Course: Mathematics for IT Professionals

Lecture 6

Functions

By
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Outline

The topics to be treated in this lecture are:

- Domain, Co-domain and Range
- One-to-one functions
- Onto functions
- Bijection
- Inverse function
- Composition
- Graphical representation
- Floor and Ceiling
- Applications of Functions

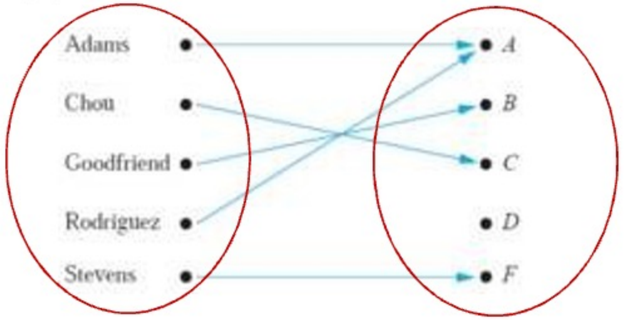
Lecture Learning Outcomes

At the end of the session, you will be able to

- understand the definition of a function
- differentiate between domain, co-domain and range
- identify functions as one-to-one, onto or both
- compute the inverse and composition of functions
- represent functions graphically
- understand the floor and ceiling of functions
- have knowledge on various applications of functions

Introduction

- A **function** is a ***mapping*** of one set of elements to another set



- Extremely important concept in Mathematics and Computer Sc.
  - Functions are used in the definition of other discrete structures such as *sequences* and *strings*.
  - Functions are used to represent the time taken by a computer to solve problem of given size
  - Computer programs and subroutines are sometimes designed based on the calculation of some function values

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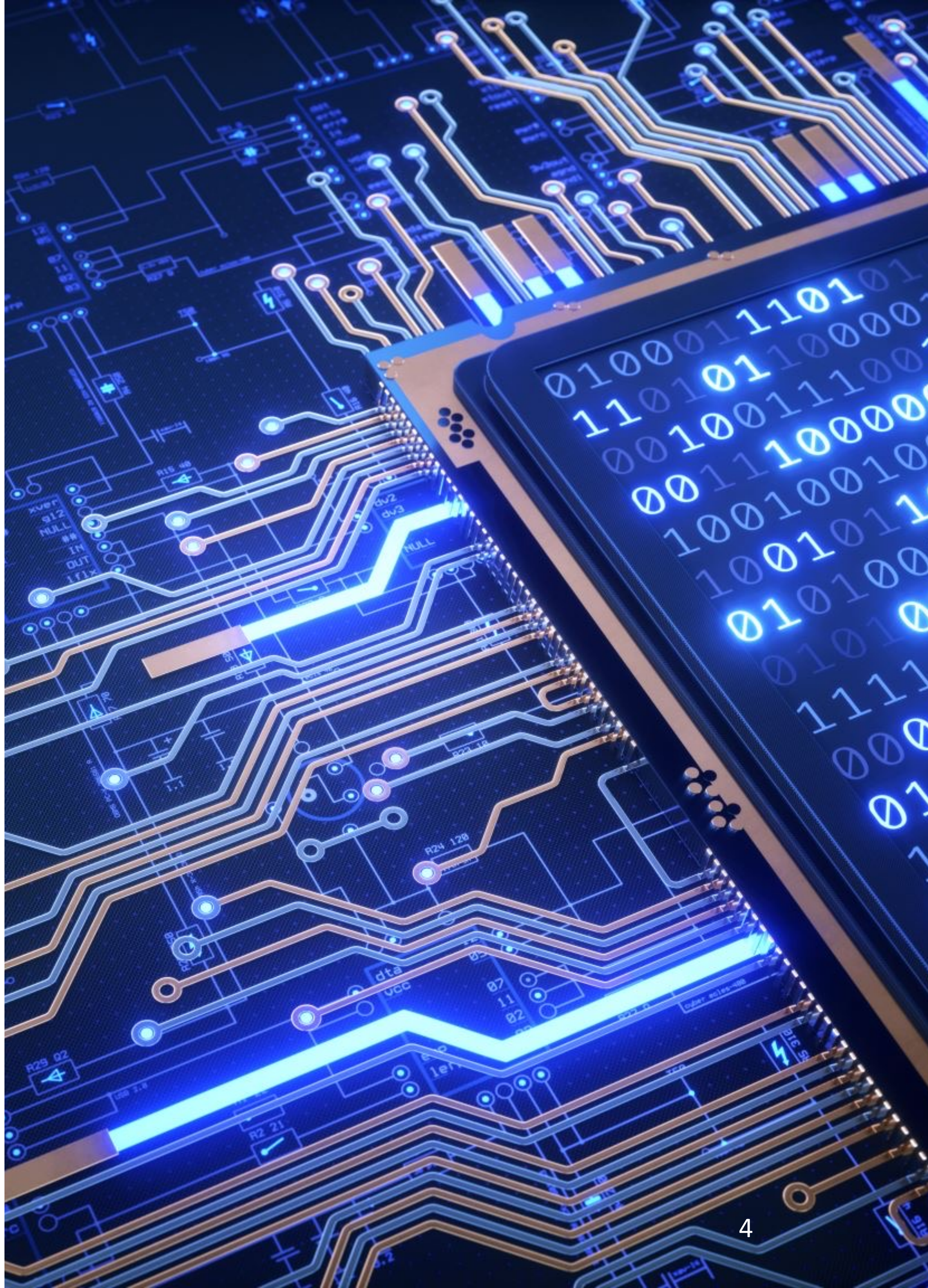

Definition of a Function

- Let A and B be *non-empty* sets. A ***function f*** from A to B is an assignment of exactly one element of B to each element of A.

  A function f from A to B, is written as $f : A \to B$.

  We write $f(a) = b$ where ***b*** is the unique element of B assigned by the function f to the element ***a*** of A
- Ways of specifying functions:
  - Showing the assignments between elements of sets explicitly
  - Giving a formula e.g., $f(x) = x + 1$
  - In terms of a relation (i.e., a subset of $A \times B$)
    - A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B
  - Graphical representation

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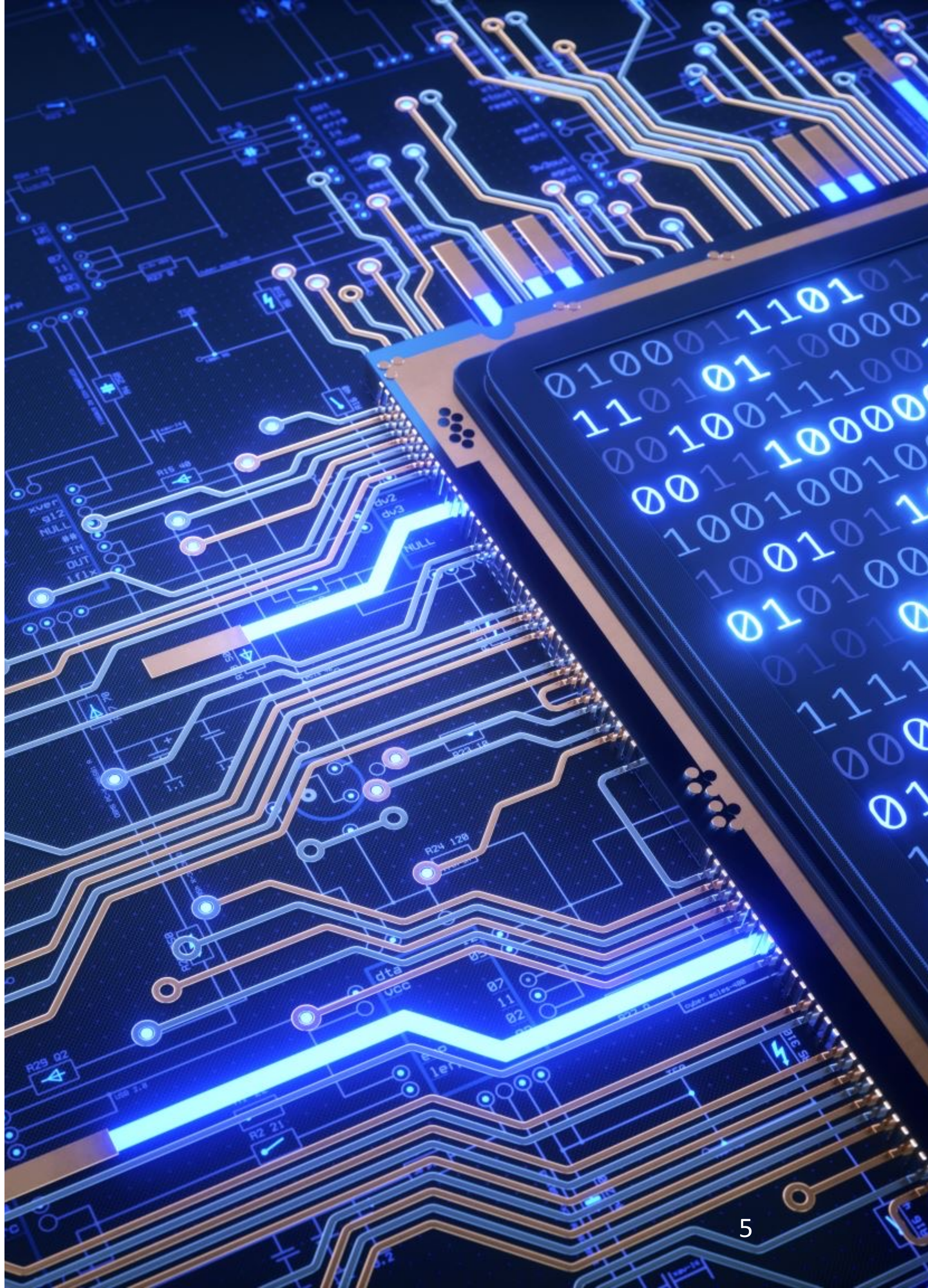

Preliminaries

- In a function $f : A \rightarrow B$,

  A is the ***domain*** of *f* and

  B is the ***codomain*** of *f*
- In $f(a) = b$, we say *b* is the ***image*** of *a* and *a* is a ***preimage*** of *b*
- The ***range of f*** (or ***image of f***) is the set of all images of elements of *A*
- Two functions are ***equal*** when they have the *same domain*, *same codomain*, and map each element of their common domain to the same element in their common codomain.
- if either the *domain* or the *codomain* or the *mapping* of elements *is changed*, we get a *different function*

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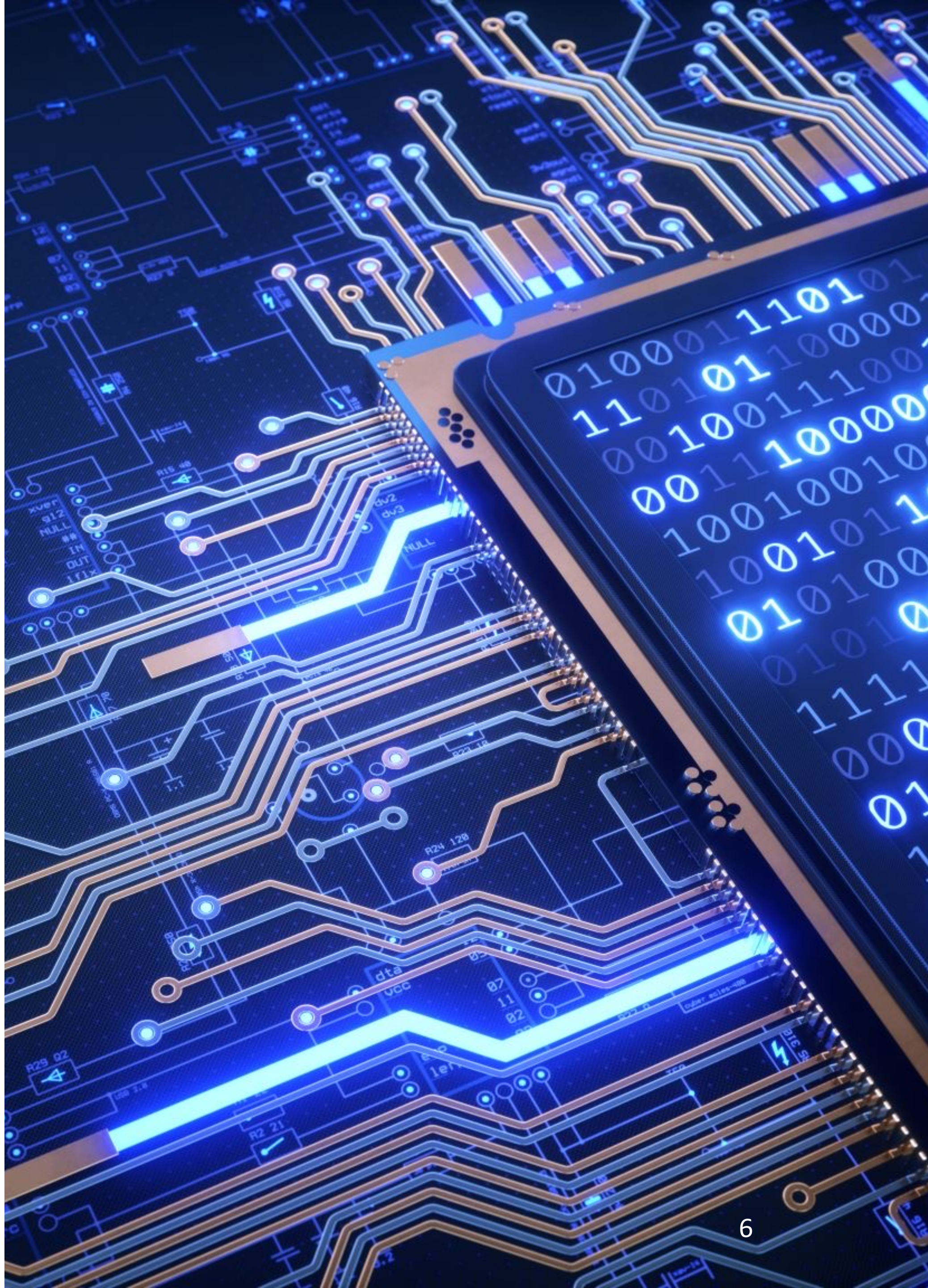
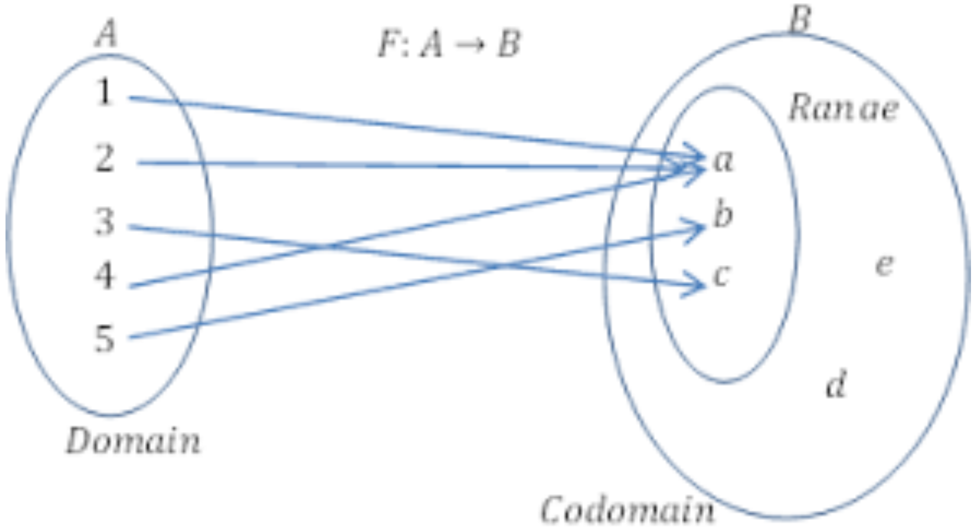


Preliminaries



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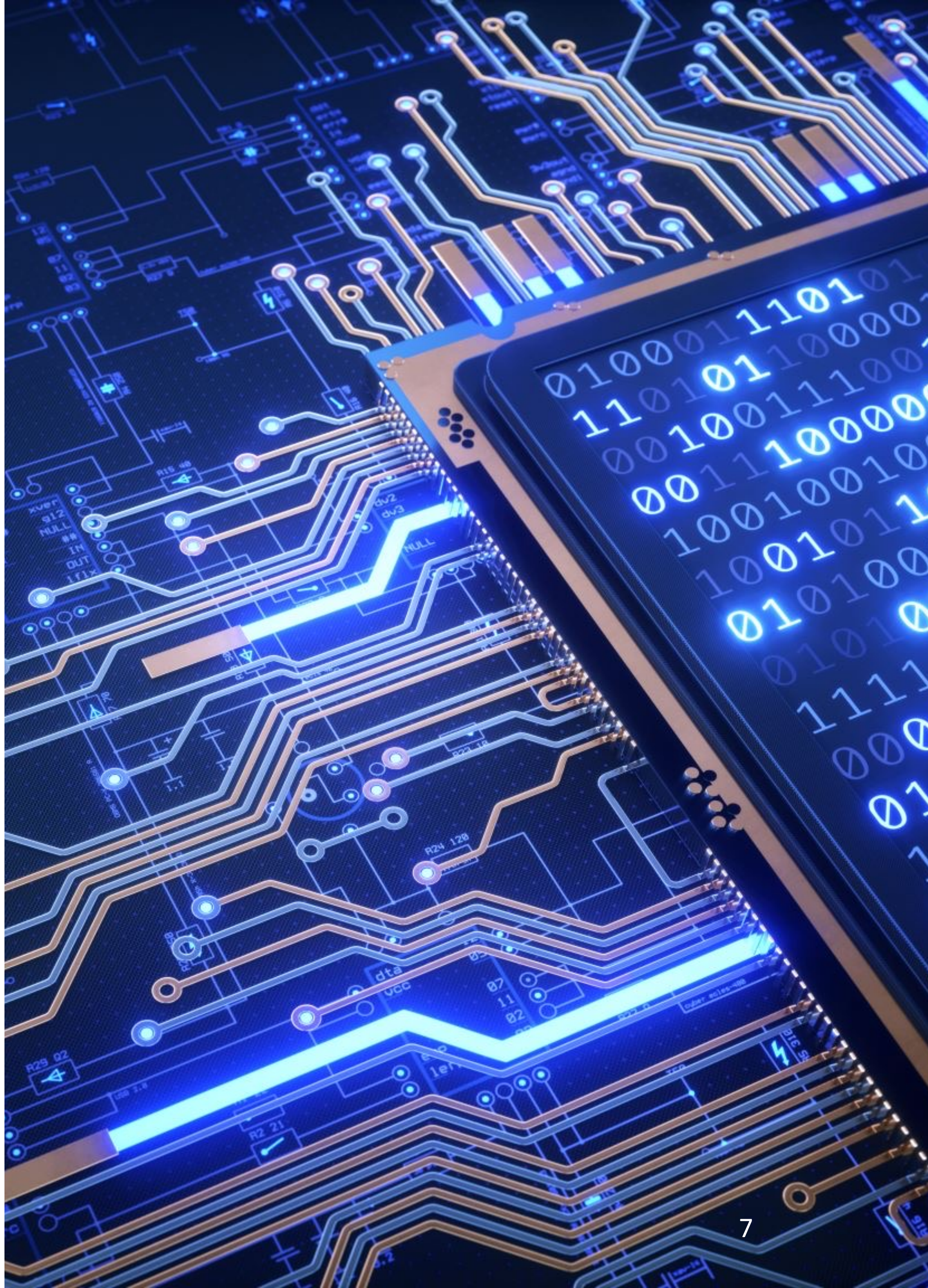
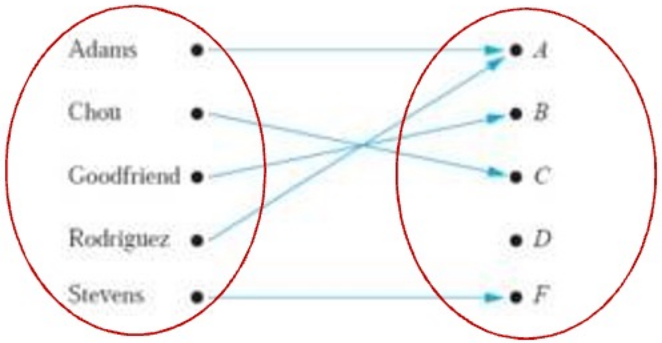


Example I

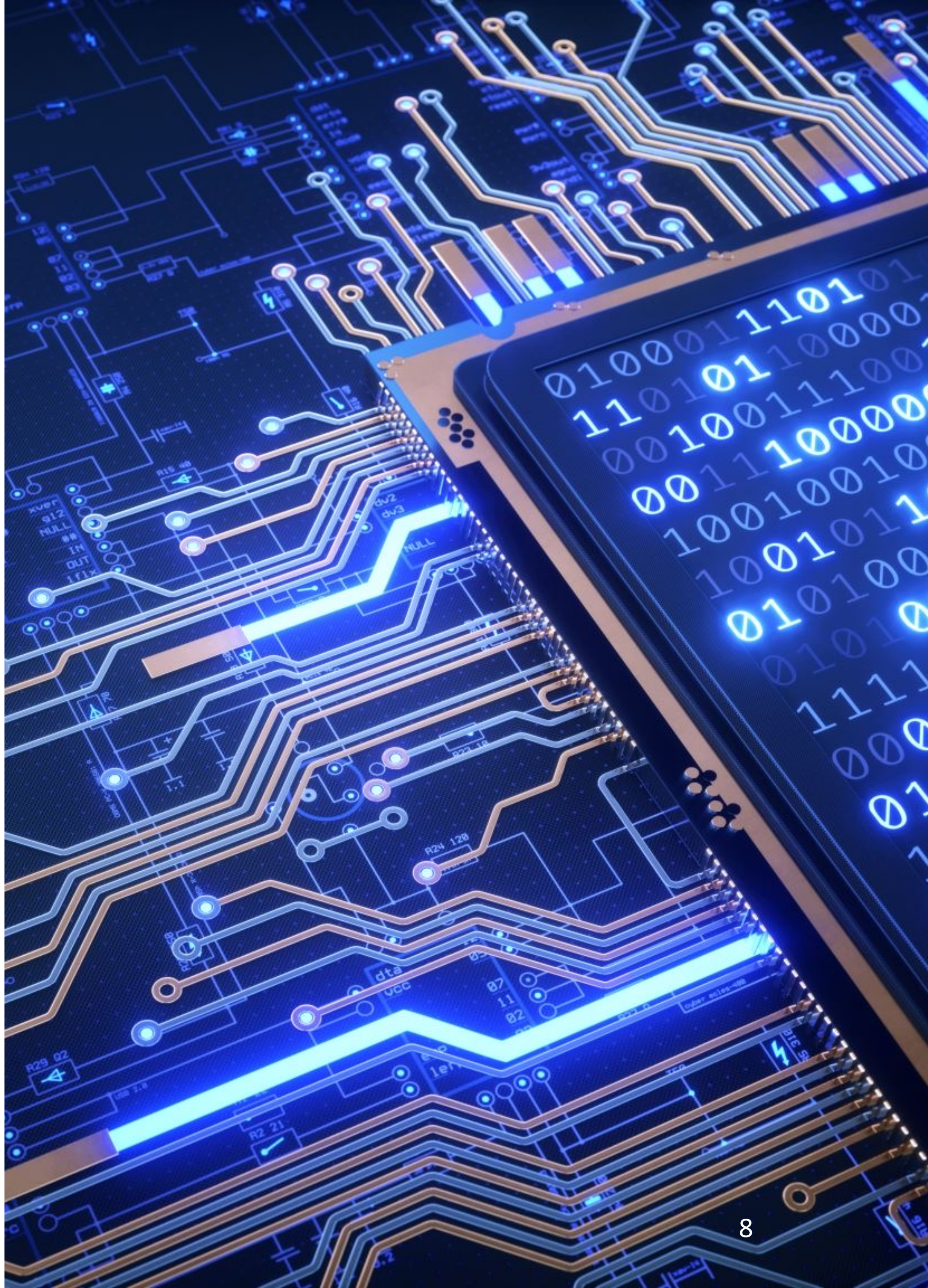
Consider the function that assigns grades to students



Domain: {Adams, Chou, Goodfriend, Rodriguez, Stevens}

Codomain: $\{A, B, C, D, F\}$

Range: $\{A, B, C, F\}$



Example II

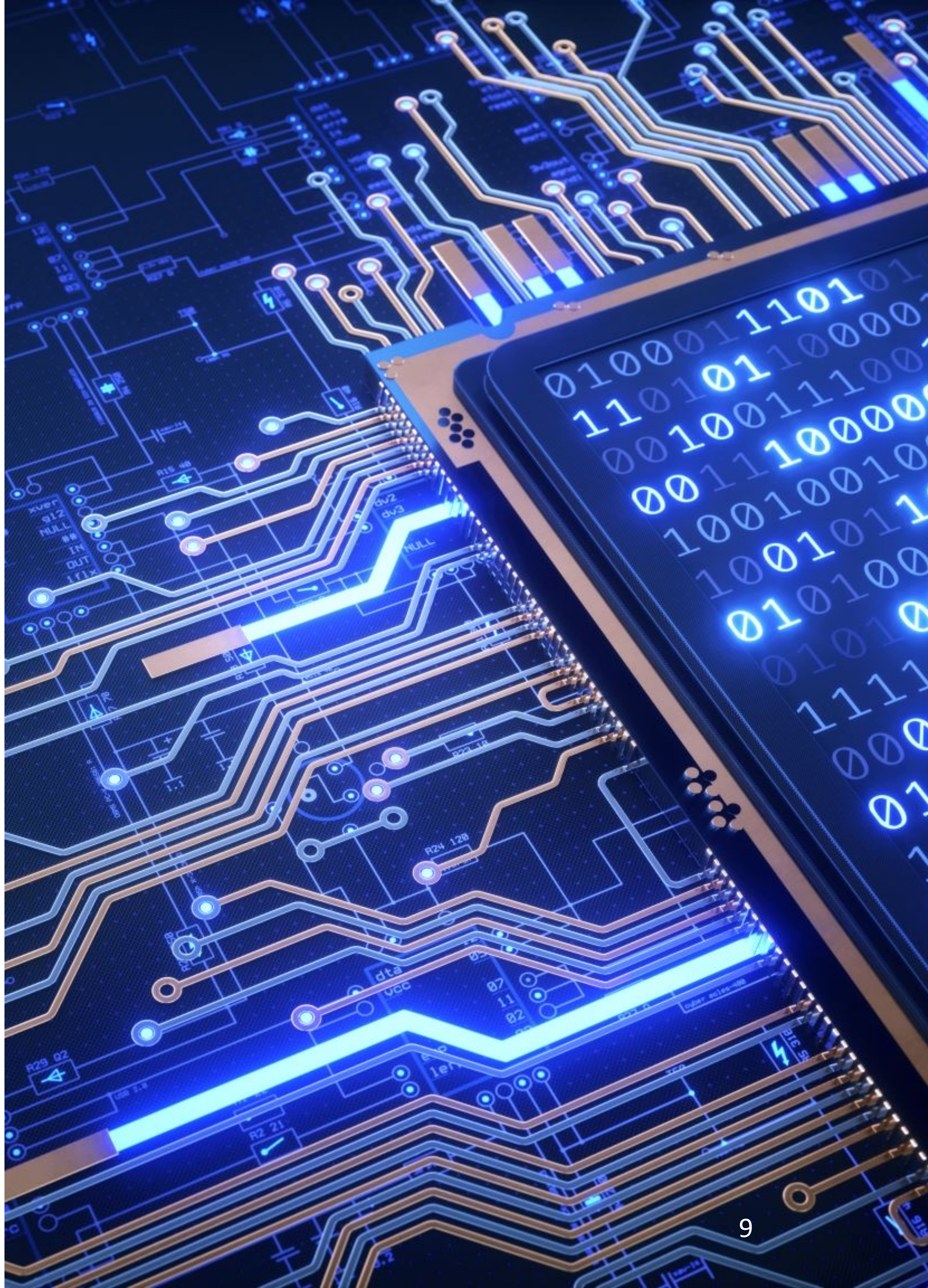
Let R be the relation with ordered pairs (Kojo, 22), (Amina, 24), (Kofi, 21), (Lizzy, 22), (Yaw, 24), and (Akua, 22), where each pair consists of a student's name and age.

Let f be the function specified by R.. Then we have,

f (Kojo) = 22
f (Amina) = 24
f (Kofi) = 21
f (Lizzy) = 22
f (Yaw) = 24 and
f (Akua) = 22

$f(x)$ = age of x, where *x is a student*

Domain: {Kojo, Amina, Kofi, Lizzy, Yaw, Akua}
Codomain: $\{x \in \mathbf{Z}^+ \mid x < 100\}$
Range: {21, 22, 24}



Example III

Let *f* be the function that assigns the last two bits of a bit string of length greater than or equal to 2, to that string.

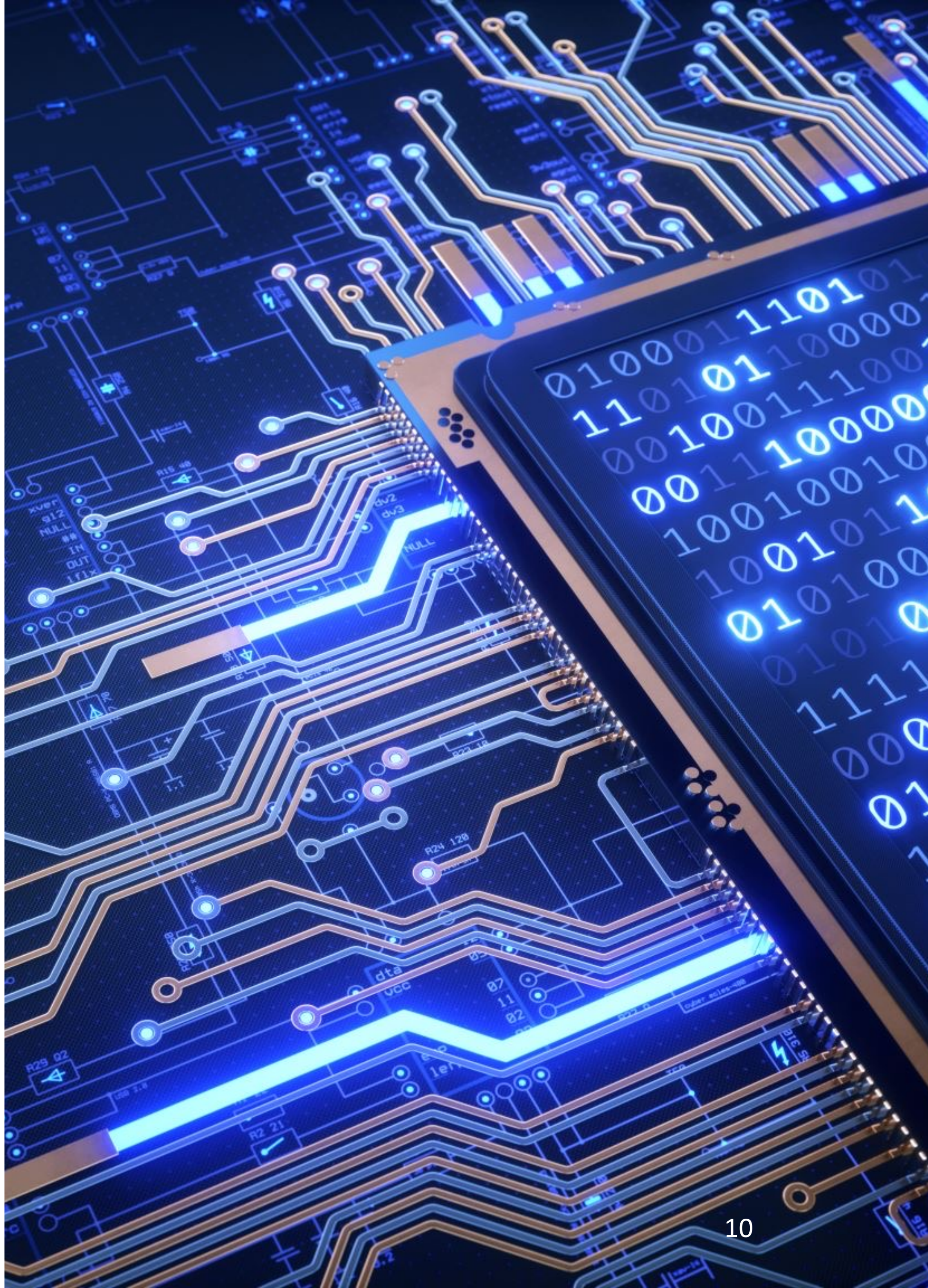
For example, *f* (11010) = 10.

For this function,

Domain: the set of all bit strings of length greater than or equal to 2

Codomain: {00, 01, 10, 11}

Range: {00, 01, 10, 11}



Example IV

- Consider the function $f : \mathbf{Z} \rightarrow \mathbf{Z}$, which assigns the square of an integer to that integer. Then, $f(x) = x^2$, where

  the domain of f is the set of all integers,

  the codomain of f is the set of all integers, and

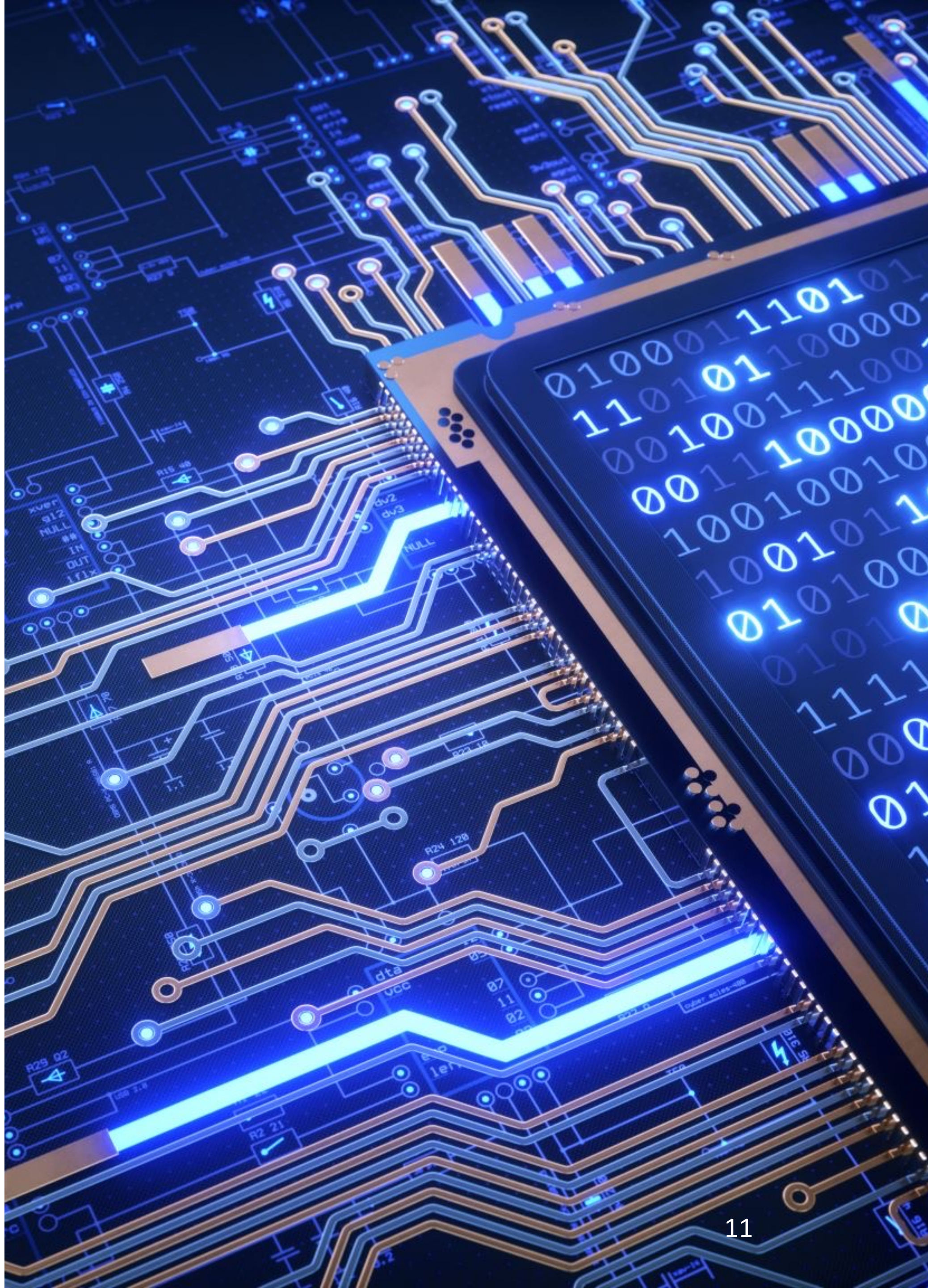
  the range of f is the set of all integers that are perfect squares

  that is, $\{0, 1, 4, 9, 16, \ldots\}$

- Consider the Java program statement

```
int area(float r) {
        . . .
}
```

The domain in this function **`area`** is the set of real numbers and the codomain is the set of …?



Addition and Multiplication

- A function having the codomain as the set of real numbers is called a **Real-valued function**
- A function having the codomain as the set of integers is called an **Integer-valued function**
- *Two real-valued functions or two integer-valued functions with the same domain can be added, as well as multiplied*

- Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined for all $x \in A$ by

  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

  $(f_1 f_2)(x) = f_1(x) f_2(x)$

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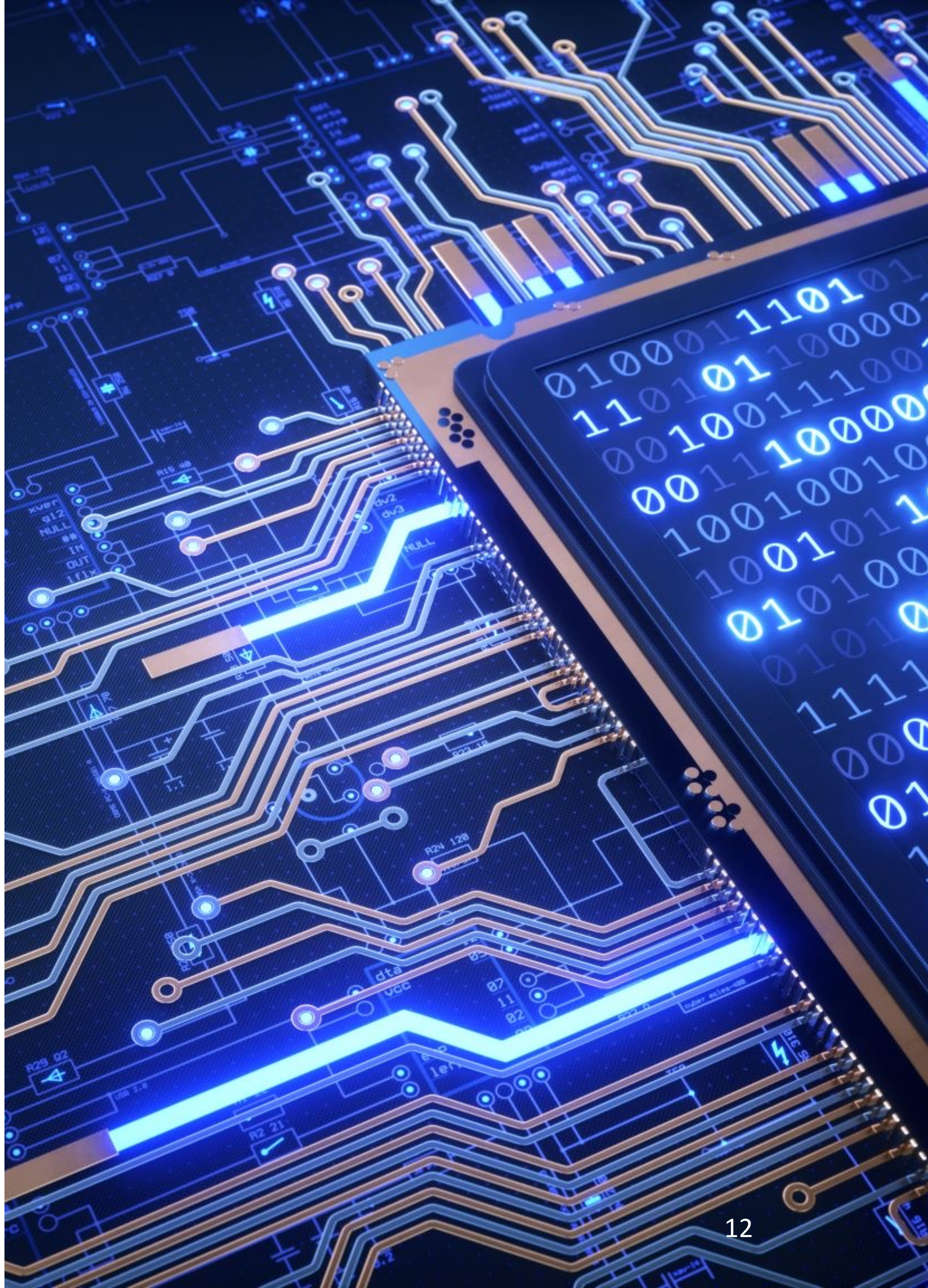
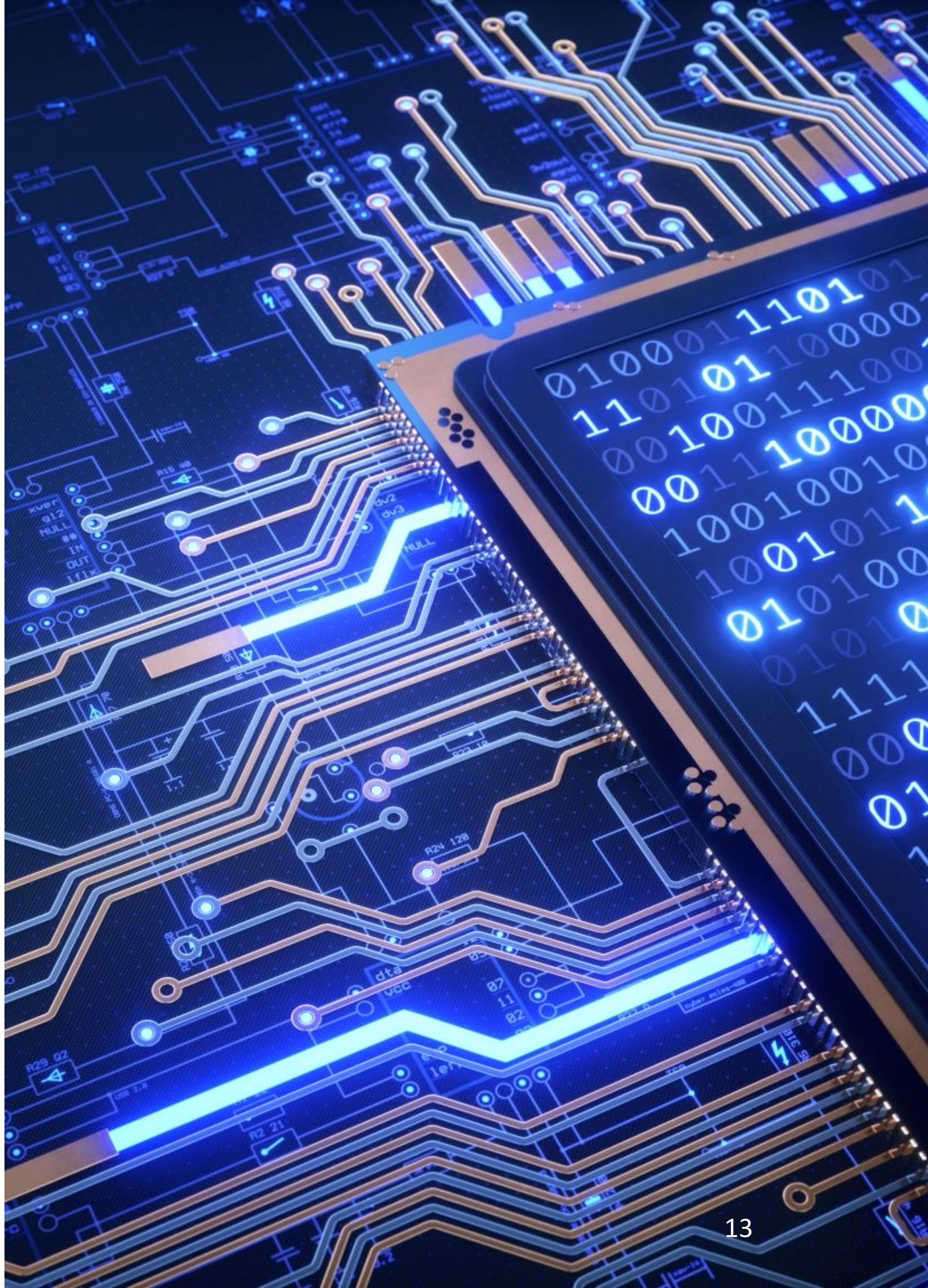


Addition and Multiplication

- Example:

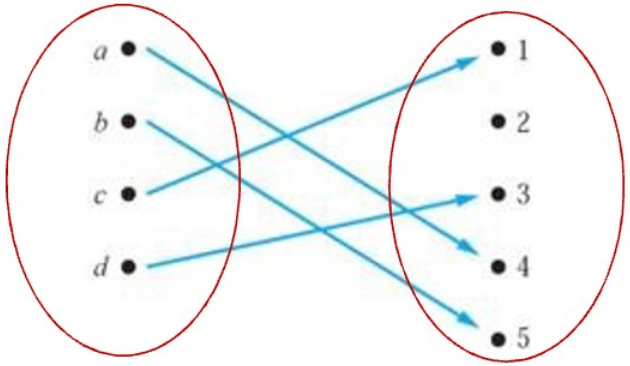
  Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. Then,

$$
\begin{aligned}
(f_1 + f_2)(x) &= f_1(x) + f_2(x) \\
&= x^2 + (x - x^2) \\
&= x \\
(f_1 f_2)(x) &= x^2(x - x^2) \\
&= x^3 - x^4
\end{aligned}
$$



One-to-one function

- A function *f* is referred to as ***one-to-one*** iff *f* (*a*) = *f* (*b*). Thus, *a* = *b* ∀*a, b* in the domain of *f.*
- The function *f* can also be described as **injective function** or an ***injunction***.



- Example:

  The function *f* from {*a, b, c, d*} to {1, 2, 3, 4, 5} with *f* (*a*) = 4, *f* (*b*) = 5, *f* (*c*) = 1, and *f* (*d*) = 3 is one-to-one.

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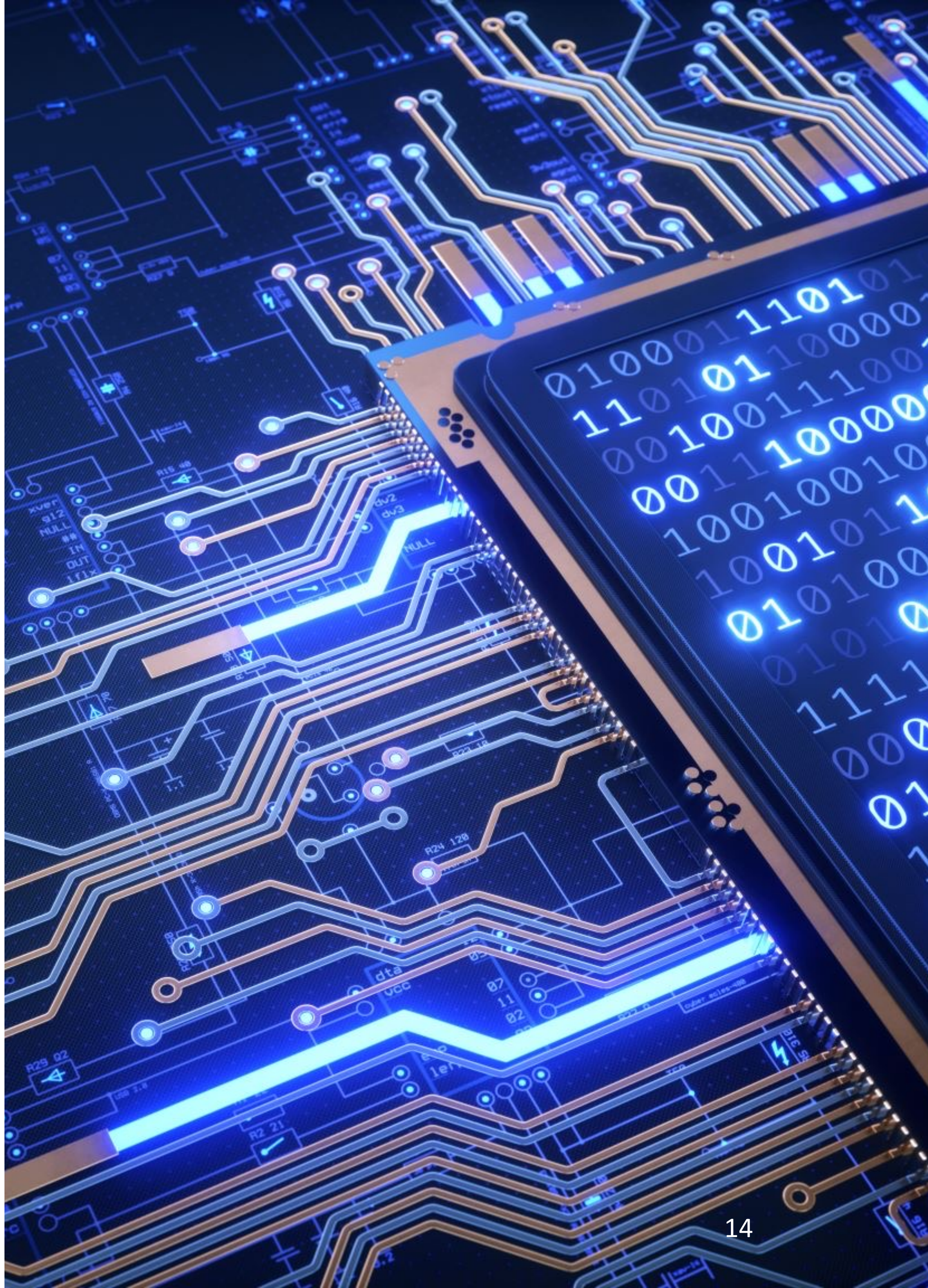


One-to-one function

- Example:

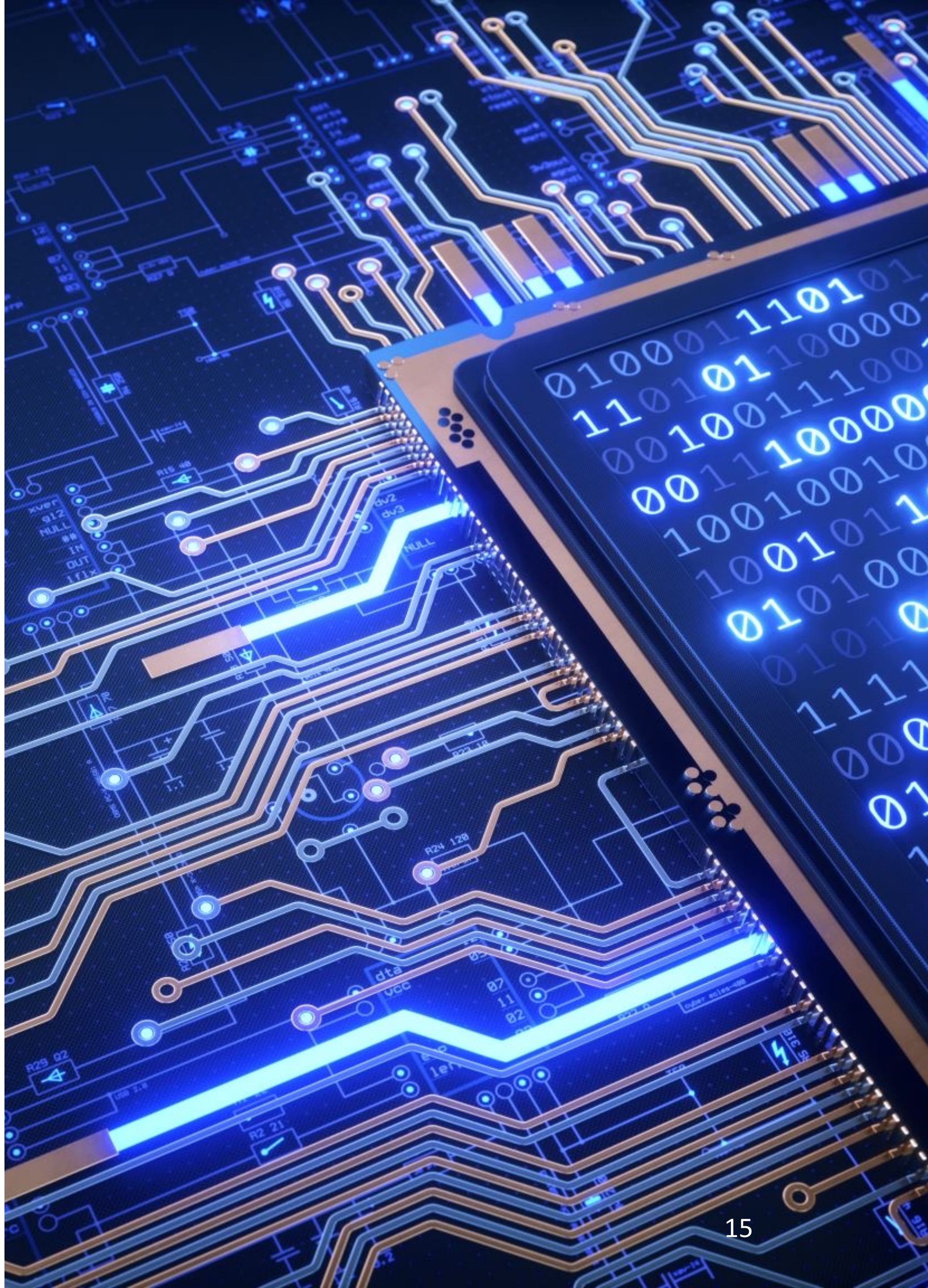
  The function $f(x) = x^2$ from the set of integers to the set of integers is **not** one-to-one because, for example, $f(1) = f(-1) = 1$ but $1 \neq -1$

- Example:

  The function $f(x) = x + 1$ from the set of positive real numbers is one-to-one

- Example:

  Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker. In this situation, the function f that assigns a job to each worker is one-to-one.



One-to-one function

- Guarantee Conditions:

  A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f.

  Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x > y$ and x and y are in the domain of f.

Note: A function that is either strictly increasing or strictly decreasing must be one-to-one.

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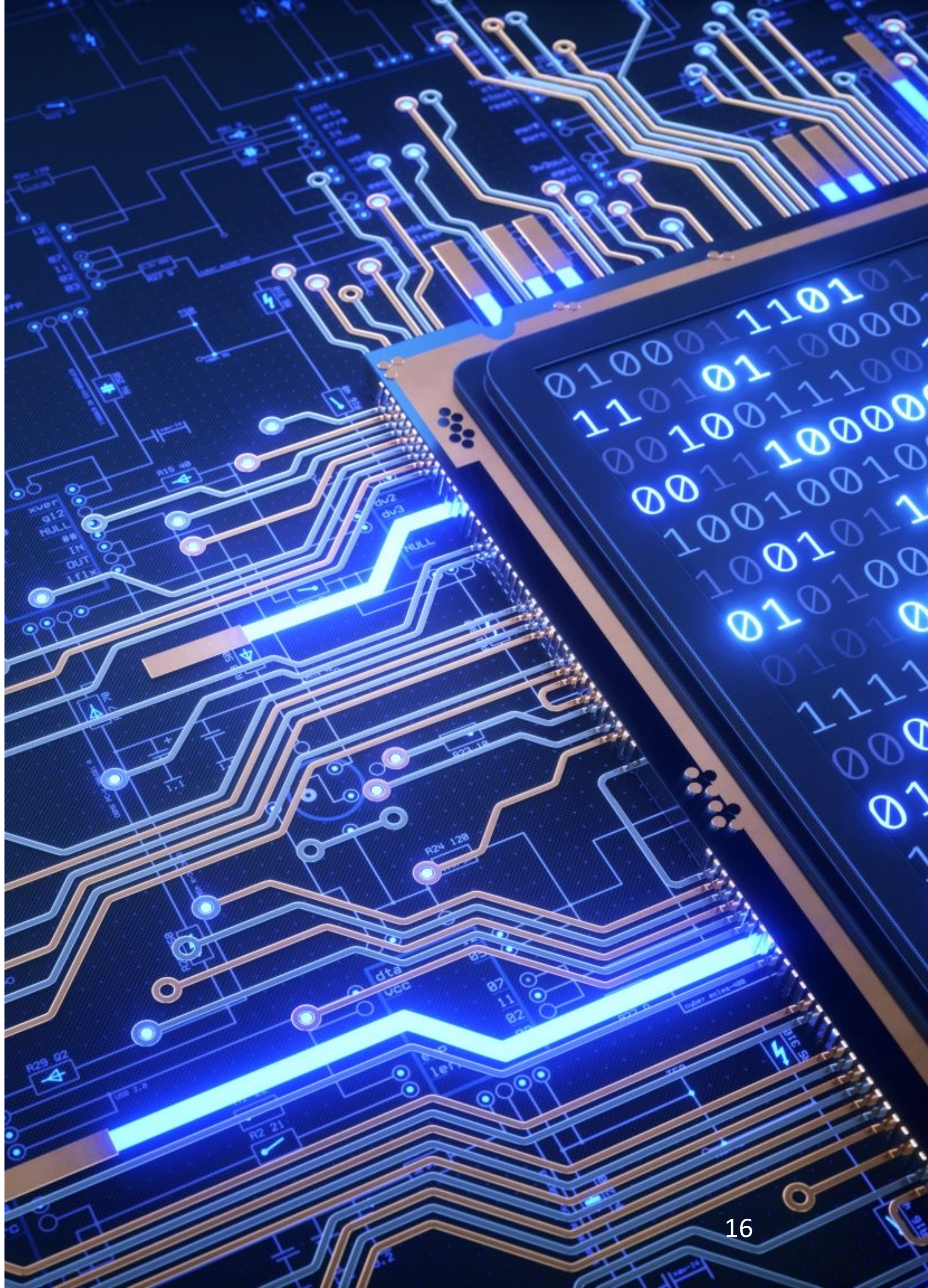
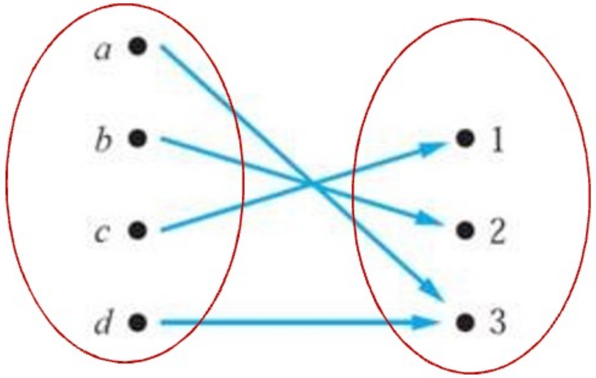


Onto Function

- A function f is called ***onto*** if its range and the codomain are equal. OR
- A function f from A to B is called ***onto****,* or a ***surjection****,* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. An *onto* function is also called ***surjective*** *function*



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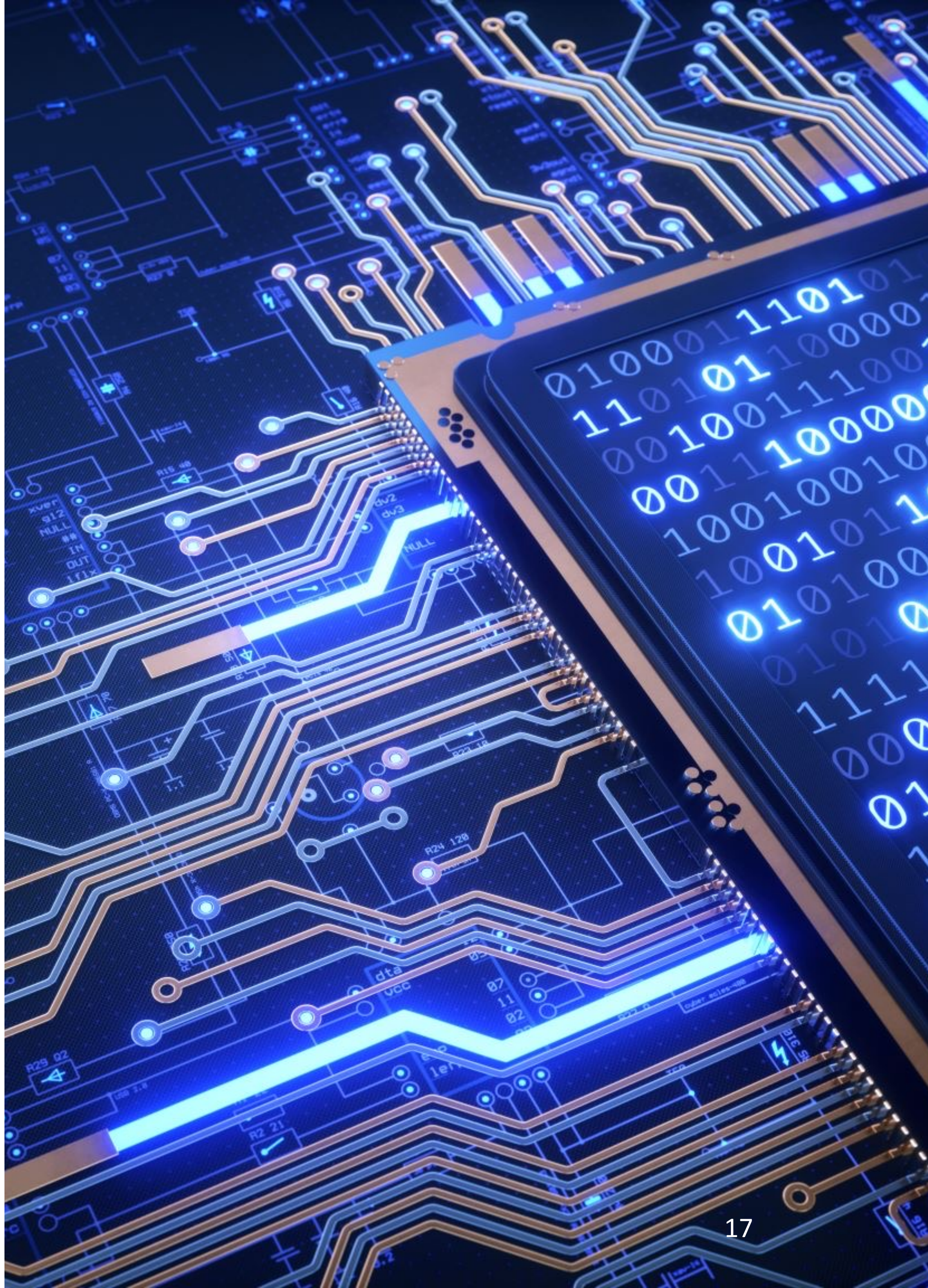


Onto Function

- Example:

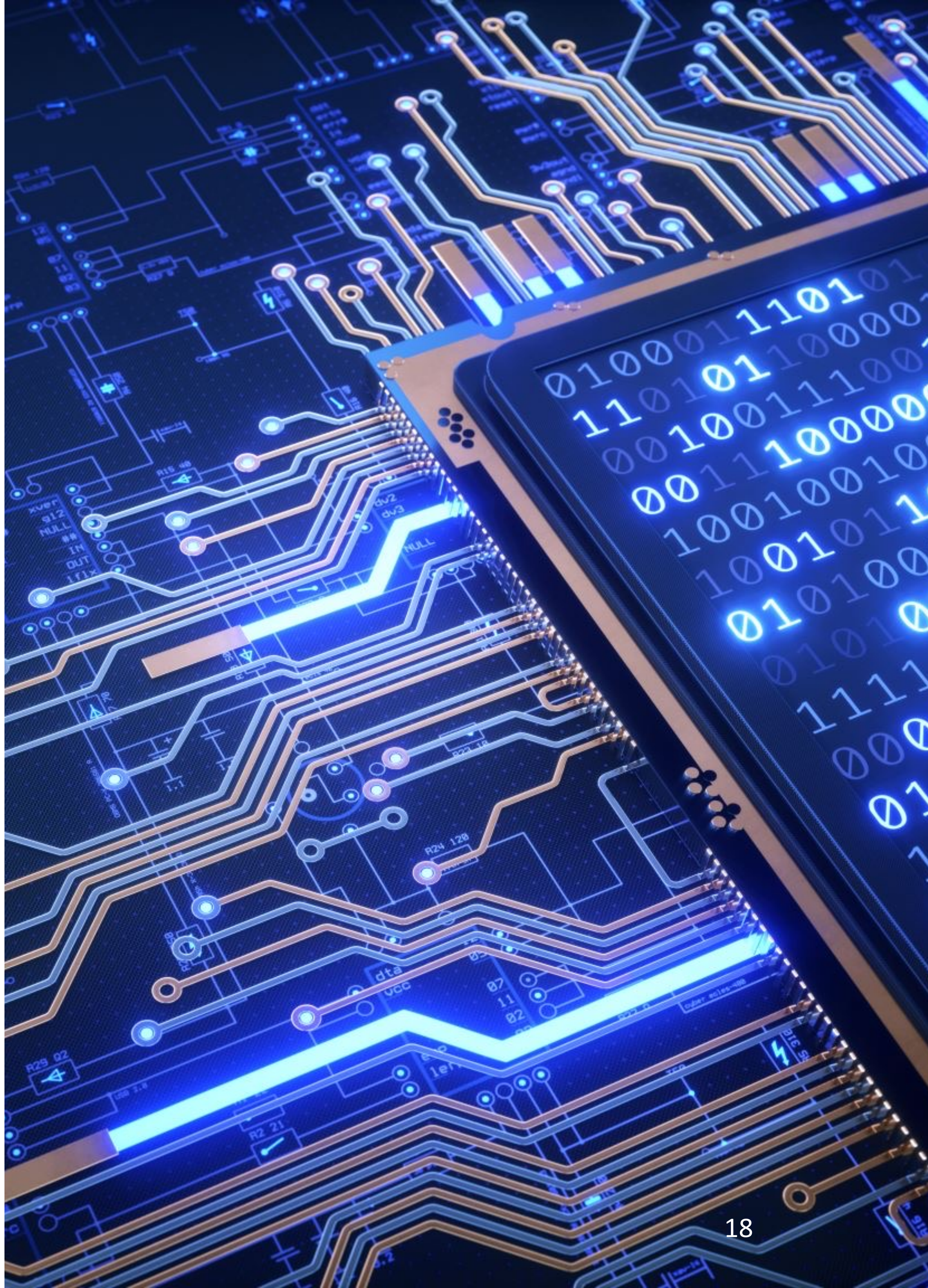
  The function $f(x) = x^2$ from the set of integers to the set of integers ***is not*** *onto*, because, for example there is no integer x with $x^2 = -1$

- Example:

  The function $f(x) = x + 1$ from the set of integers to the set of integers ***is*** onto, because, for every integer y there is an integer x such that $f(x) = y$

- Example:

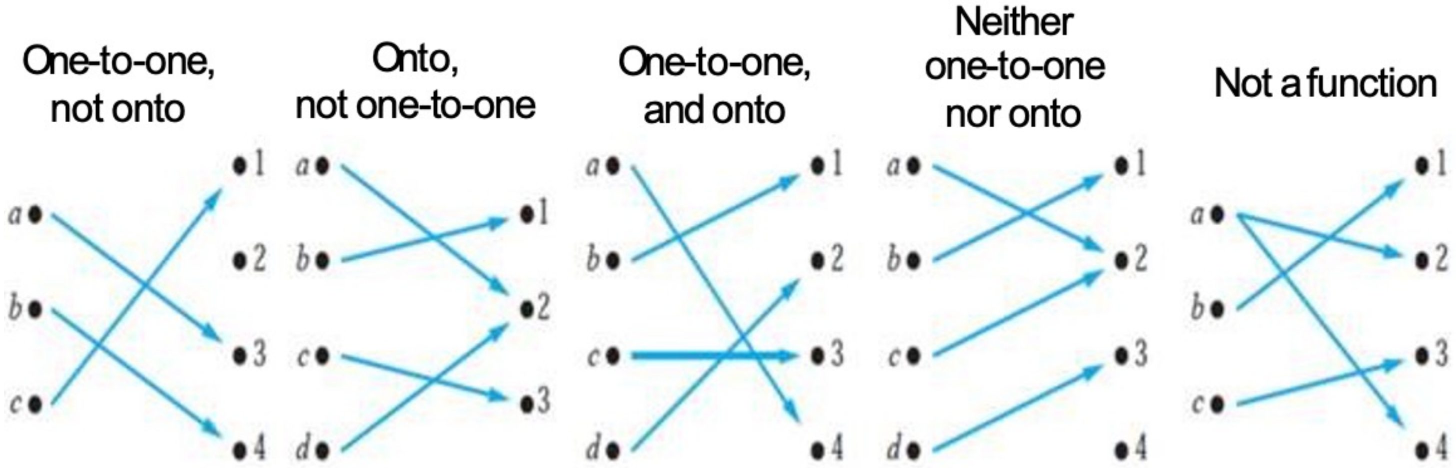
  Consider the function f that assigns jobs to workers. The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned

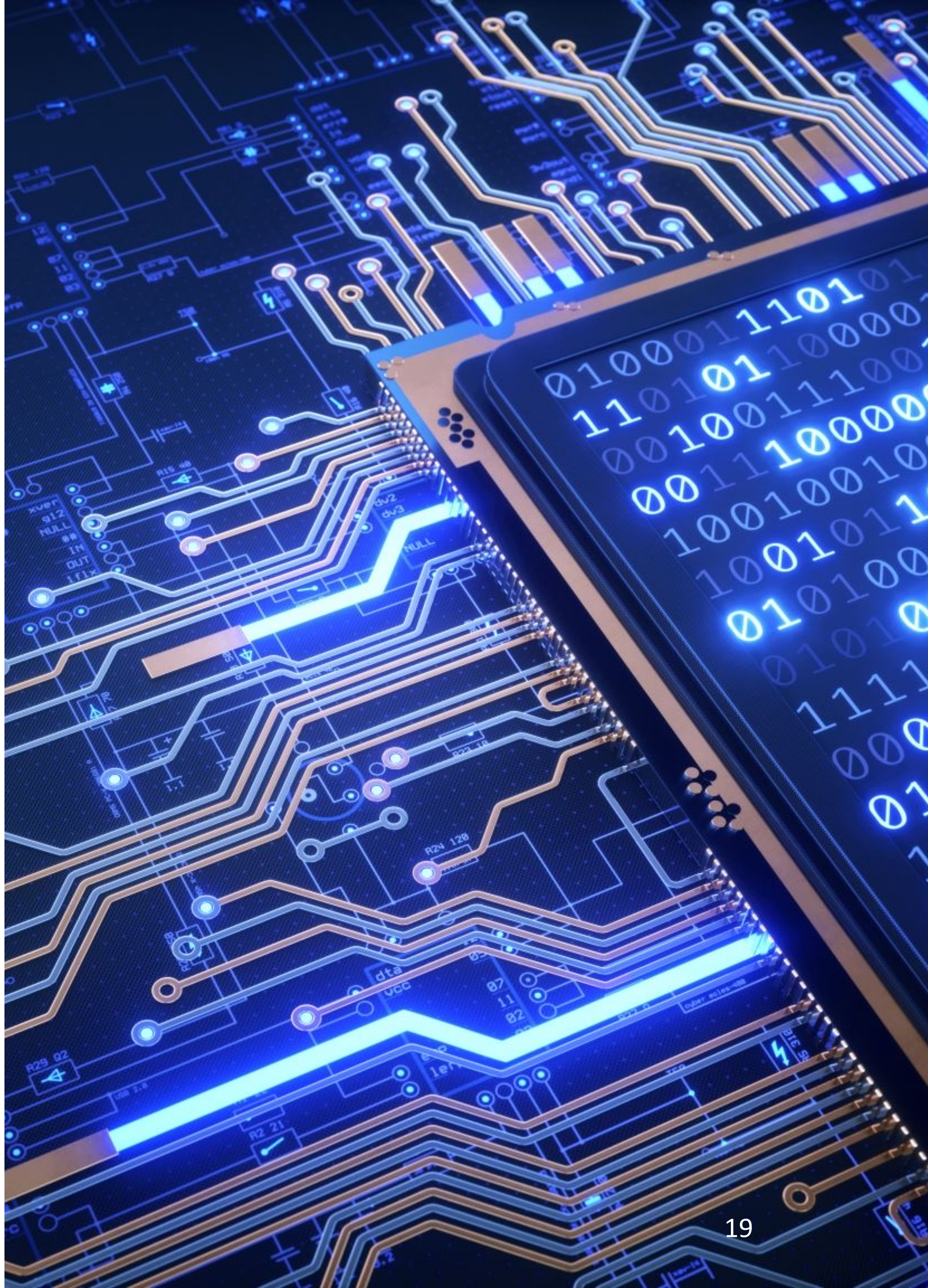


Bijection

- The function *f* is a ***one-to-one correspondence***, or a ***bijection*** or a ***bijective function*** if it is both one-to-one and onto

Different types of correspondences





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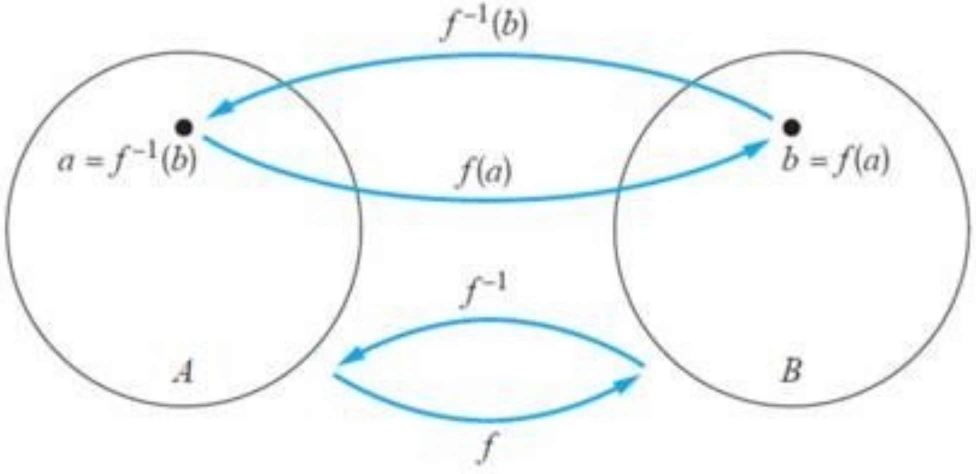
Inverse Function

- Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$



Note: f^{-1} is not same as $1/f$

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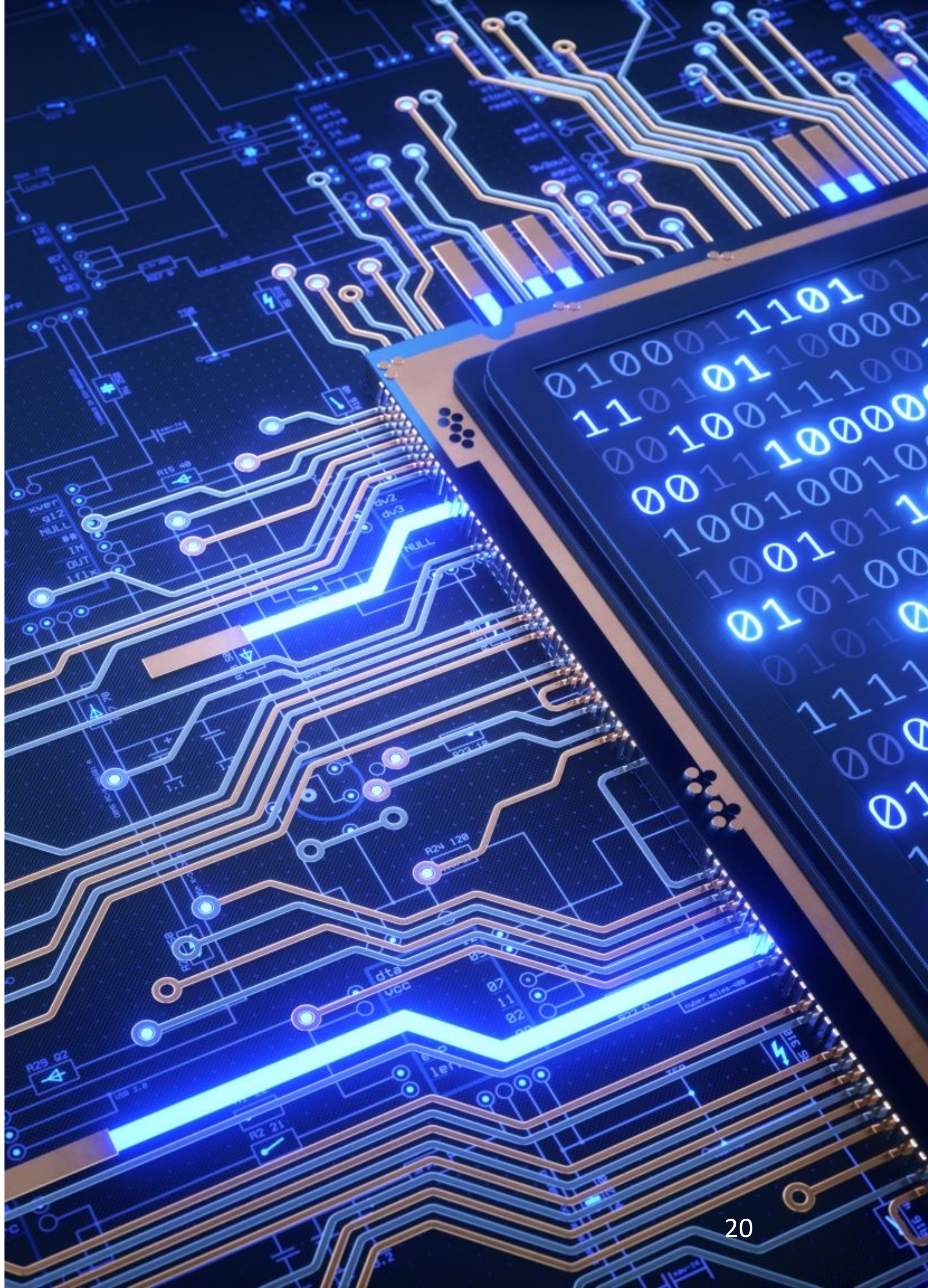


Inverse Function

- If a function f is not a one-to-one correspondence, we cannot define an inverse function of f

- Example:

  Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

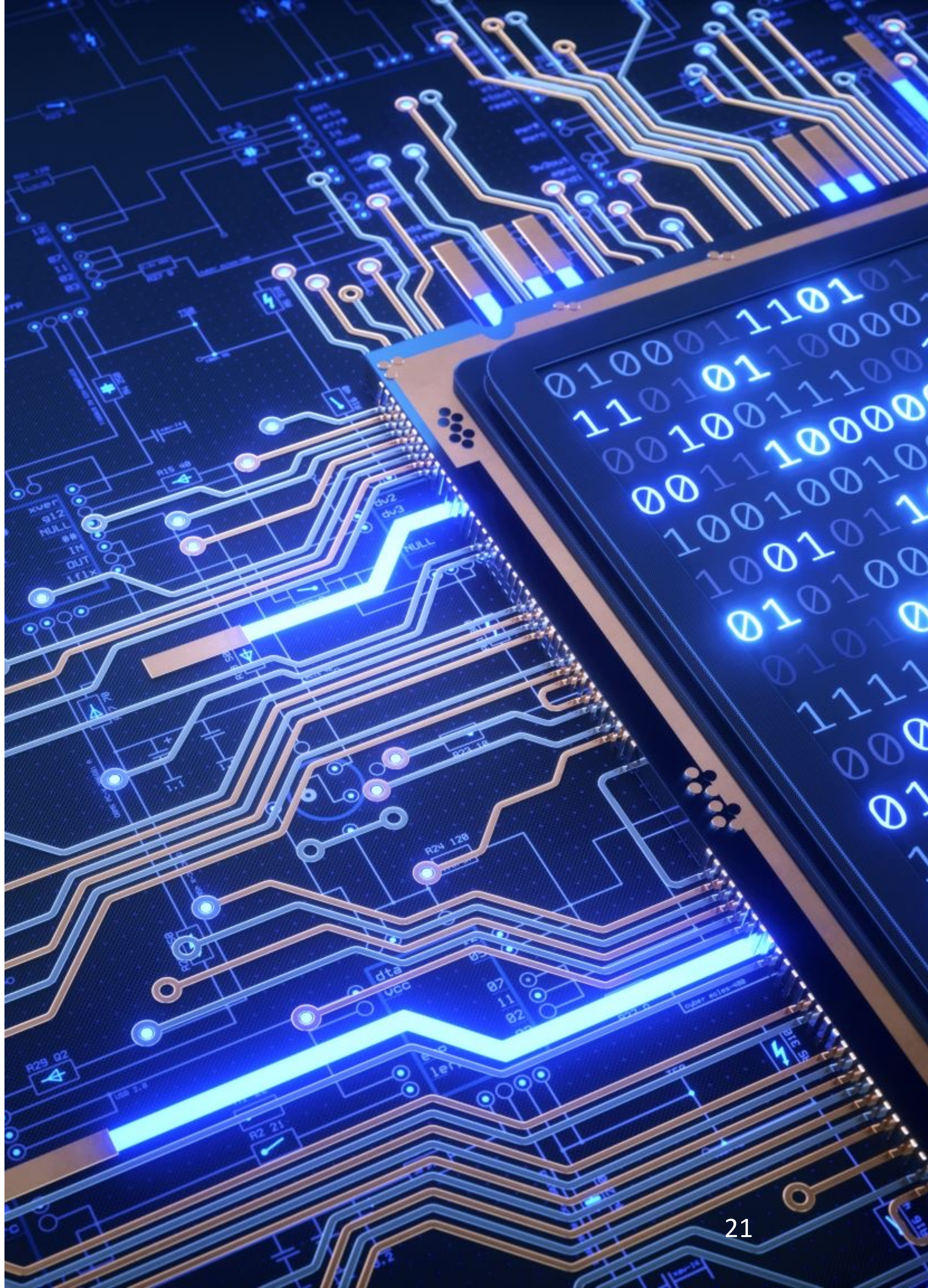
  The function f is invertible because it is a one-to-one correspondence.

  The inverse function f^{-1} reverses the correspondence given by f so

  $f^{-1}(1) = c$,

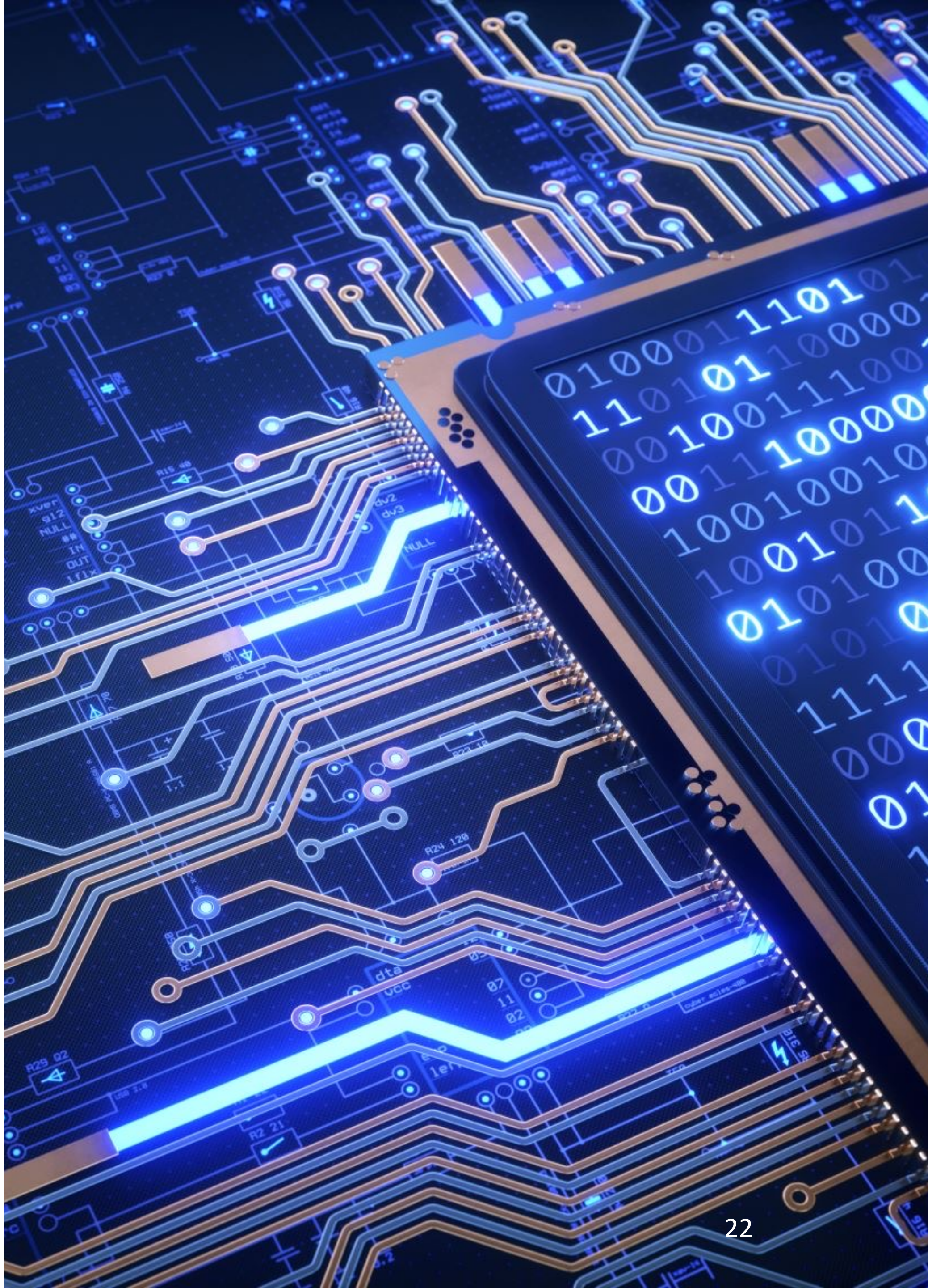
  $f^{-1}(2) = a$, and

  $f^{-1}(3) = b$



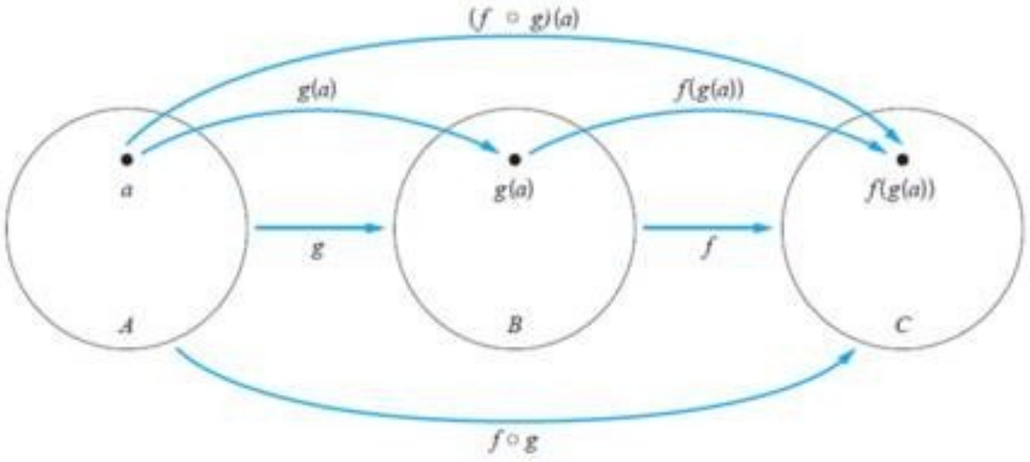
Inverse Function

- Example:
  - Let $f : Z \rightarrow Z$ be such that $f(x) = x + 1$
  - The function f has an inverse because it is a one-to-one correspondence.
  - To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of Z that is sent to y by f .
  - Consequently, $f^{-1}(y) = y - 1$.
- Example:
  - The function f from R to R with $f(x) = x^2$ is not invertible
  - Because, for example, $f(-2) = f(2) = 4$, so f is not one-to-one Also, an inverse function of f has to assign two elements to 4



Composition

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The ***composition*** of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$



Note : The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

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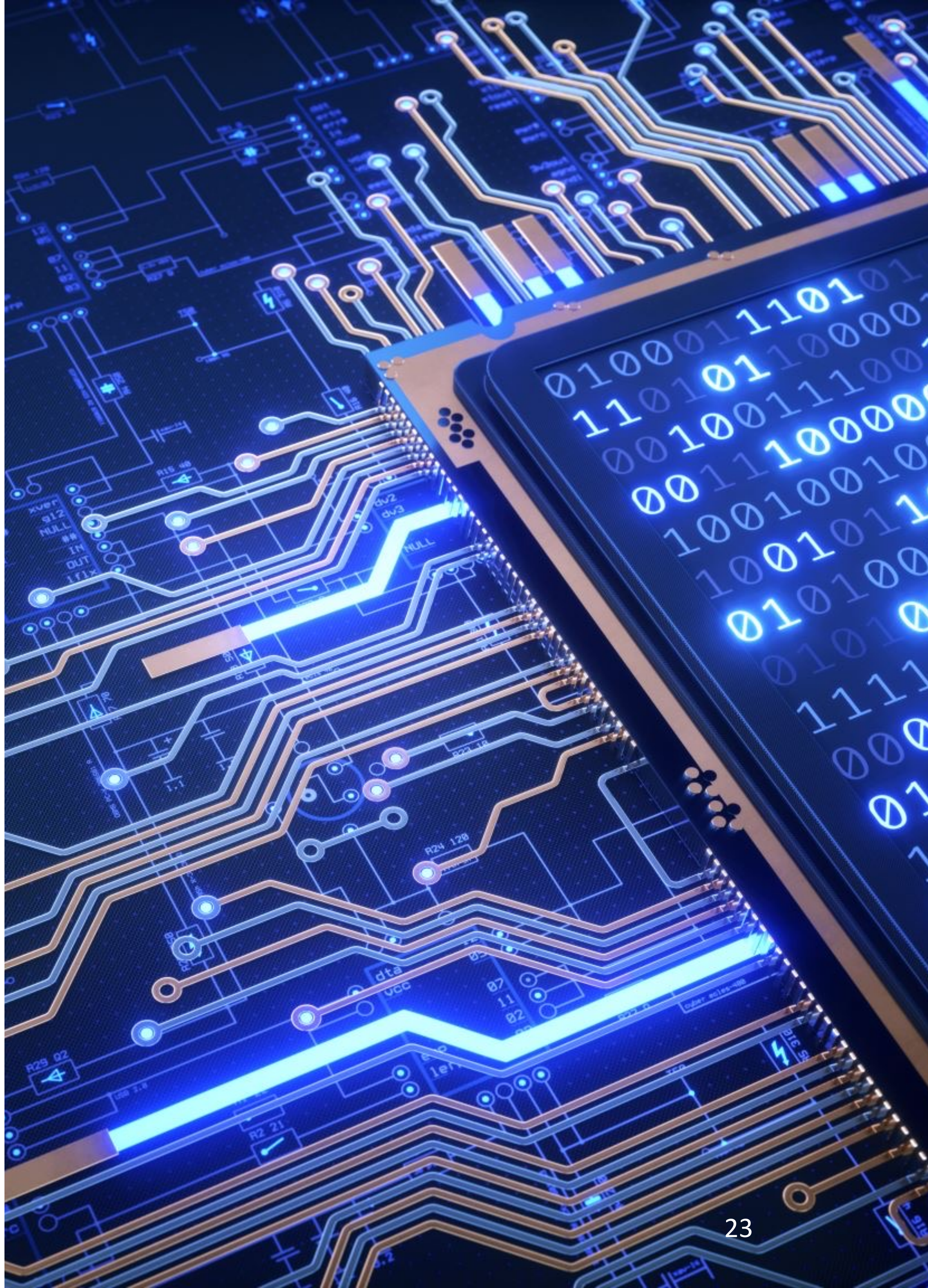

Composition

- Example:

  Let g be the function from the set $\{a, b, c\}$ to itself such that, $g(a) = b$, $g(b) = c$, and $g(c) = a$.

  Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

  The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$

  $(g \circ f)$ is not defined, because, range of f is not a subset of domain of g

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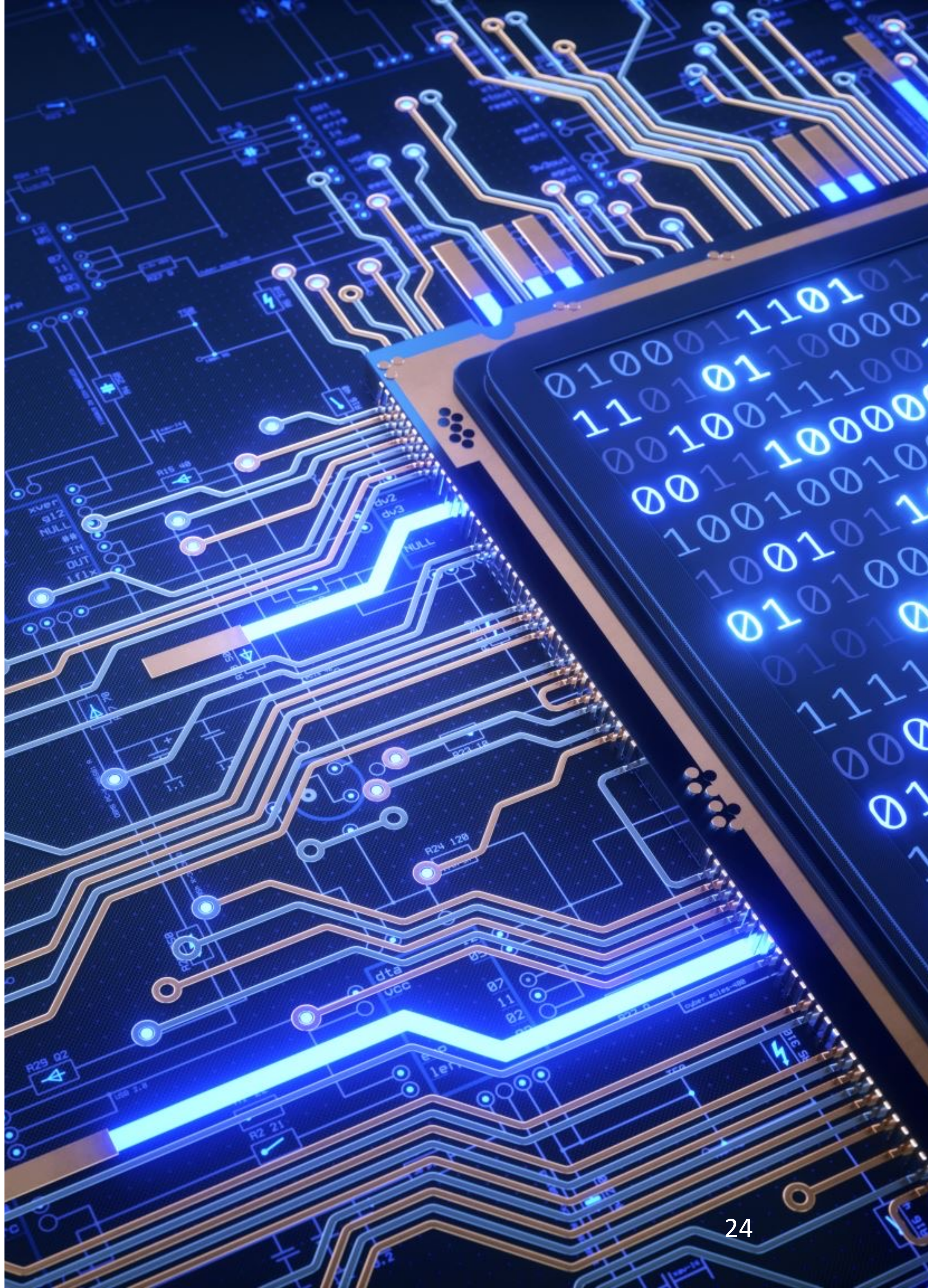


Composition

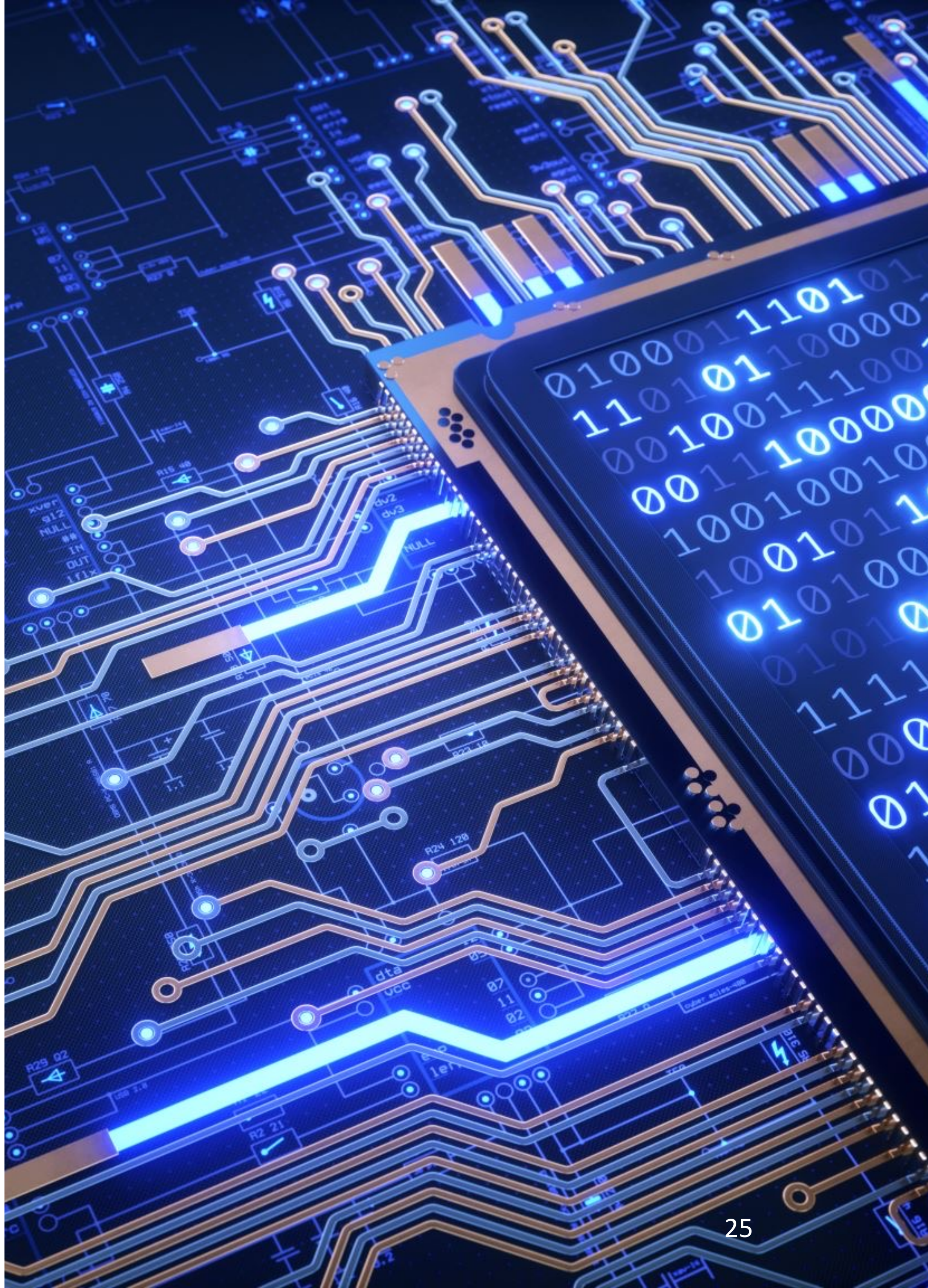
- Example:

  Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$

  $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

  $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

- *Note*: $(f \circ g) \neq (g \circ f)$



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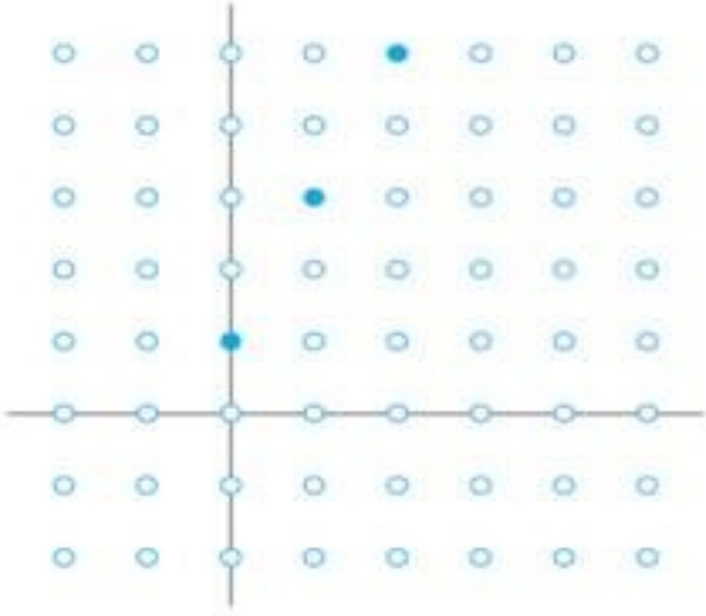
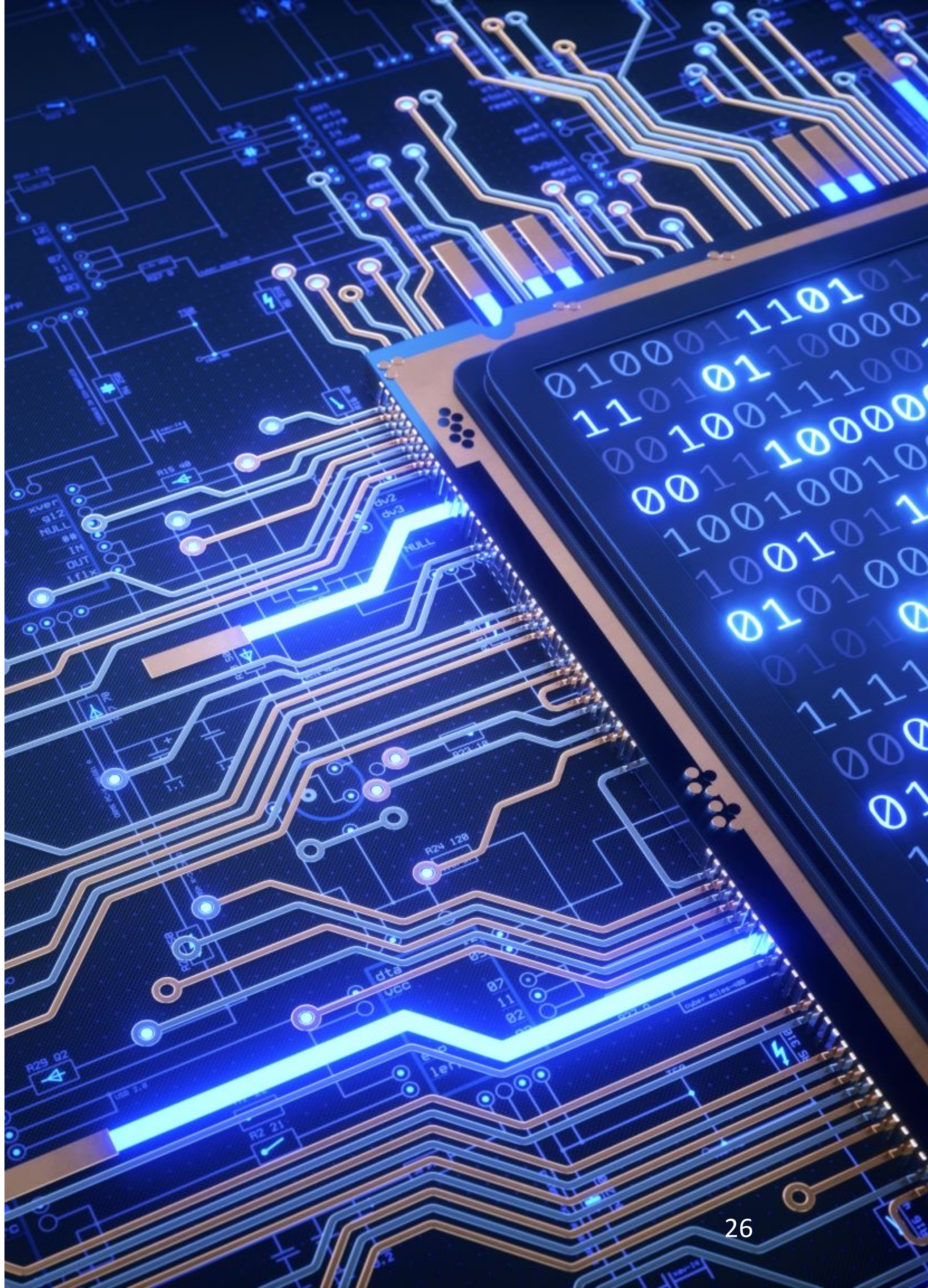
Graph Representation

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$
- Example:

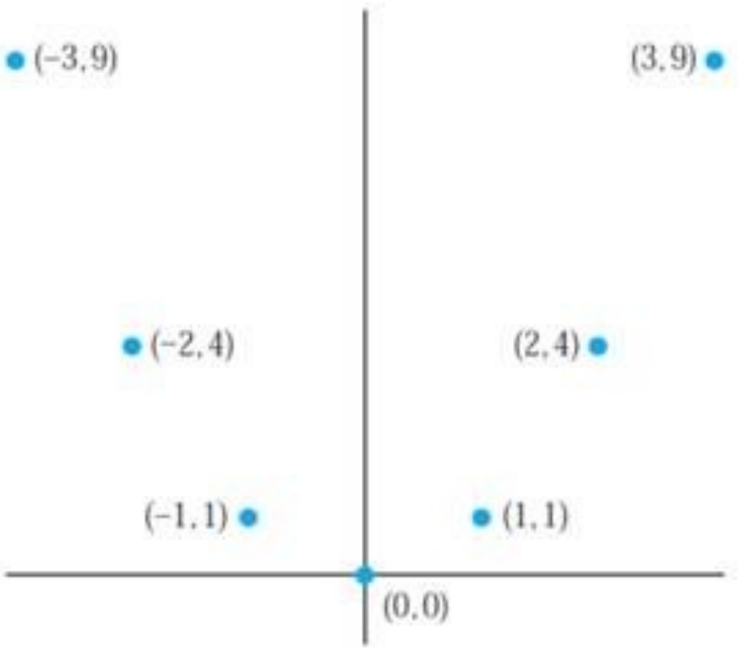
  The graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers is the set of ordered pairs of the form $(n, 2n + 1)$

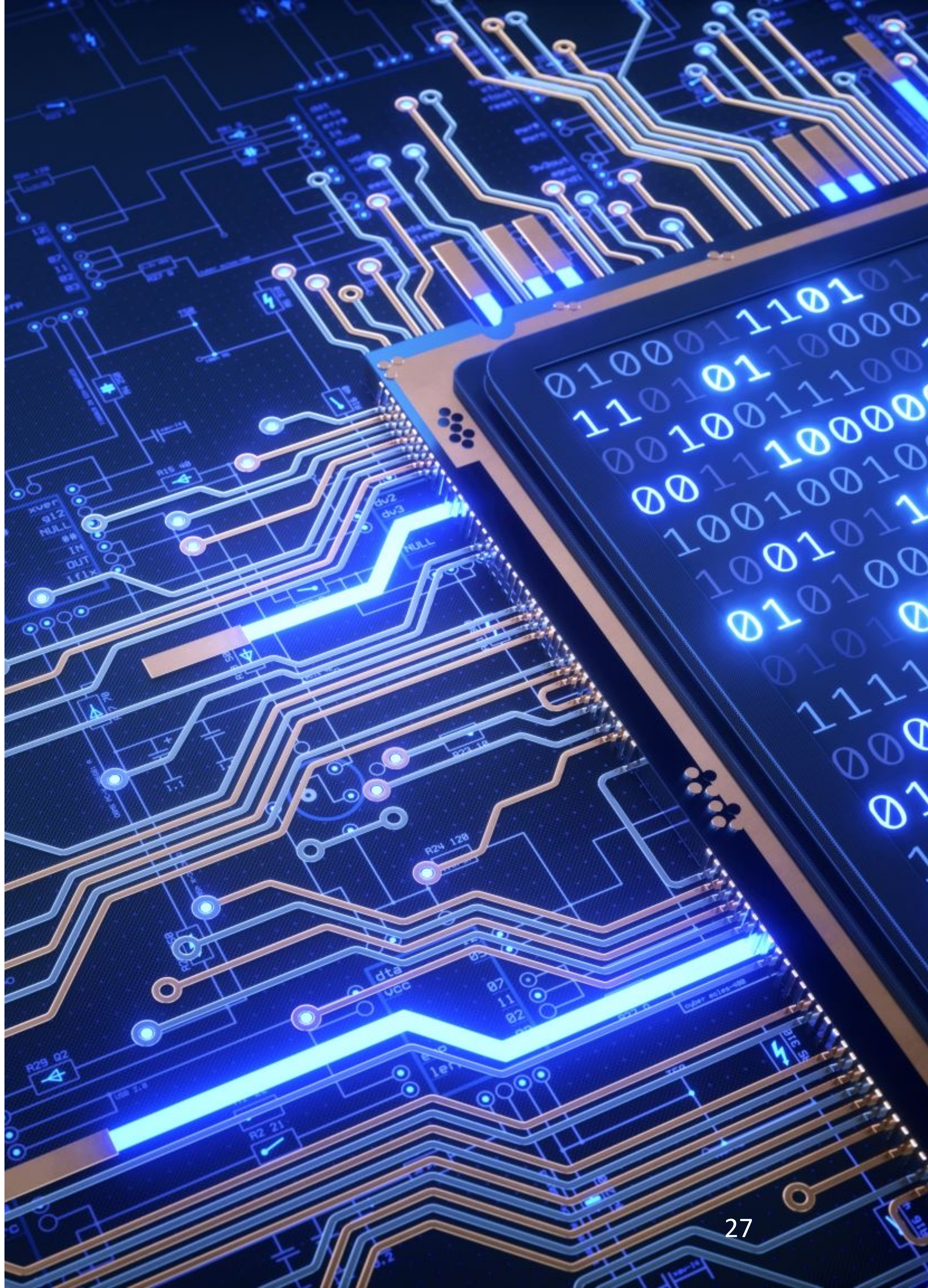




Graph Representation

- Example: The graph of the function $f(x) = x^2$ from the set of integers to the set of integers is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$





Floor and Ceiling

- **Floor** and **Ceiling** are the two important functions in discrete mathematics
- The ***floor function*** assigns to the real number *x* the largest integer that is less than or equal to *x*
- The value of the floor function at *x* is denoted by $\lfloor x \rfloor$
- The ***ceiling function*** assigns to the real number *x* the smallest integer that is greater than or equal to *x*
- The value of the ceiling function at *x* is denoted by $\lceil x \rceil$
- Examples:

$$\lfloor \tfrac{1}{2} \rfloor = 0, \quad \lceil \tfrac{1}{2} \rceil = 1, \quad \lfloor -\tfrac{1}{2} \rfloor = -1, \quad \lceil -\tfrac{1}{2} \rceil = 0$$

$$\lfloor 3.1 \rfloor = 3, \quad \lceil 3.1 \rceil = 4, \quad \lfloor 7 \rfloor = 7, \quad \lceil 7 \rceil = 7$$

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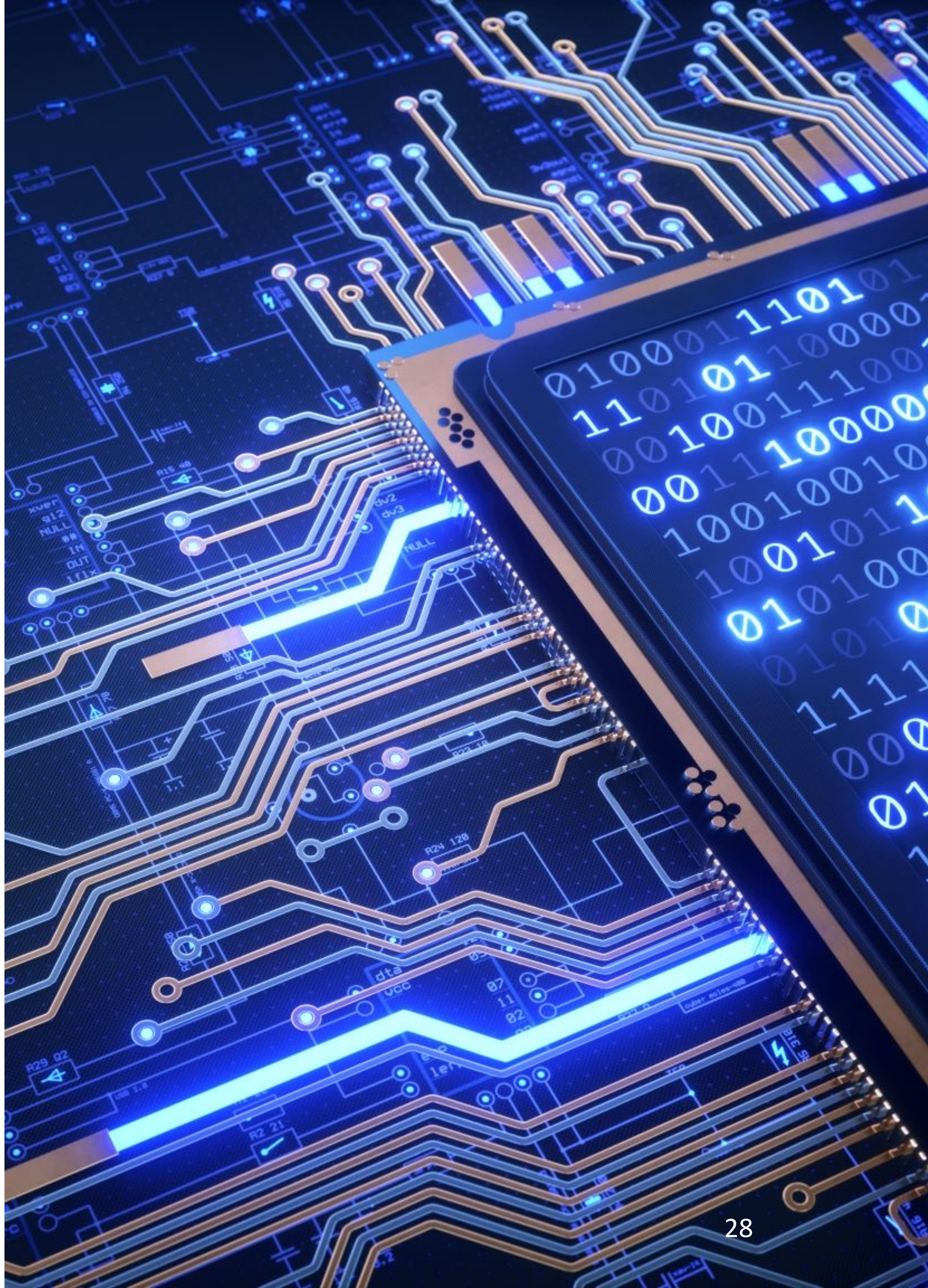


Floor and Ceiling

- Useful properties:

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$

(1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x + 1$

(2) $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

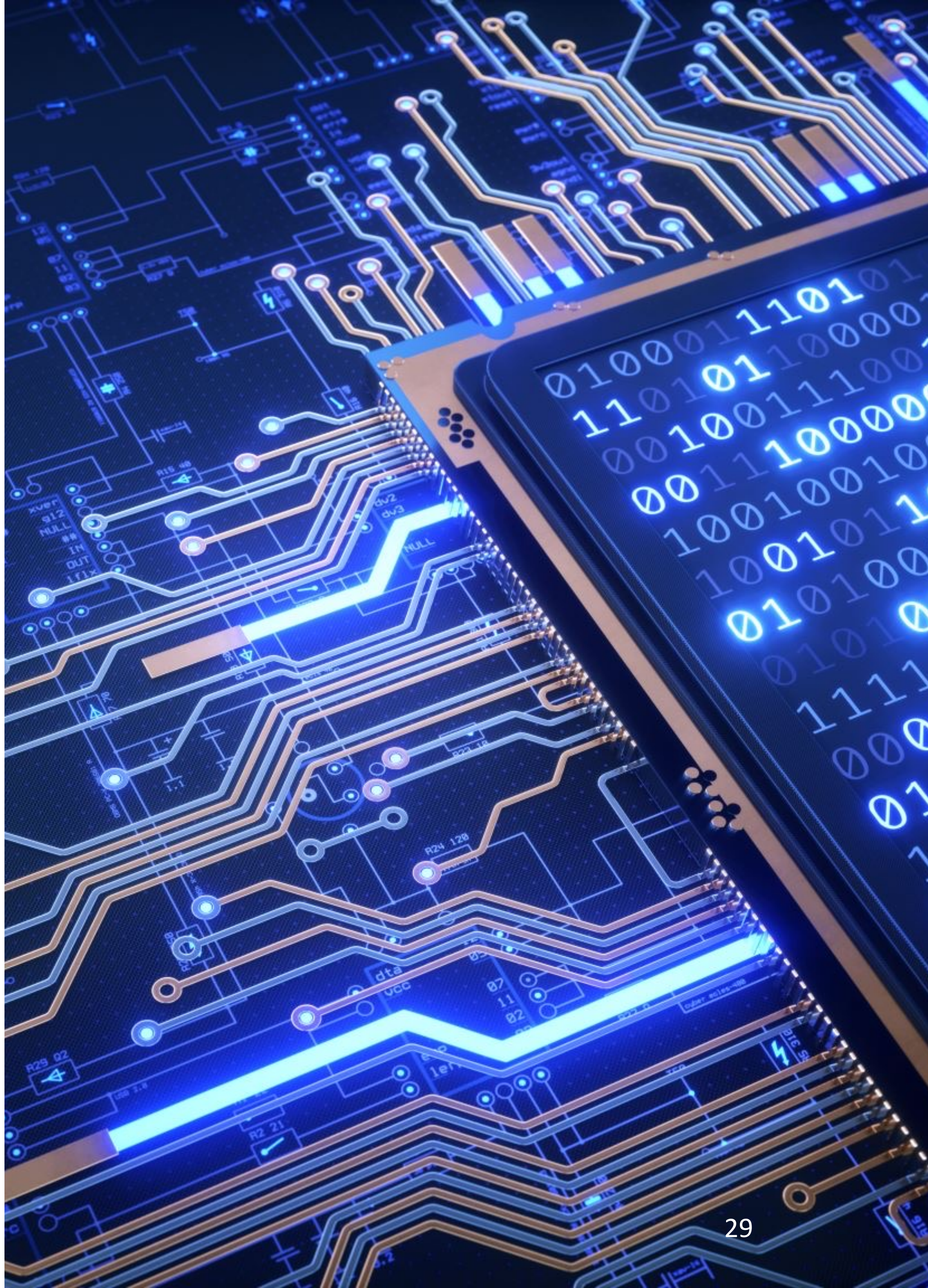
(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Note:
n is an integer
x is a real number

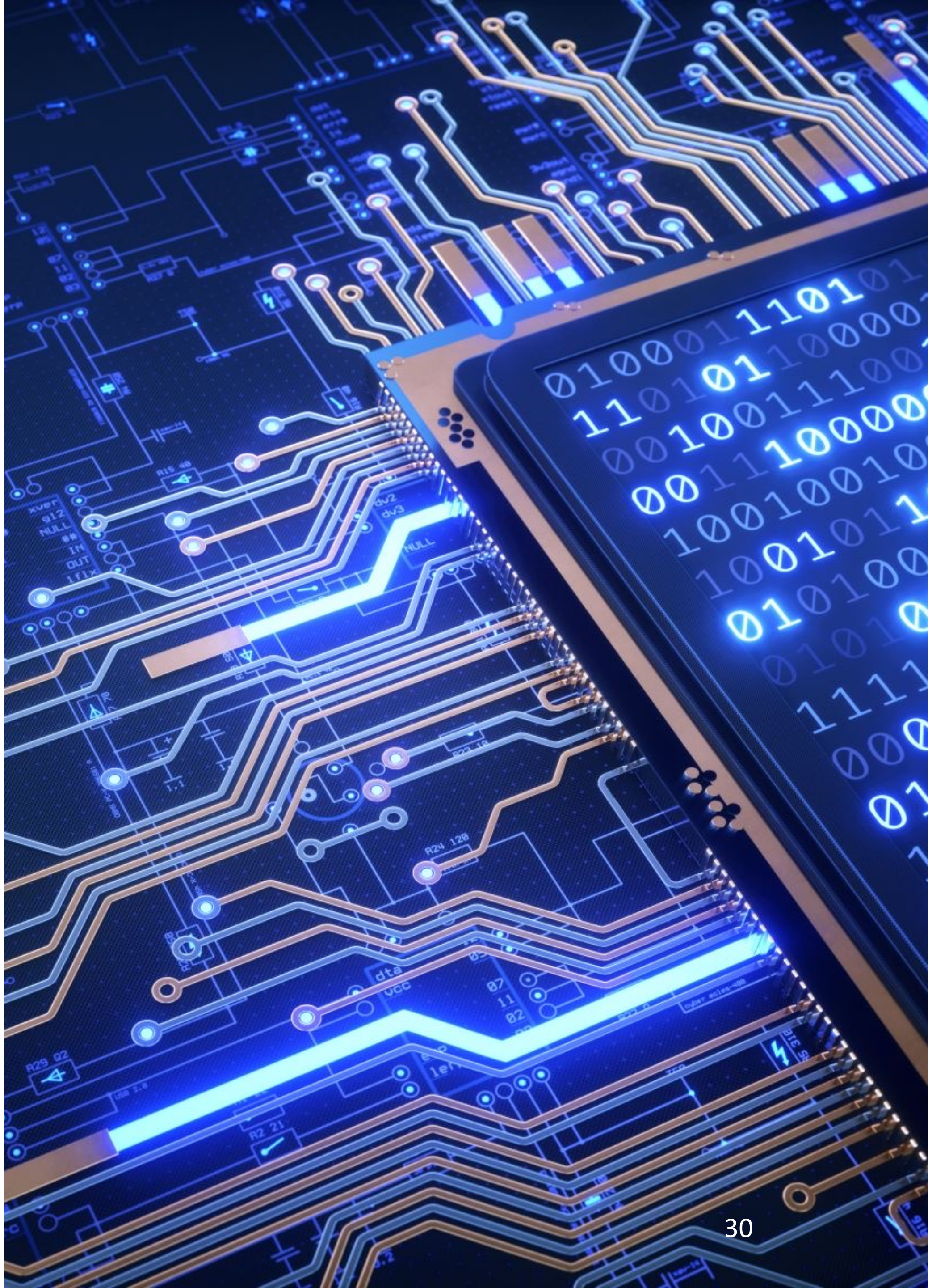
Rosen, K. H. (2012). *Discrete mathematics and its applications (7th Edition)*. McGraw-Hill.



Applications of Functions

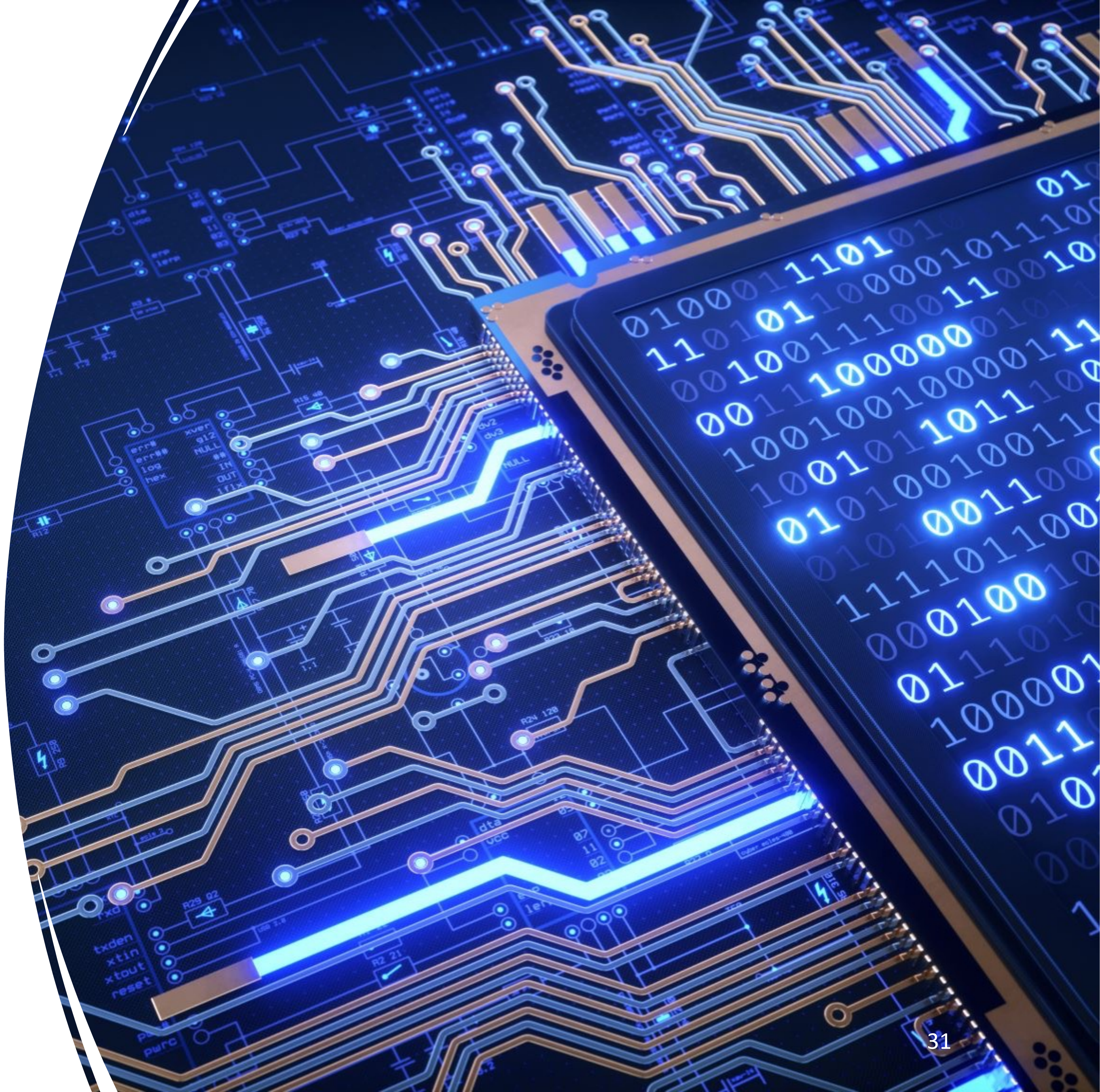
- Algorithm Design and Implementation
- Concurrency and Parallelism
- Web/Mobile app Development
- Recursive Algorithms
- Object-Oriented Programming
- Data Transformation and Processing
- Procedural Programming
- Event Handling and Callbacks
- Modularization and Abstraction

Rosen, K. H. (2012). *Discrete mathematics and its applications (7th Edition)*. McGraw-Hill.



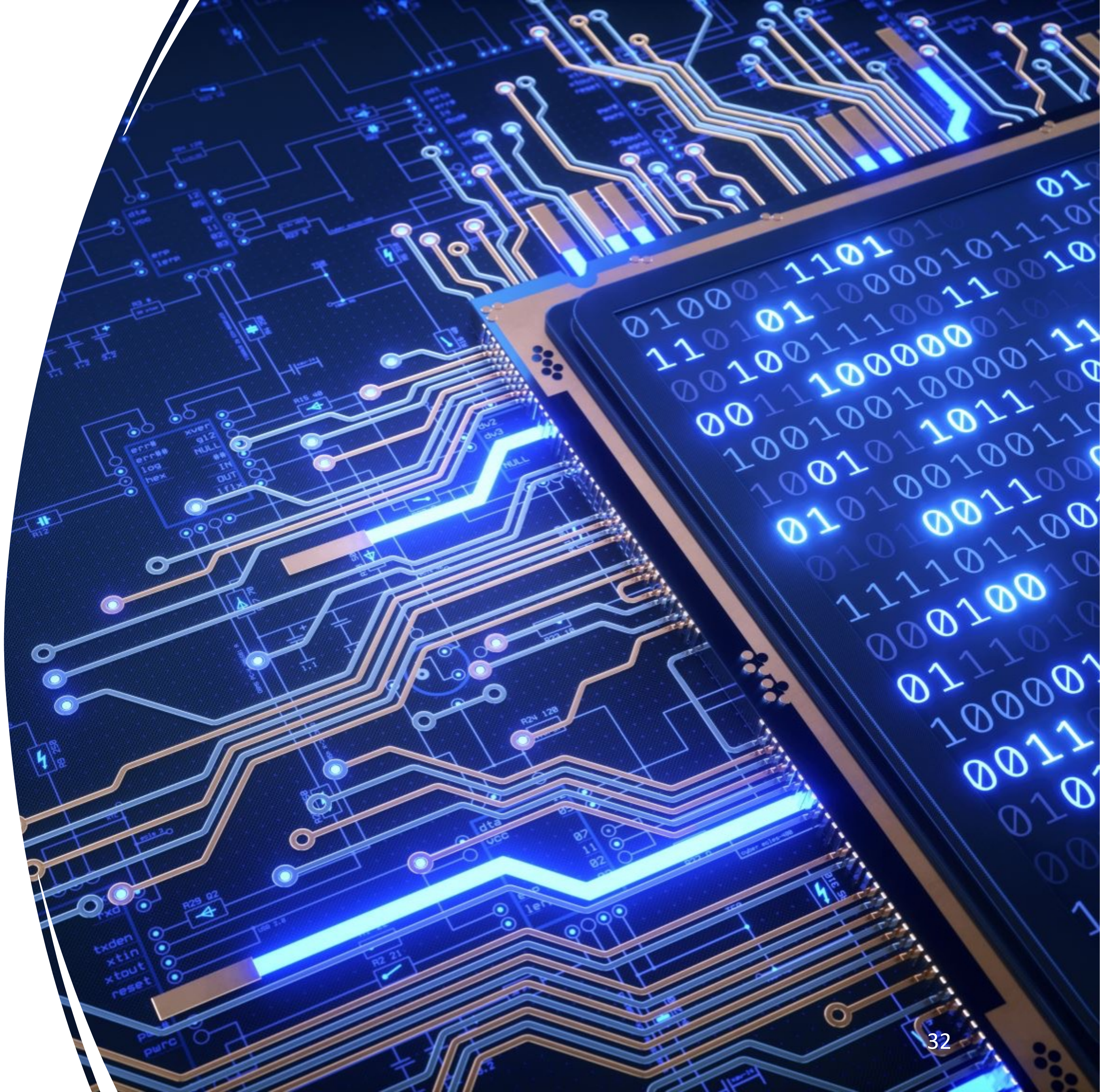
Summary

- Functions are a fundamental building block in computer science
- Forms of functions:
  - One-to-one
  - Onto
  - Bijective
- Inverse and Composite functions
- Applications of functions



Reference

Rosen, K. H. (2012). *Discrete mathematics and its applications (7th Edition)*. McGraw-Hill.
Chapter 2



See you next time!