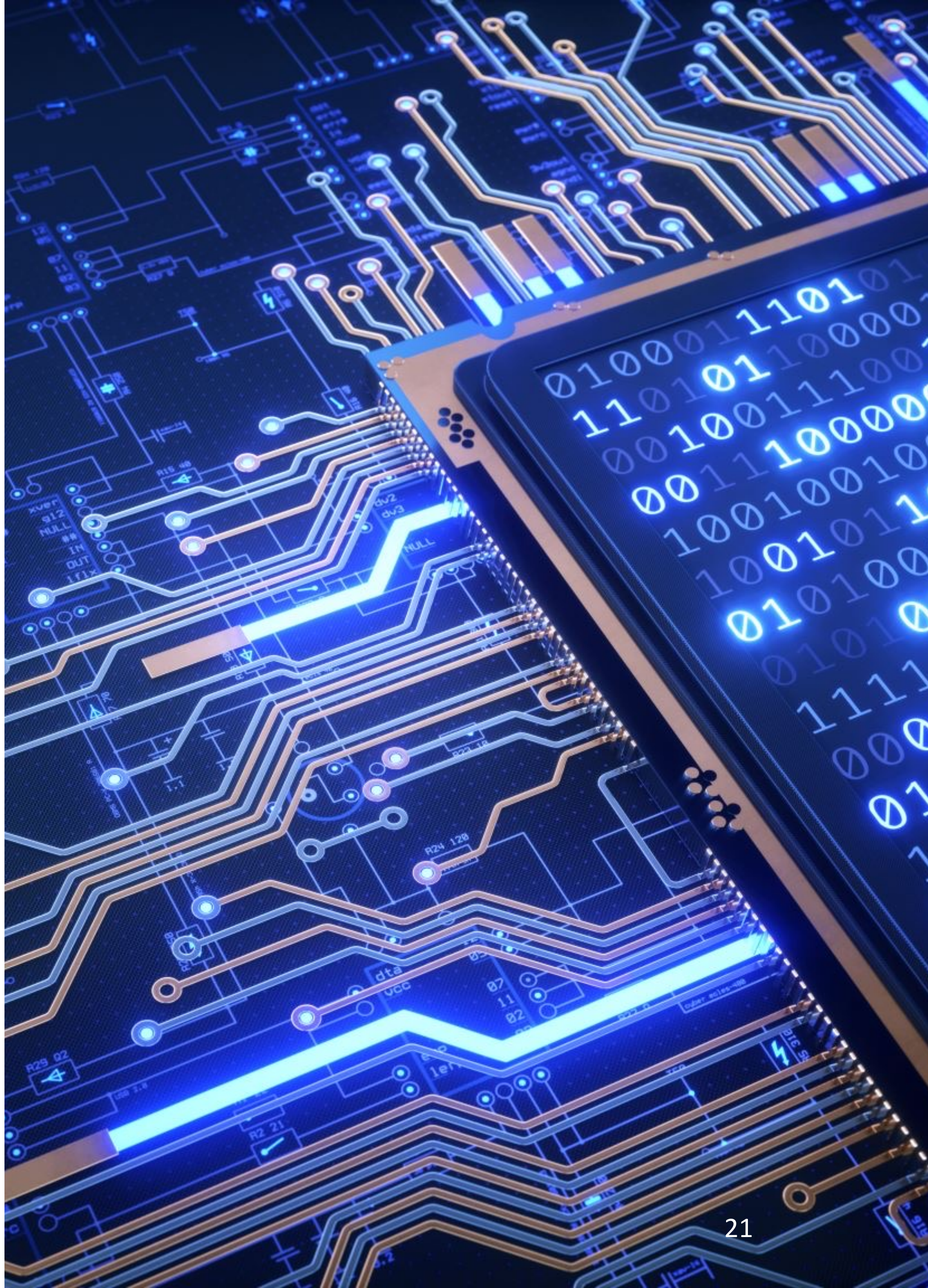
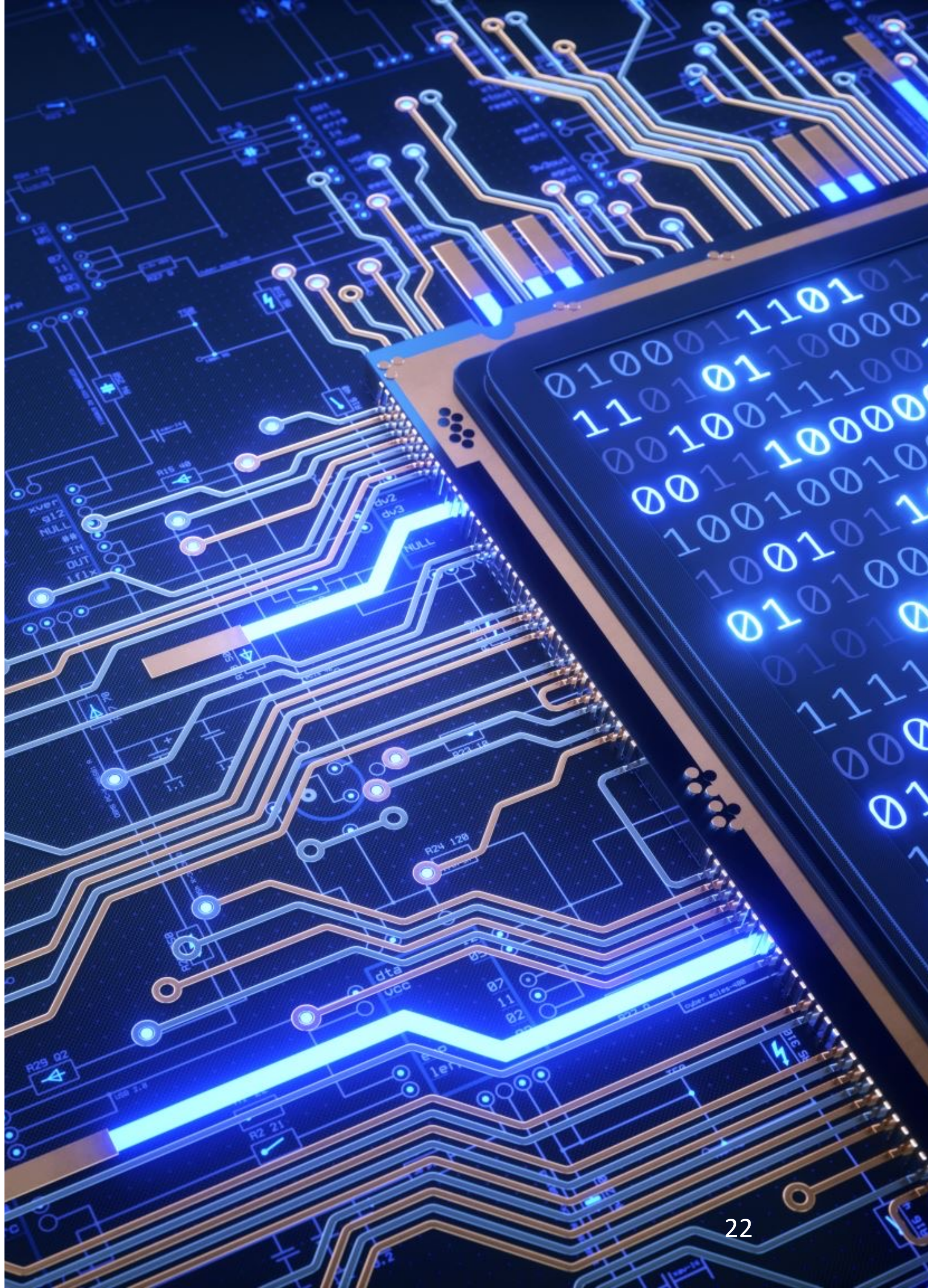


Power set

- The ***power set*** of a set S is ***the set of all subsets*** of the set S
- It is denoted by $P(S)$
- Examples:
  - $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
  - $P(\emptyset) = \{\emptyset\}$
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

Note: If a set has n elements, then its power set has 2^n elements



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Ordered tuple

- The ***ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its n^{th} element
- Ordered 2-tuples are called **ordered pairs**
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$
- $(a, b) \neq (b, a)$ unless $a = b$
- For example, the ordered pair <1, 2> is not equal to the ordered pair <2, 1>

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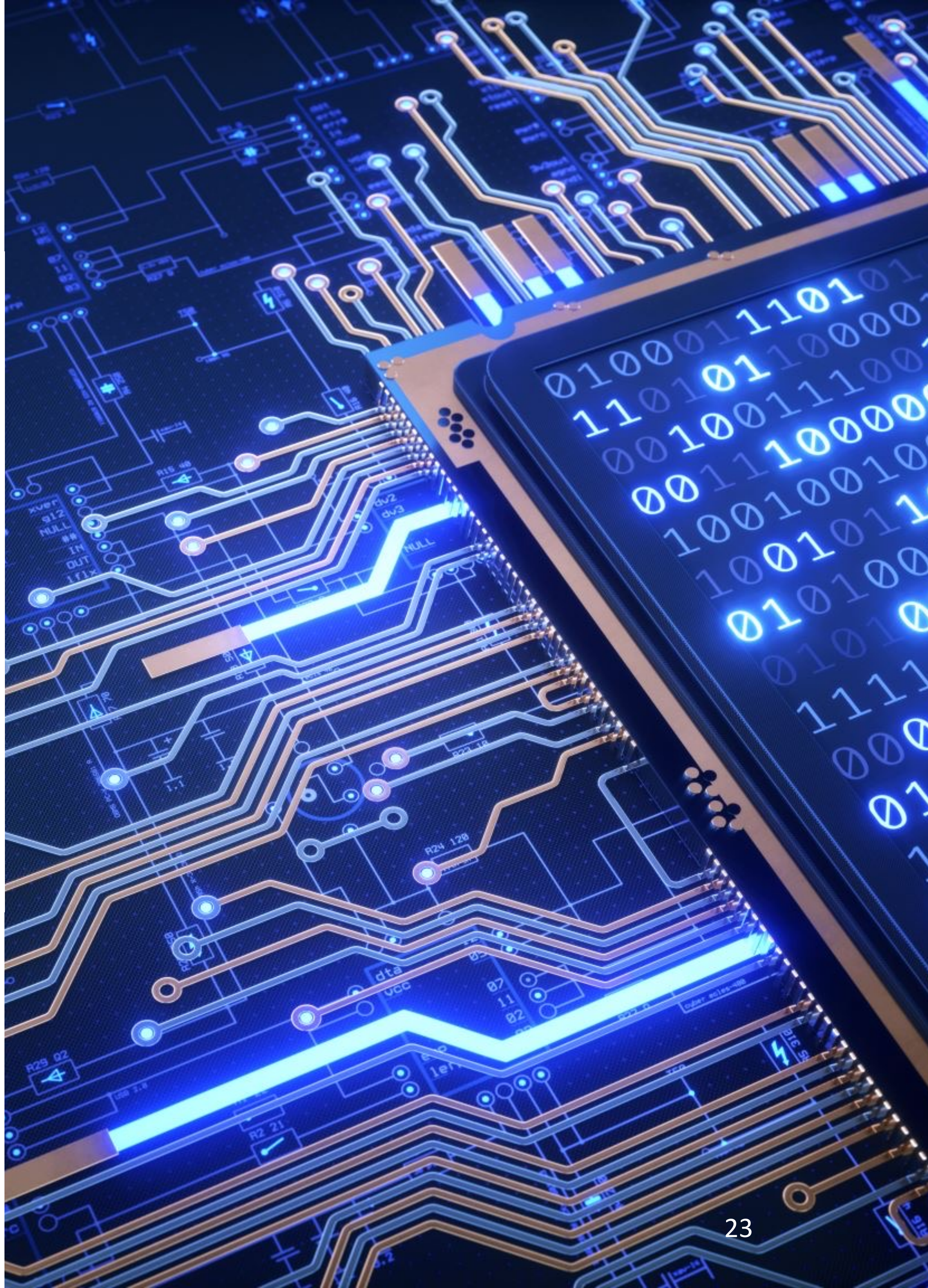

Cartesian product

- The *Cartesian product* of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. That is, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$
- Examples:
  - The Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$ is

    $$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

  - Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product of $A \times B$ and how can it be used?

    Answer : $A \times B$ represents all possible enrolments of students in courses at the university

- *Note:* $A \times B \neq B \times A$

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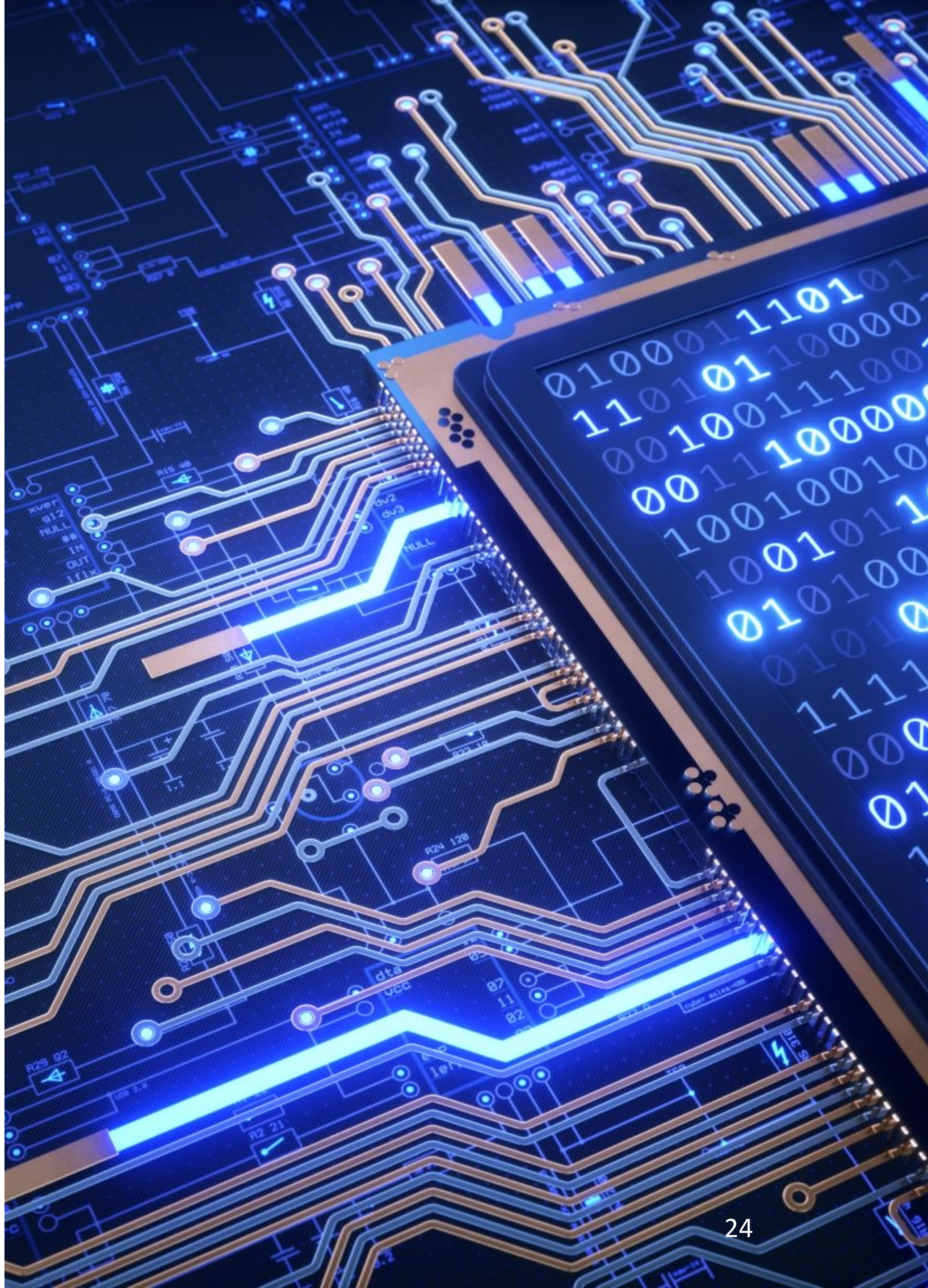


Cartesian product

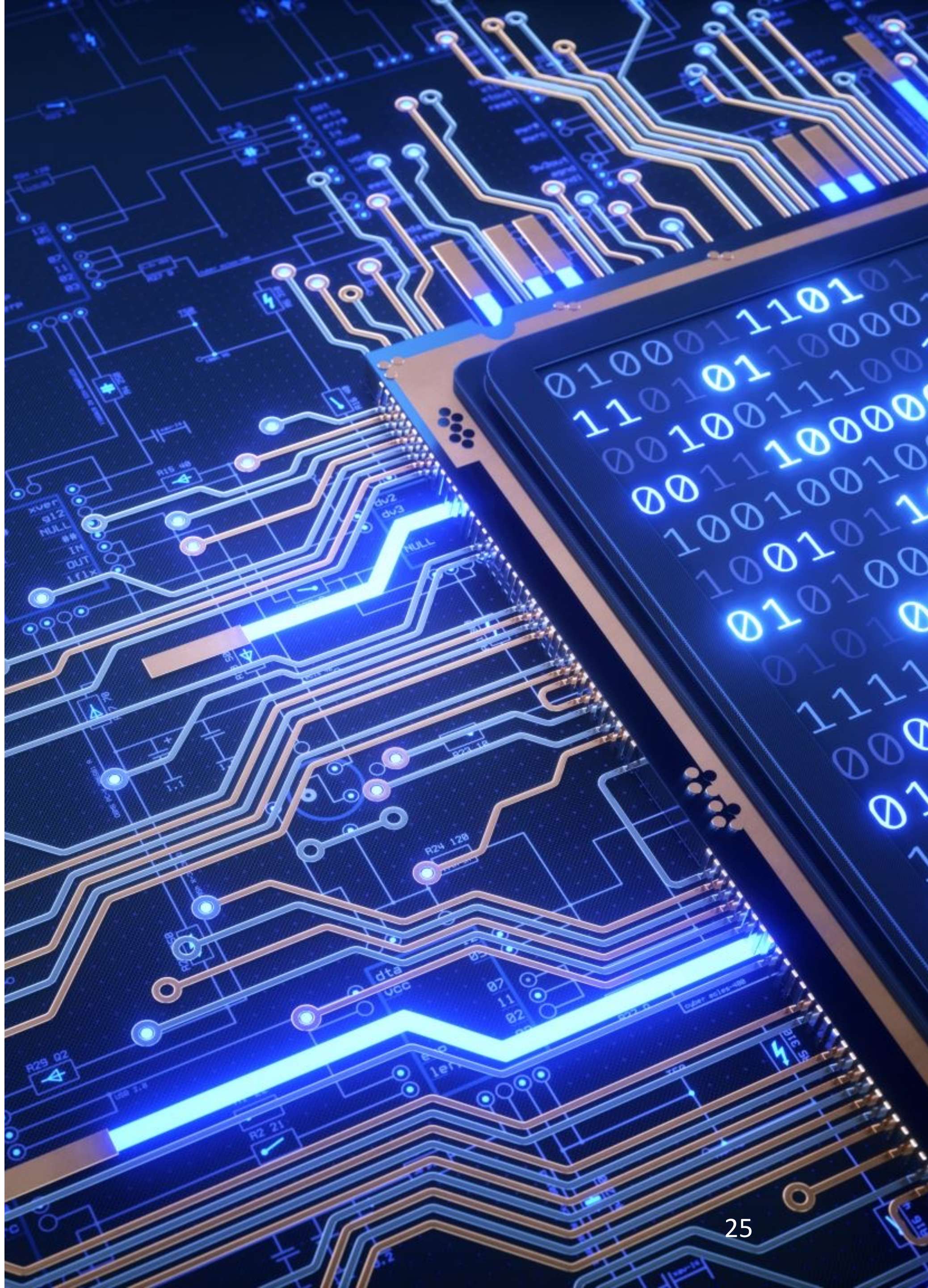
- The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$
- Example:
  - Let $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$ then

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$$

- *Note*: $(A \times B) \times C$ is not the same as $A \times B \times C$

- $A^2 = A \times A$, and $A^3 = A \times A \times A$ and so on

  More generally, $A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$



Cartesian product

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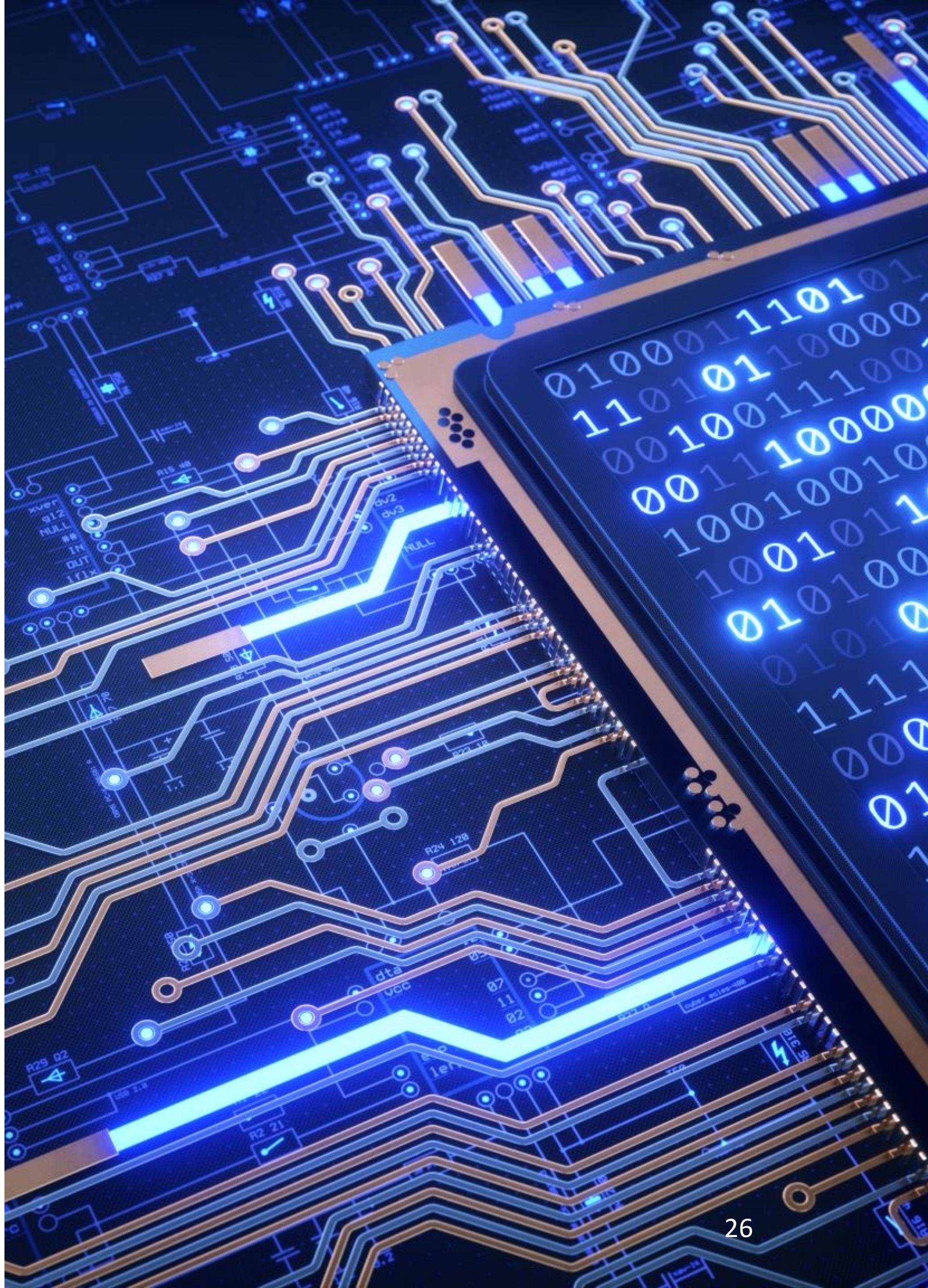
- Example:
  - Let $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$ then

$$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$$

- *Note*: $(A \times B) \times C$ is not the same as $A \times B \times C$

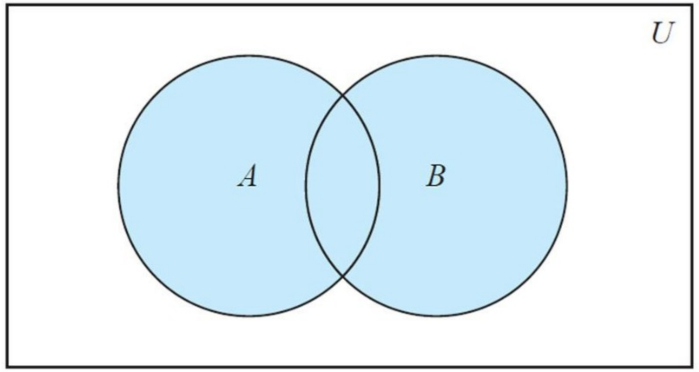
First get A x B = {(0,1), (0,2), (1,1), (1,2)}

(AxB) x C = {([0,1],0), ([0,1],1), ([0,1],2), ([0,2],0), ([0,2],1), ([0,2],2), ([1,1],0), ([1,1],1), ([1,1],2), ([1,2],0), ([1,2],1), ([1,2],2)}



Set Operations

- The ***union*** of two sets A and B, denoted by $A \cup B$, is the set that contains the elements that are **either in *A* or in *B*, or in both**.
  $A \cup B = \{x \mid x \in A \vee x \in B\}$
- *Examples*:
  - $\{a, e, o\} \cup \{i, u, 3\} = \{a, e, i, o, u, 3\}$
  - $\{1, 3, 5\} \cup \{1, 2, 3, 5\} = \{1, 2, 3, 5\}$



$A \cup B$ is shaded.

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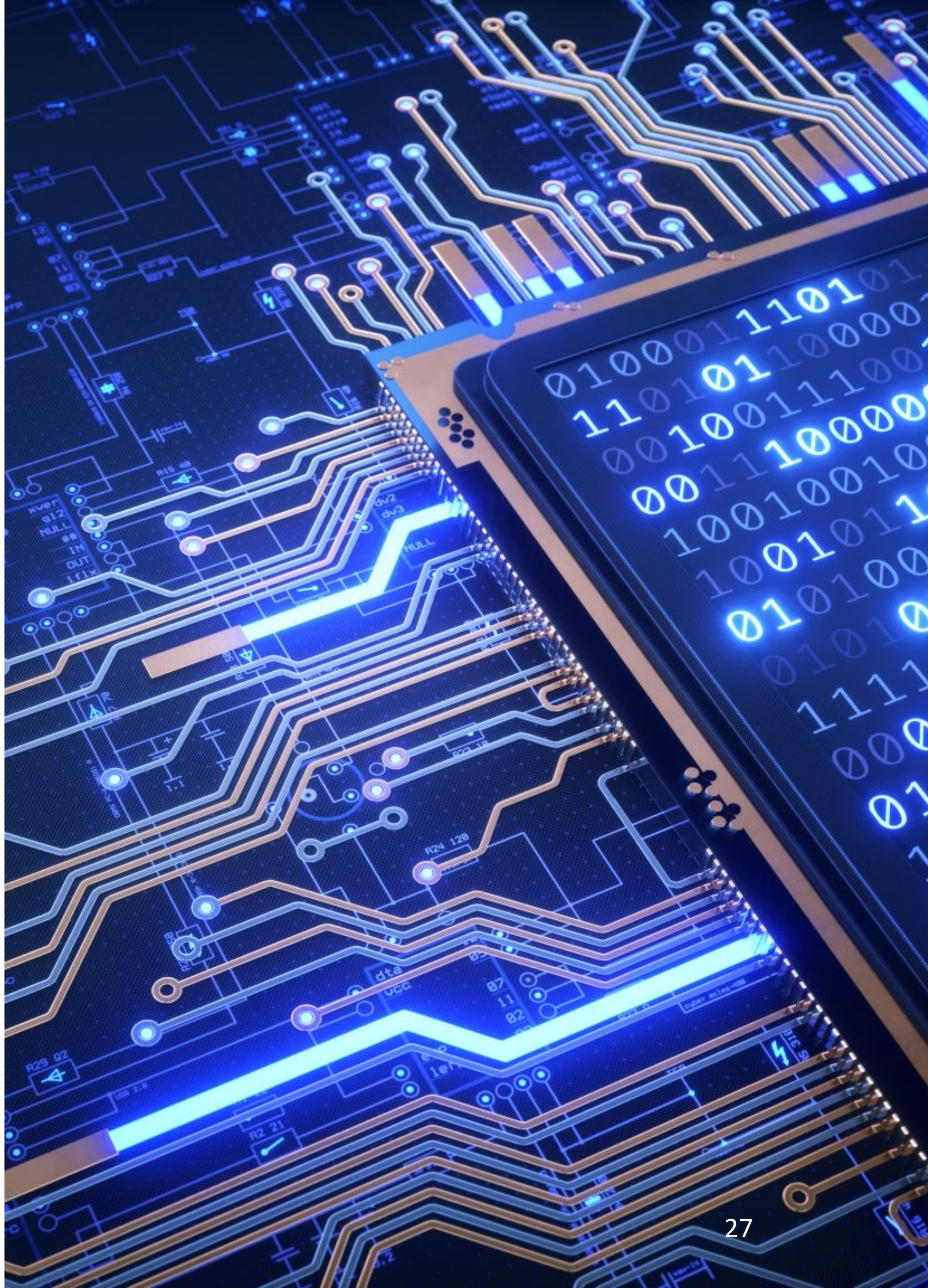


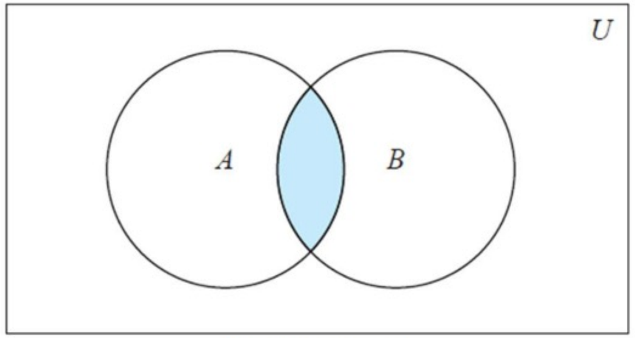
Set Operations

- The ***intersection*** of two sets A and B, denoted by $A \cap B$, is the set that contains the elements **both in *A* and in *B***

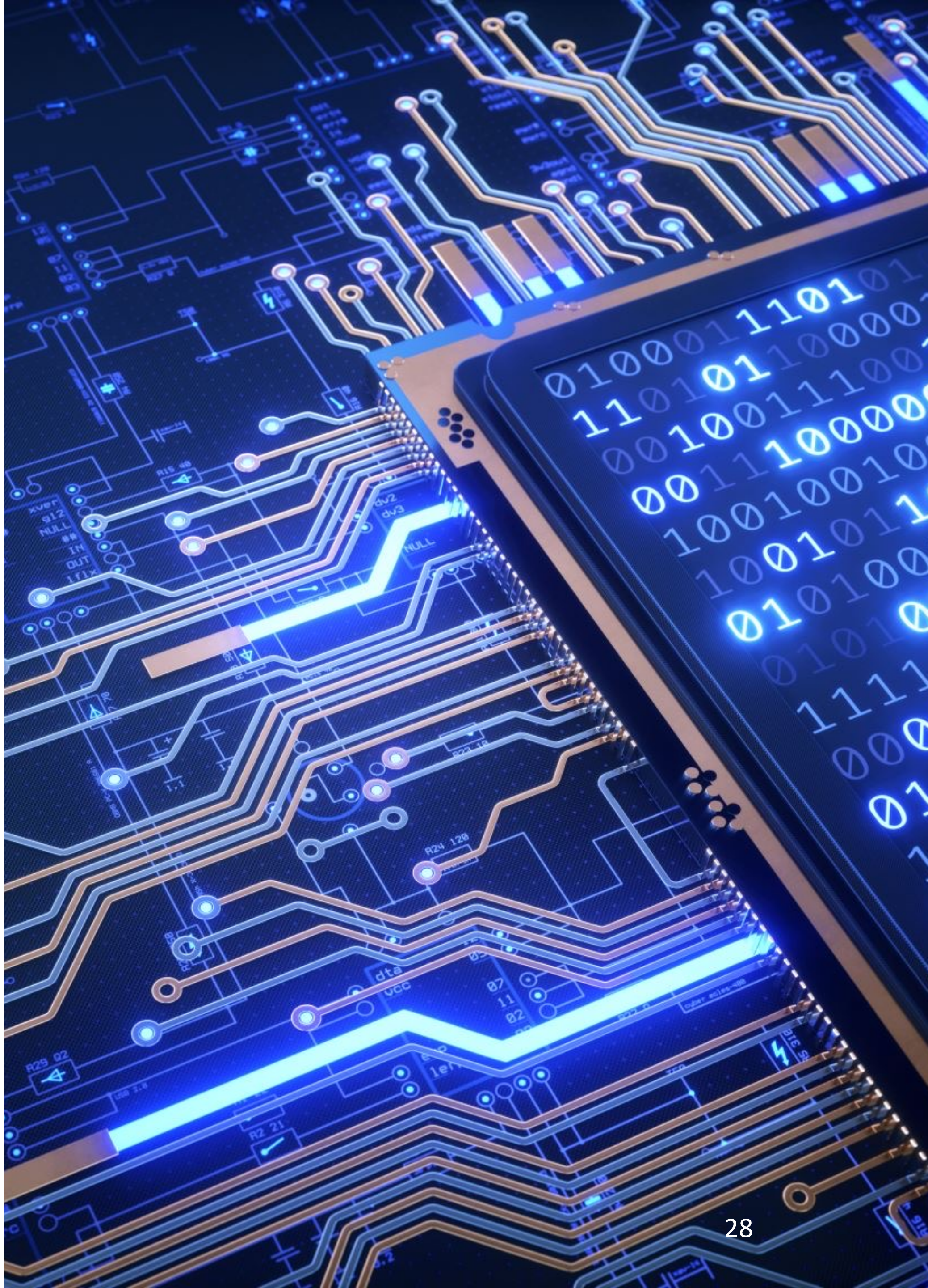
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

- *Examples*:
  - $\{a, e, o\} \cap \{i, u, 3\} = \{\ \} = \emptyset$
  - $\{1, 3, 5\} \cap \{1, 2, 5\} = \{1, 5\}$

Two sets are called ***disjoint***, if their intersection is the empty set



$A \cap B$ is shaded.



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Set Operations

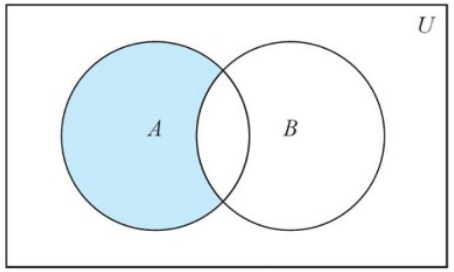
- The ***difference*** of two sets A and B, denoted by $A - B$ (or $A \backslash B$) is the set that contains the elements that are **in A but not in B**

$$A - B = \{x \mid x \in x\, A \wedge x \notin B\}$$

- Difference is also called the *complement of B with respect to A*
- *Examples*:

  $\{a, e, o\} - \{i, u, 3\} = \{a, e, o\}$

  $\{1, 3, 5\} - \{1, 2, 5\} = \{3\} \neq \{2\} = \{1, 2, 5\} - \{1, 3, 5\}$



$A - B$ is shaded.

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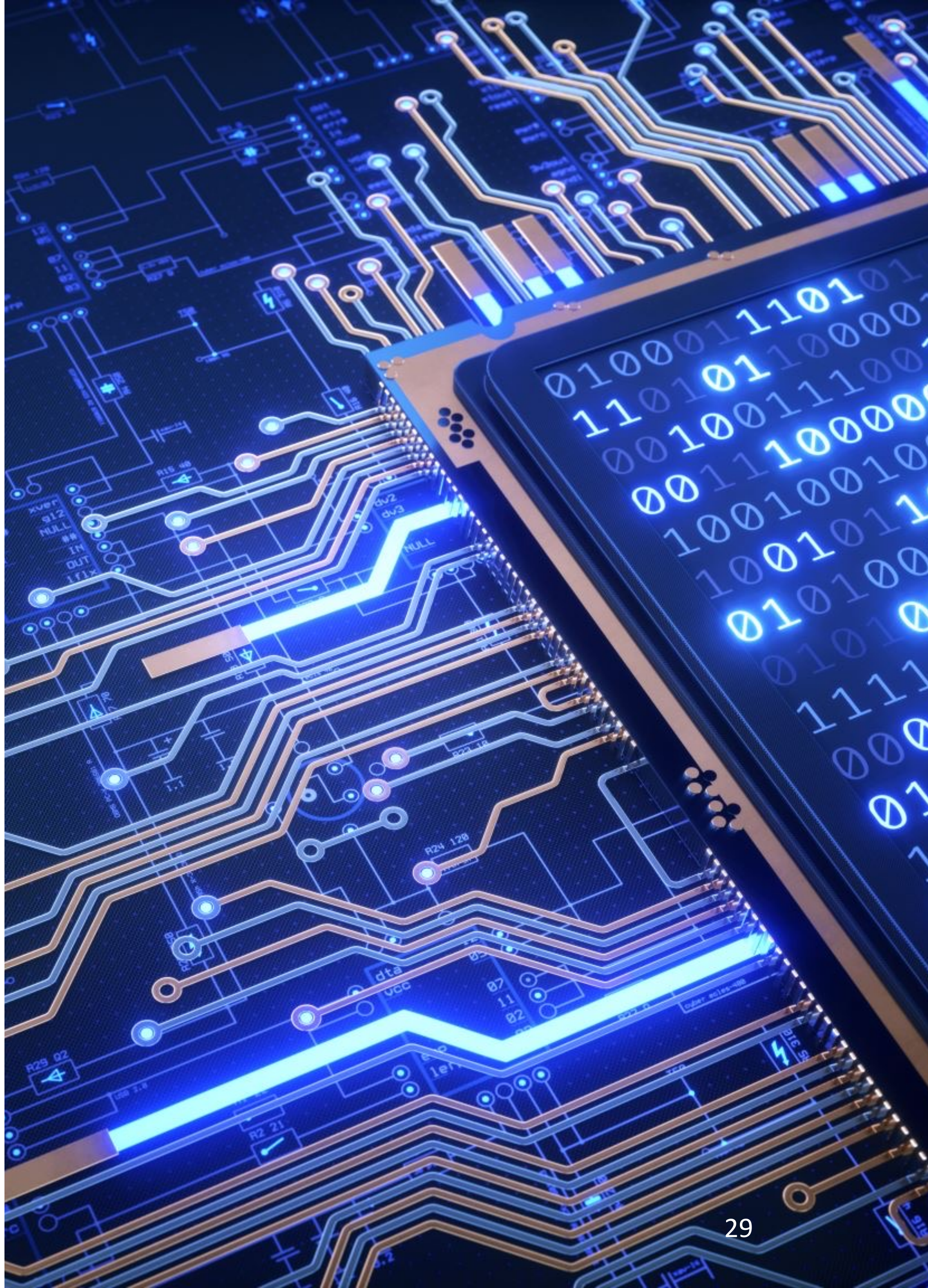


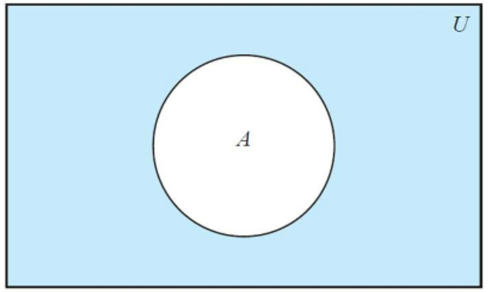
Set Operations

- The ***complement*** of set A denoted by $\bar{A}$ (or A') is the *complement of A with respect to U.* That is, $U - A$

$$\bar{A} = \{x \in U \mid x \notin A\}$$

- *Example*:
  - Let A be the set of positive integers greater than 10, with universal set as the set of all positive integers.

Then $\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$



$\bar{A}$ is shaded.

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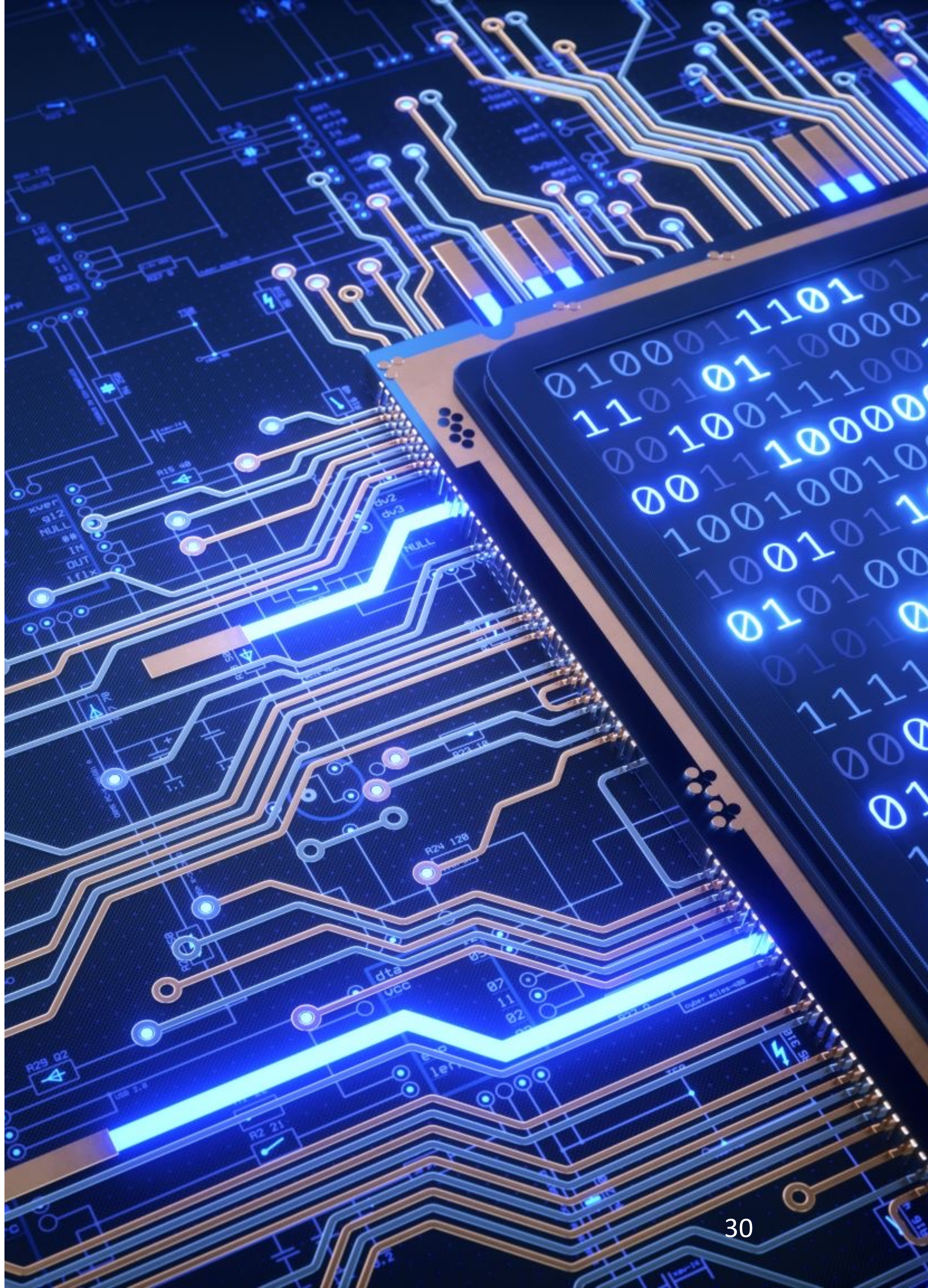

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

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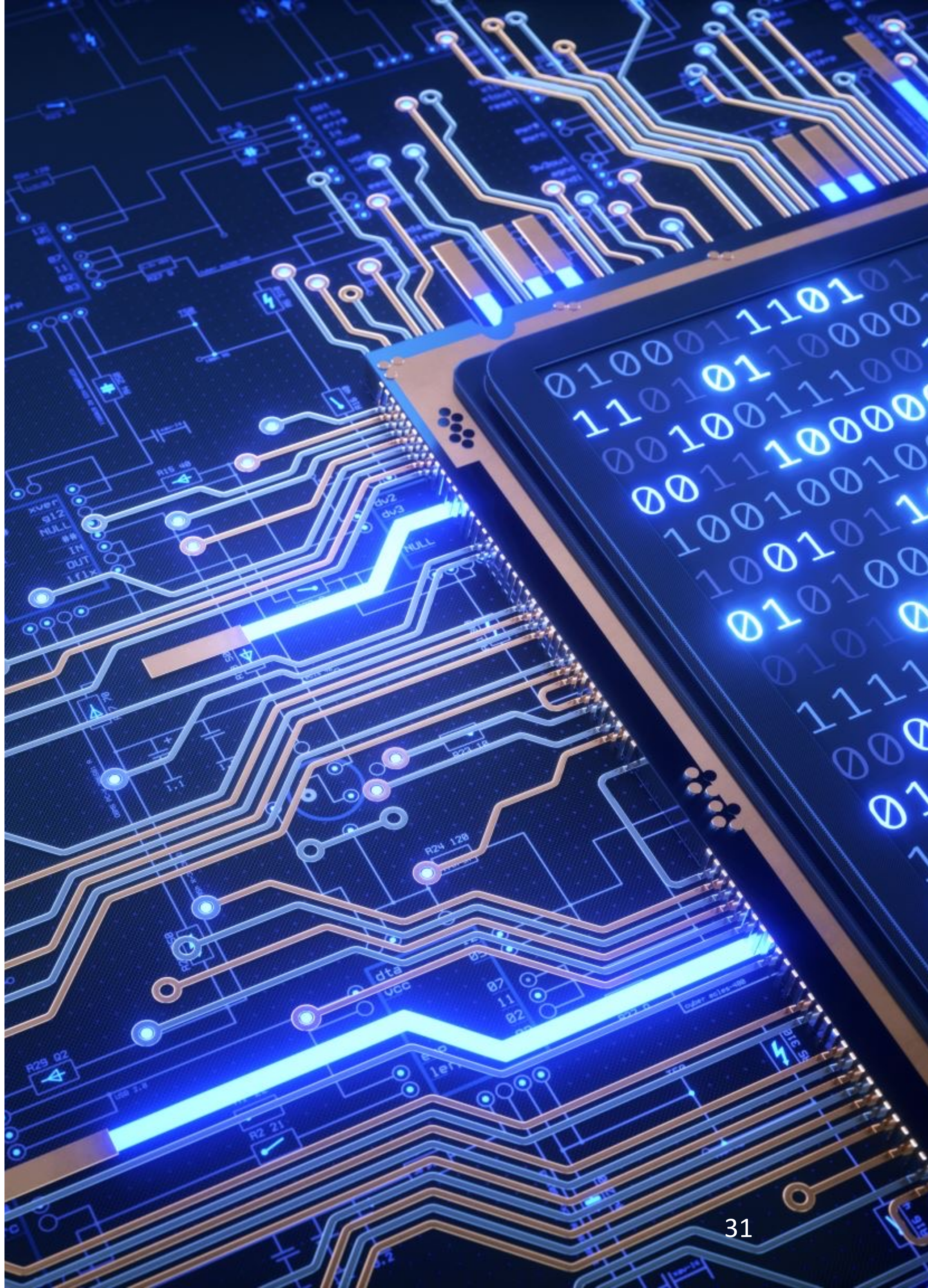

Set Identities

Identity	*Name*
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

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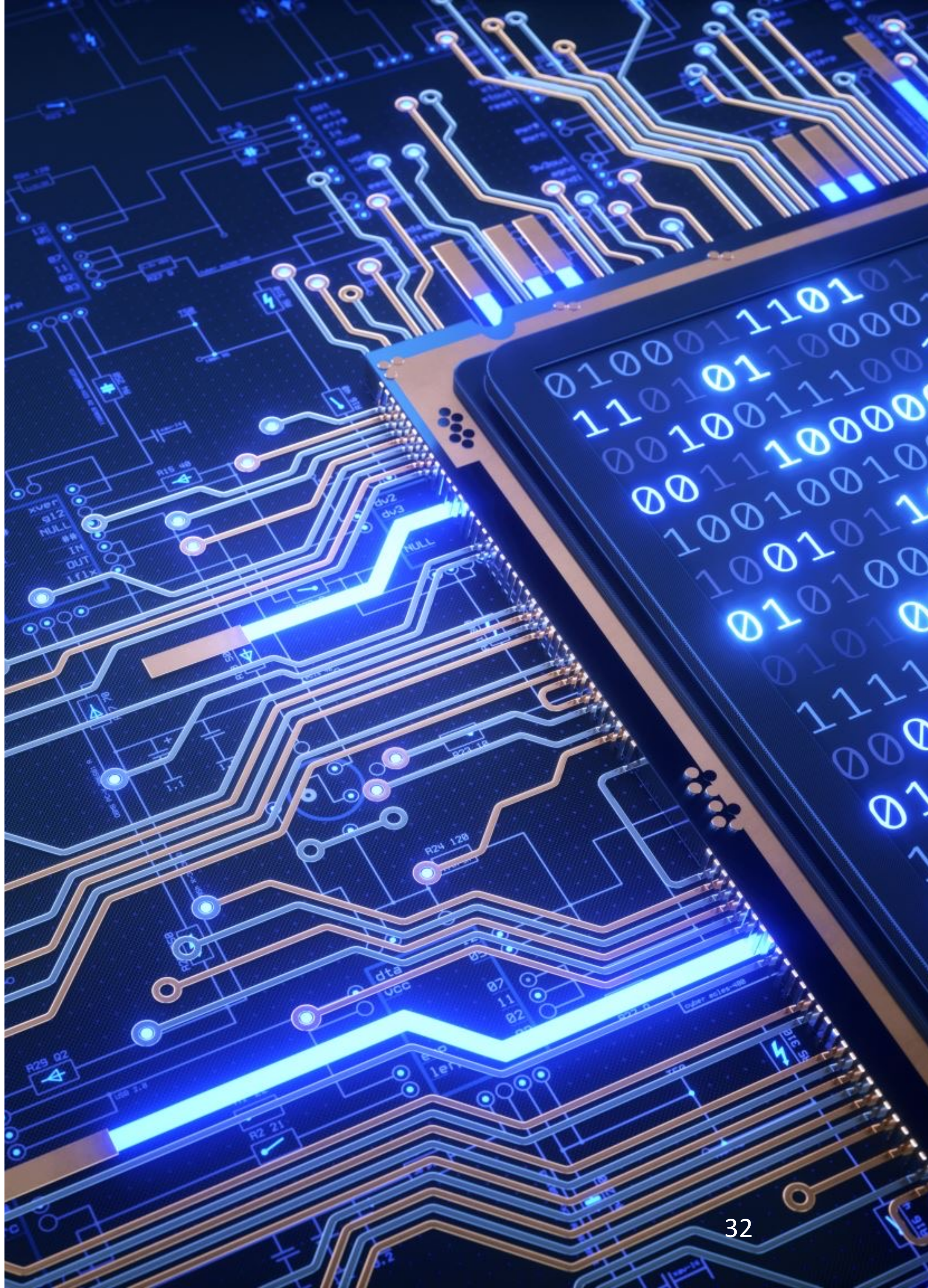
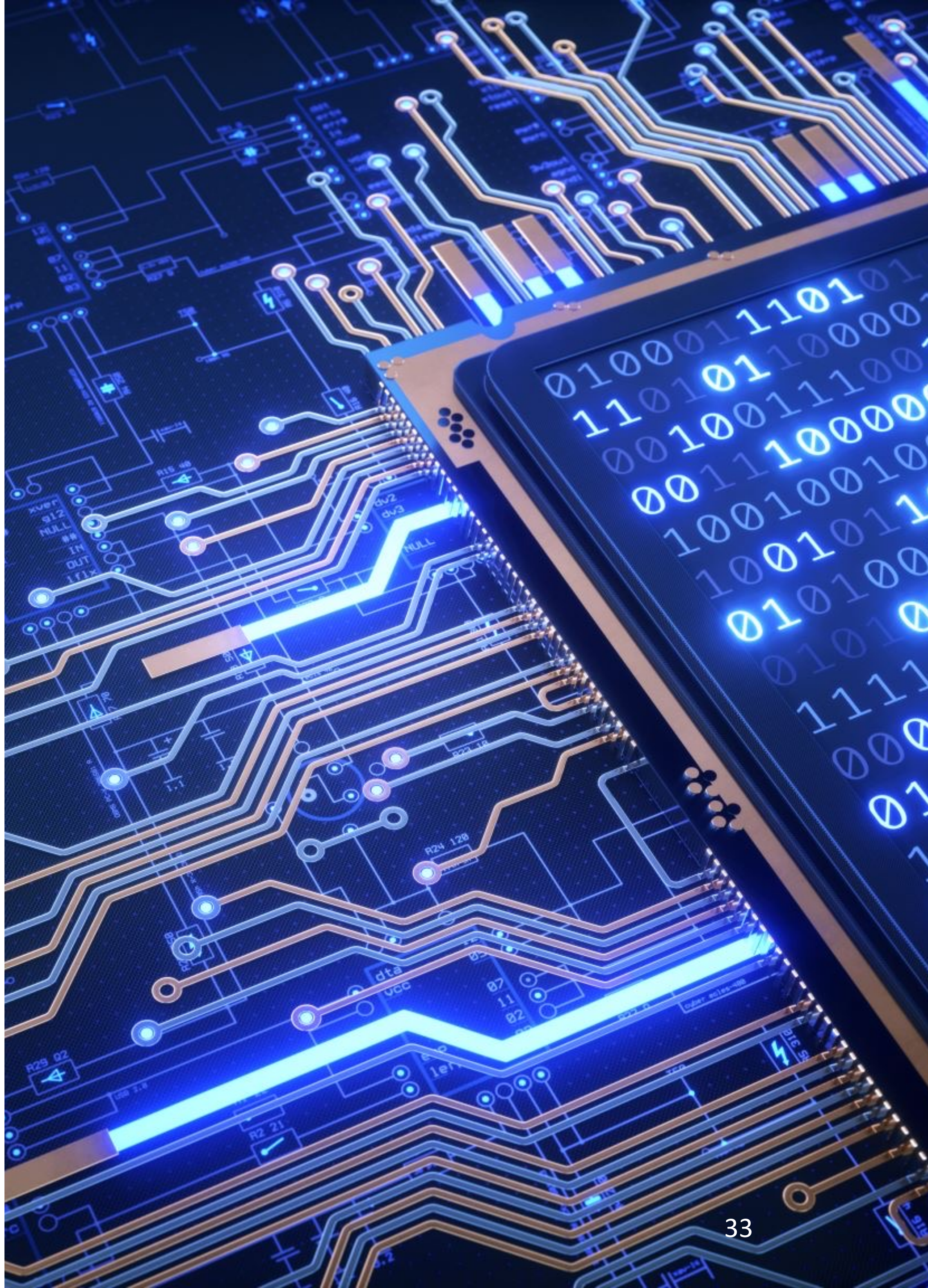


Computer Representation

- The elements of a set can be represented in the computer memory using an arbitrary ordering of the elements of the *finite* universal set U.
- Let $a_1, a_2, \ldots, a_n$ represent the elements of the universal set in some chosen order
- A subset A of U can be represented as a *bit string of length n,* where the i^{th} bit in it will be 1 if a_i belongs to A and 0 otherwise



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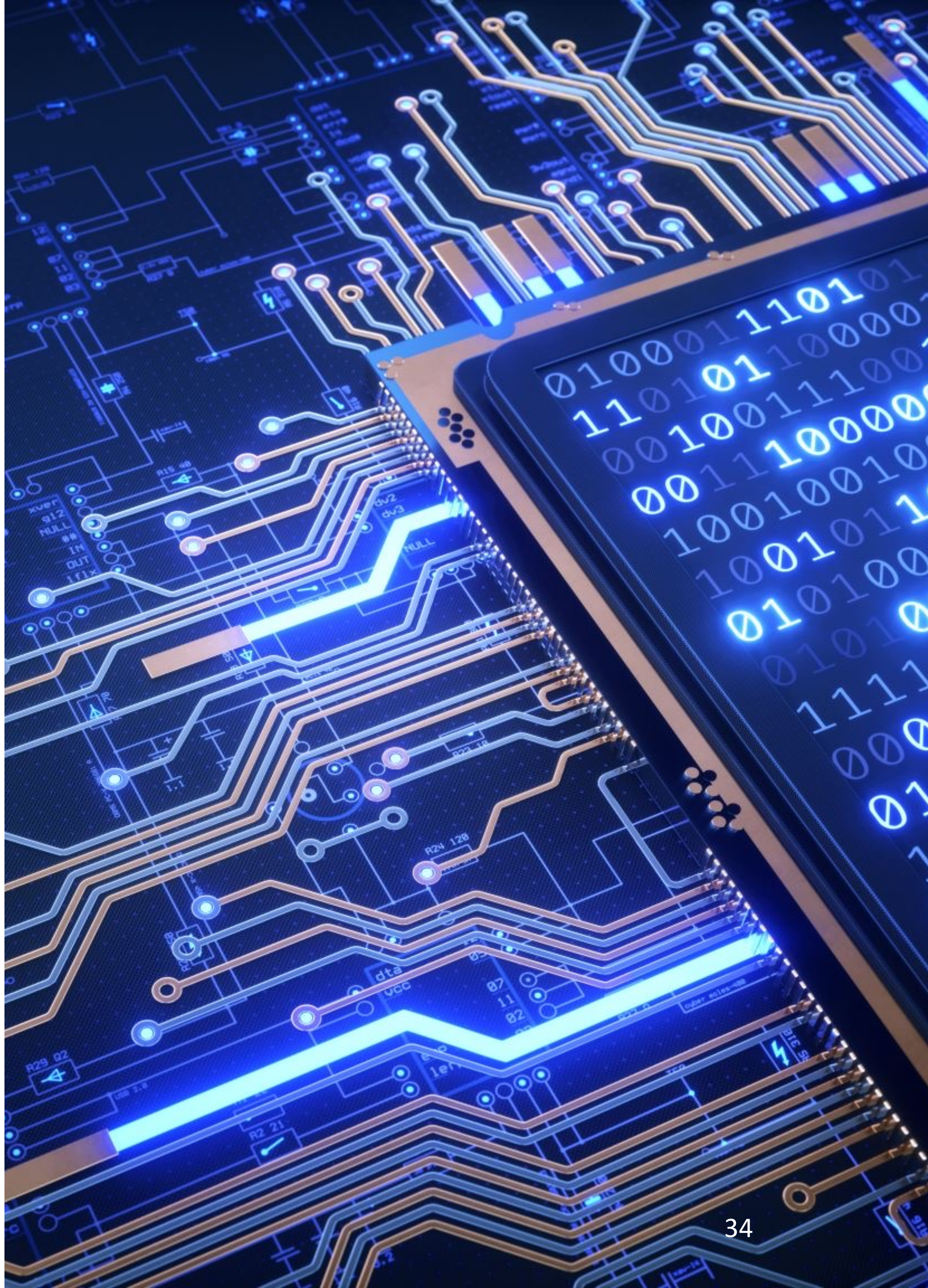
Worked Examples

1) Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Answer: 10 1010 1010, 01 0101 0101, *and* 11 1110 0000

2) What is the bit string for the complement of the set of all odd integers in U above?

Answer: 01 0101 0101

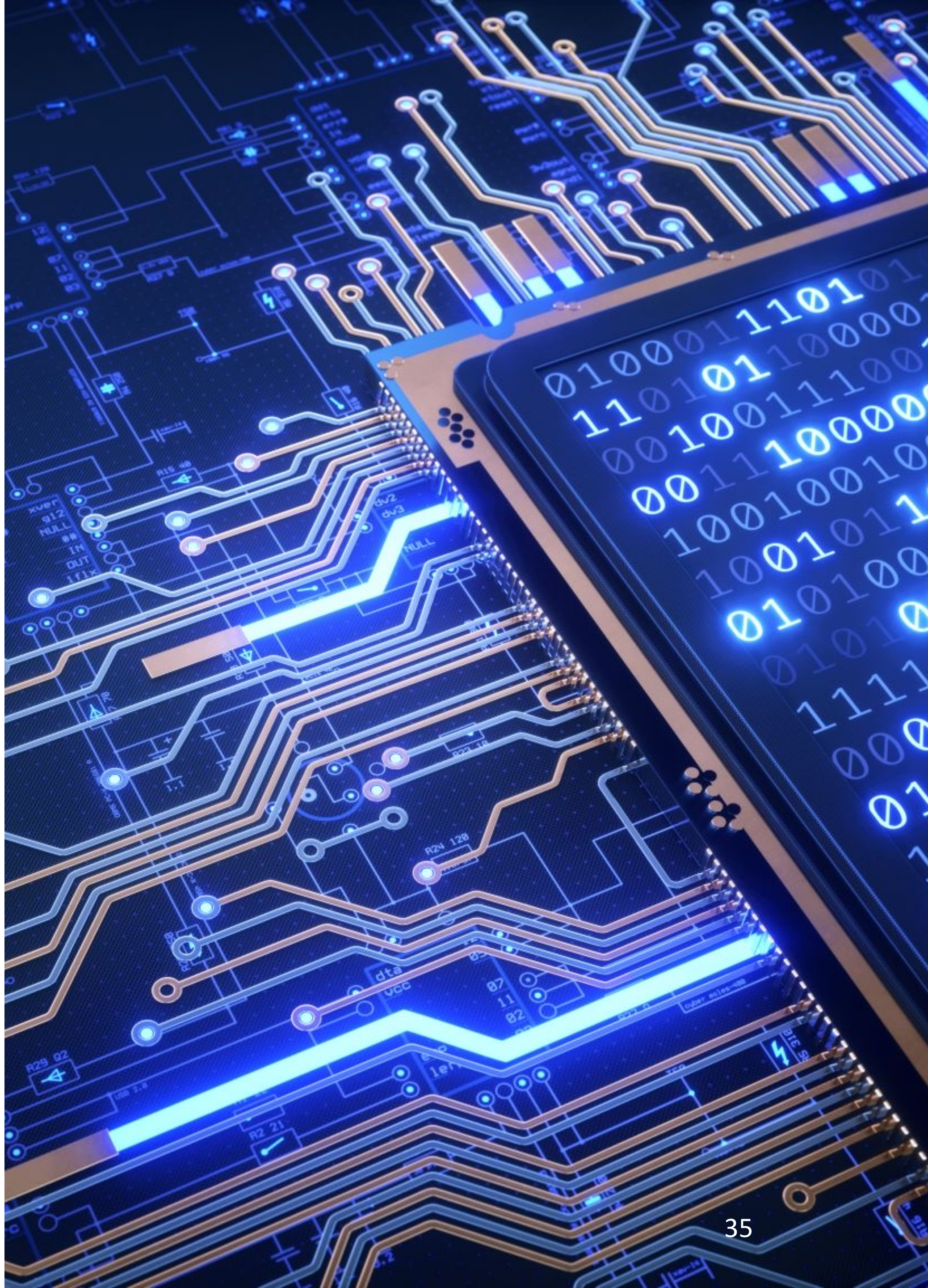


Computer Representation

- To obtain the bit string for the **union and intersection** of two sets, we perform bitwise Boolean operations on the bit strings representing the two sets.
- Bit string for the union is the bitwise *OR* of the bit strings for the two sets
- The bit string for the intersection is the bitwise *AND* of the bit strings for the two sets
- *Example*:
  - Find the Union and intersection of {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} , where U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

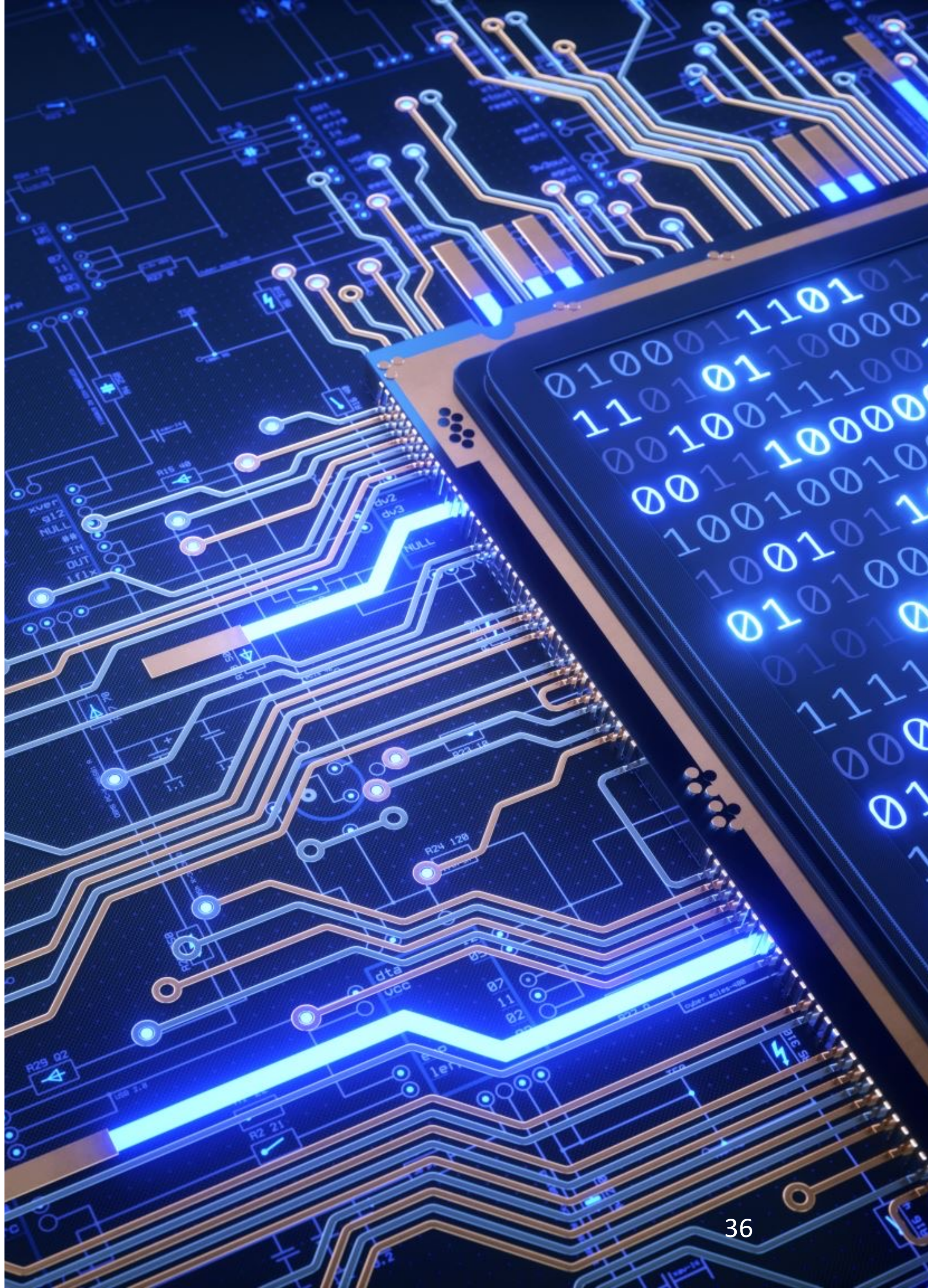
Answer: Union:- 11 1110 0000 $\vee$ 10 1010 1010 = 11 1110 1010

Intersection:- 11 1110 0000 $\wedge$10 1010 1010 = 10 1010 0000



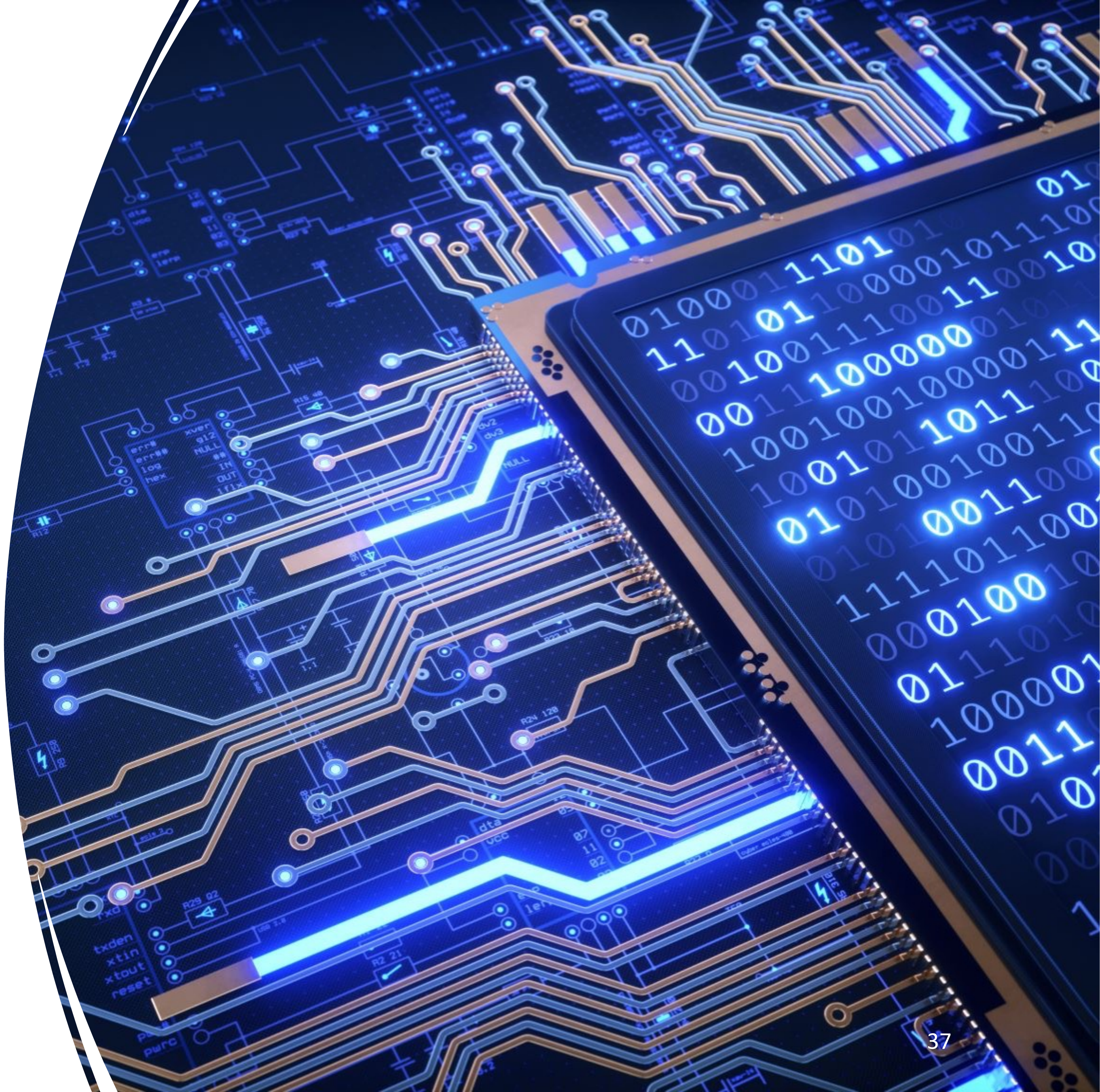
Applications of Sets

- Machine learning and Data mining
- Database Management
- Economics and Finance
- Geometry and Topology
- Graph theory
- Engineering and Operations Research
- Game theory
- Education and Pedagogy
- Biology and Genetics

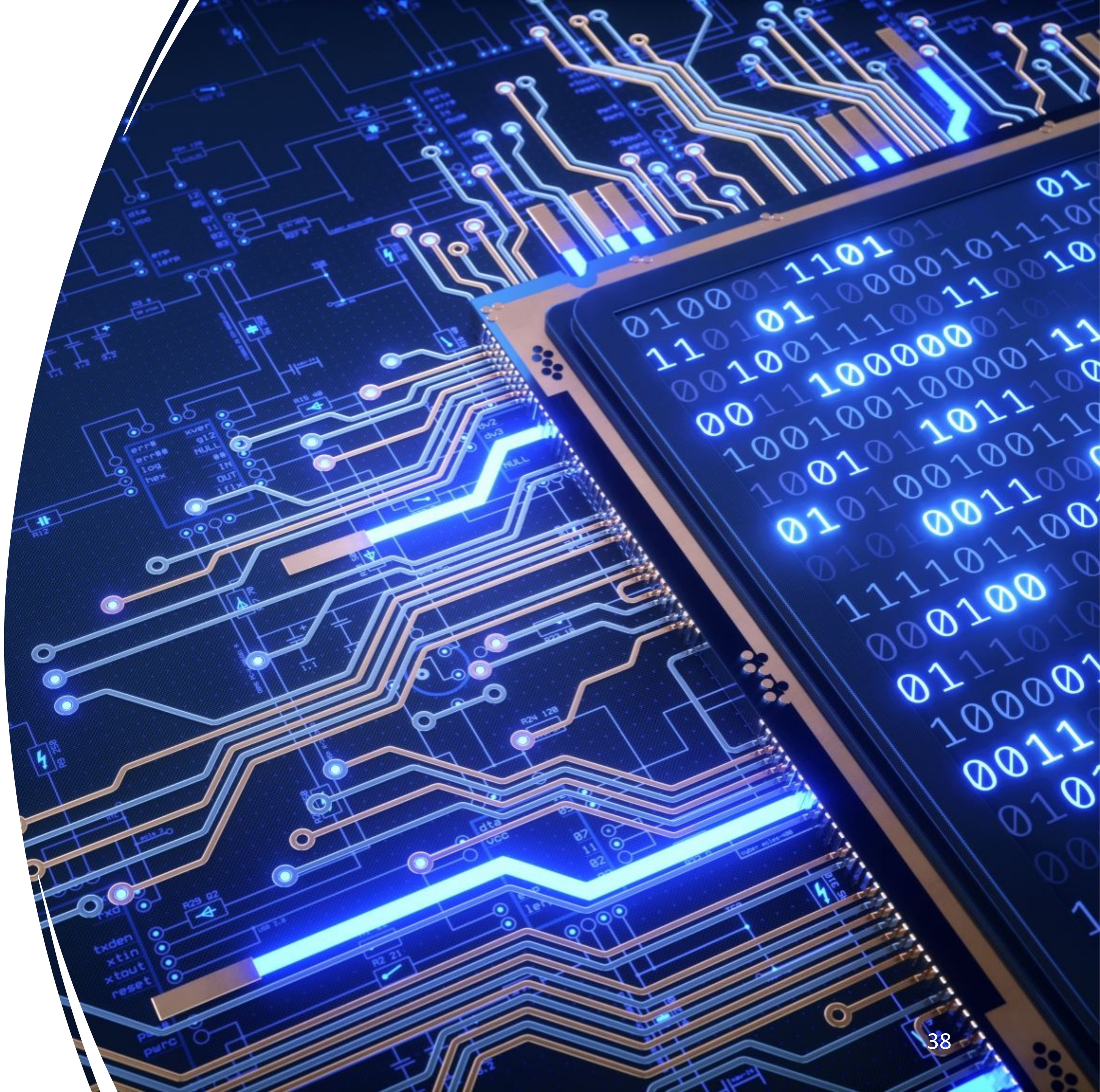


Summary

- Set deals with the
  - collection of objects,
  - studying properties and relationships,
  - solving problems in diverse domains.
- Cardinality
- Cartesian product
- Set operations
- Computer representation



Reference



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Chapter 2

See you next time!