Course: Mathematics for IT Professionals

Lecture 8

Recursion and Relation

By

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Outline

The topics to be treated in this lecture are:

- Recursive functions
- Binary relations
- Reflexivity property
- Symmetric property
- Antisymmetric property
- Transitive property
- Equivalence relation
- Applications

Lecture Learning Outcomes

At the end of the session, you will be able to

- define functions recursively
- define basis and recursive steps
- understand the properties of relation
- understand equivalence relation
- know some applications of recursion and relation

Introduction

- **Recursion** is a process of defining an object in terms of itself

(e.g., discrete structures like functions, sets, sequences etc.)

- Example:

  Consider the *sequence* of powers of 2

  The terms of this sequence can be specified using a formula defined as:

  $a_n = 2^n$ for $n = 0, 1, 2, 3, \ldots$

  This *sequence* can be defined recursively as:
  $a_0 = 1$ (*Basis*)
  $a_{n+1} = 2a_n$ for $n = 0, 1, 2, 3, \ldots$ (*Recursive definition*)

  - Recursive definition is *also called **inductive*** definition

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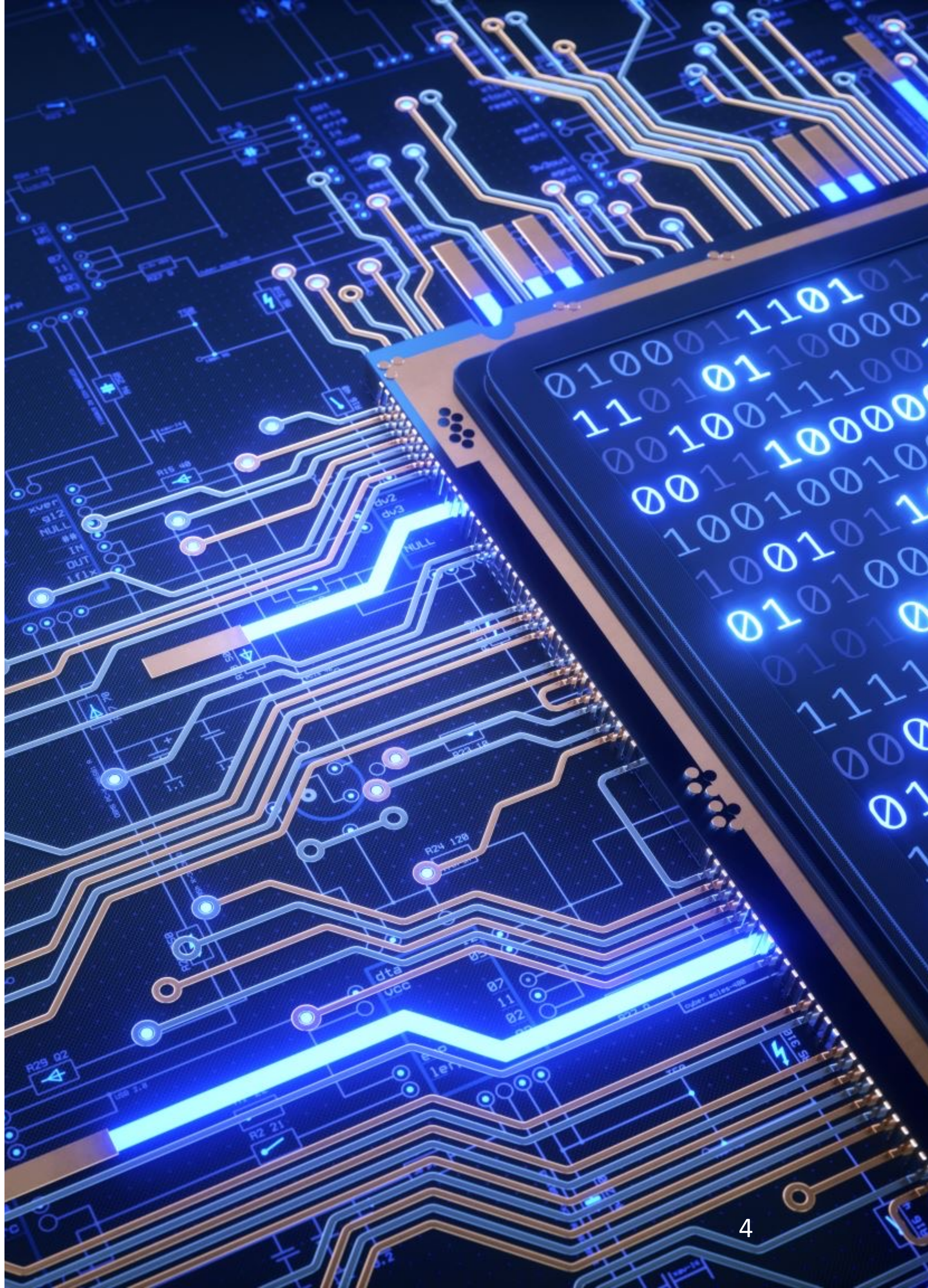

Defining functions recursively

- A function with the set of positive integers as its domain can be defined recursively using two steps
- ***BASIS STEP***: Specify the value of the function at zero.
- ***RECURSIVE STEP:*** Give a rule for finding its value at an integer from its values at smaller integers
- Recursively defined functions are ***well-defined***

– The value of the function is determined in an unambiguous way.

- Example :

Give a recursive definition of a^n, where a is a non-zero real number and n is a non-negative integer

- ***Answer***: $a^0 = 1$ *(Basis)*

$a^{n+1} = a \cdot a^n$ for $n = 0, 1, 2, 3 \ldots$ *(Recursive)*

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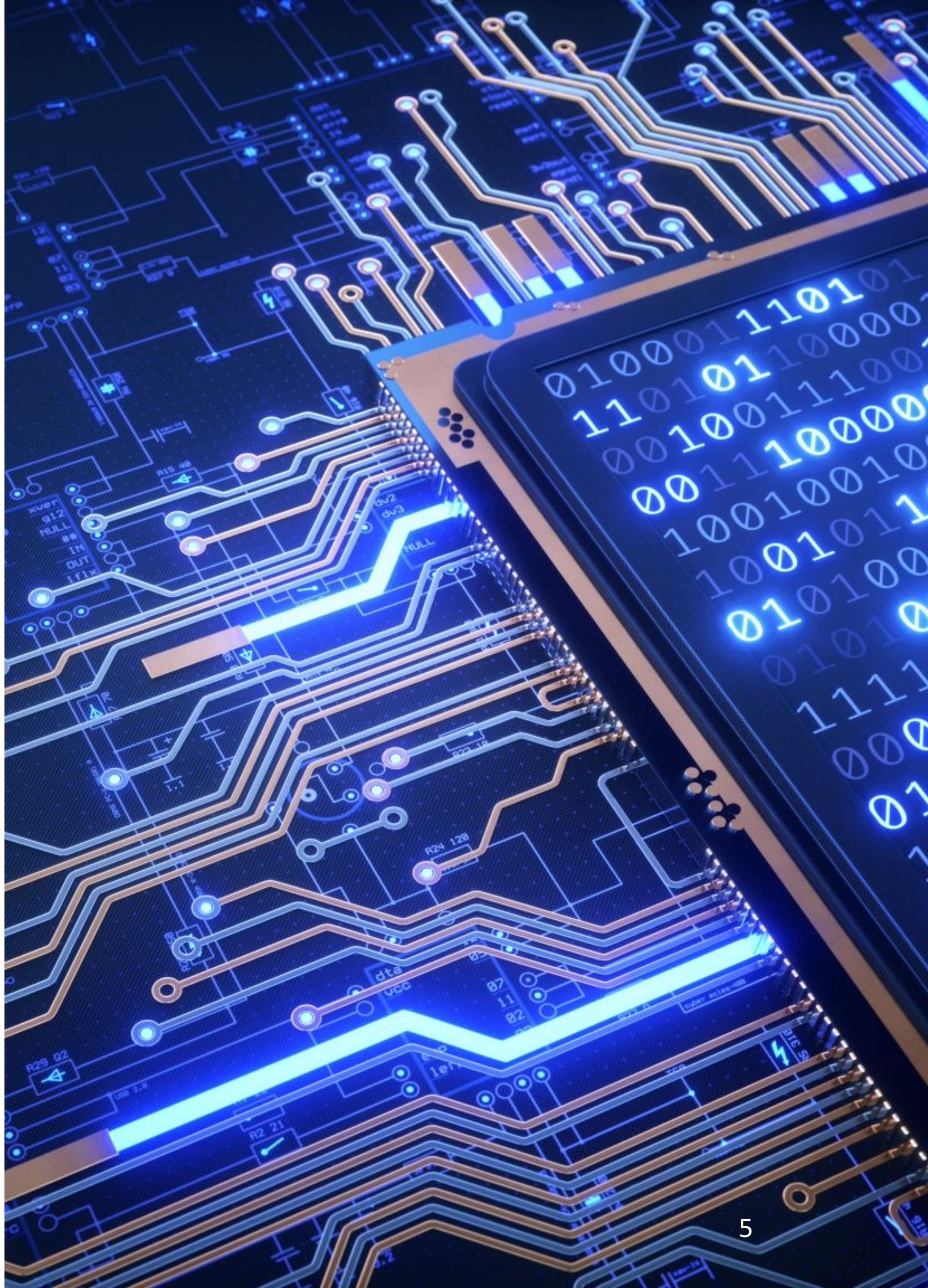


Defining functions recursively

- **Example :**

  Consider the function *f* defined recursively as,

  $f(0) = 3$

  $f(n+1) = 2f(n) + 3$

  Find $f(1)$, $f(2)$, $f(3)$ and $f(4)$

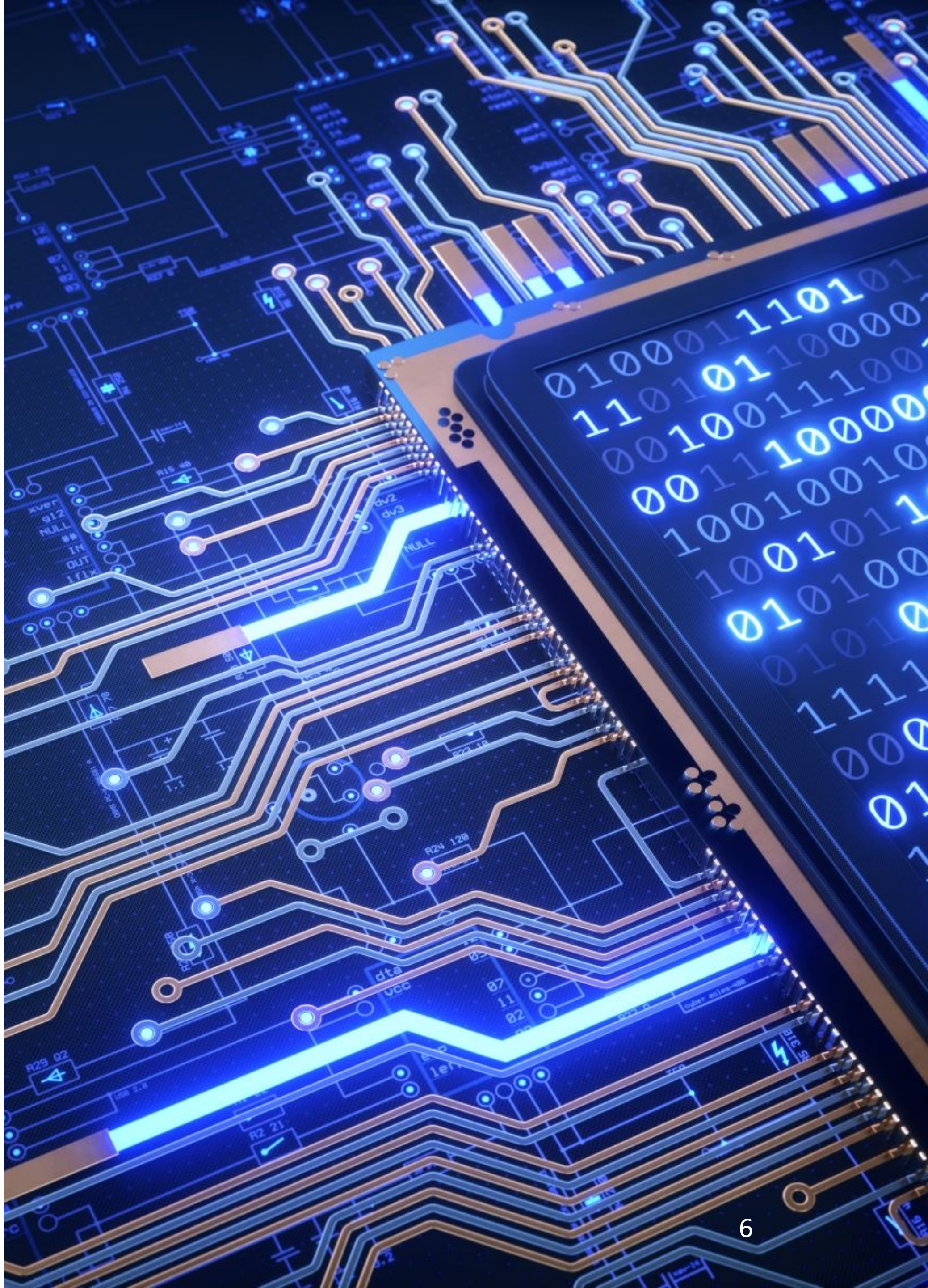
  Answer:

  $f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$

  $f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$

  $f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$

  $f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$



Defining functions recursively

- **Try this:**
  Find f (2), f (3), f (4), and f (5) if f is defined recursively by

  $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, \ldots$

  $$f(n+1) = f(n-1)/f(n)$$

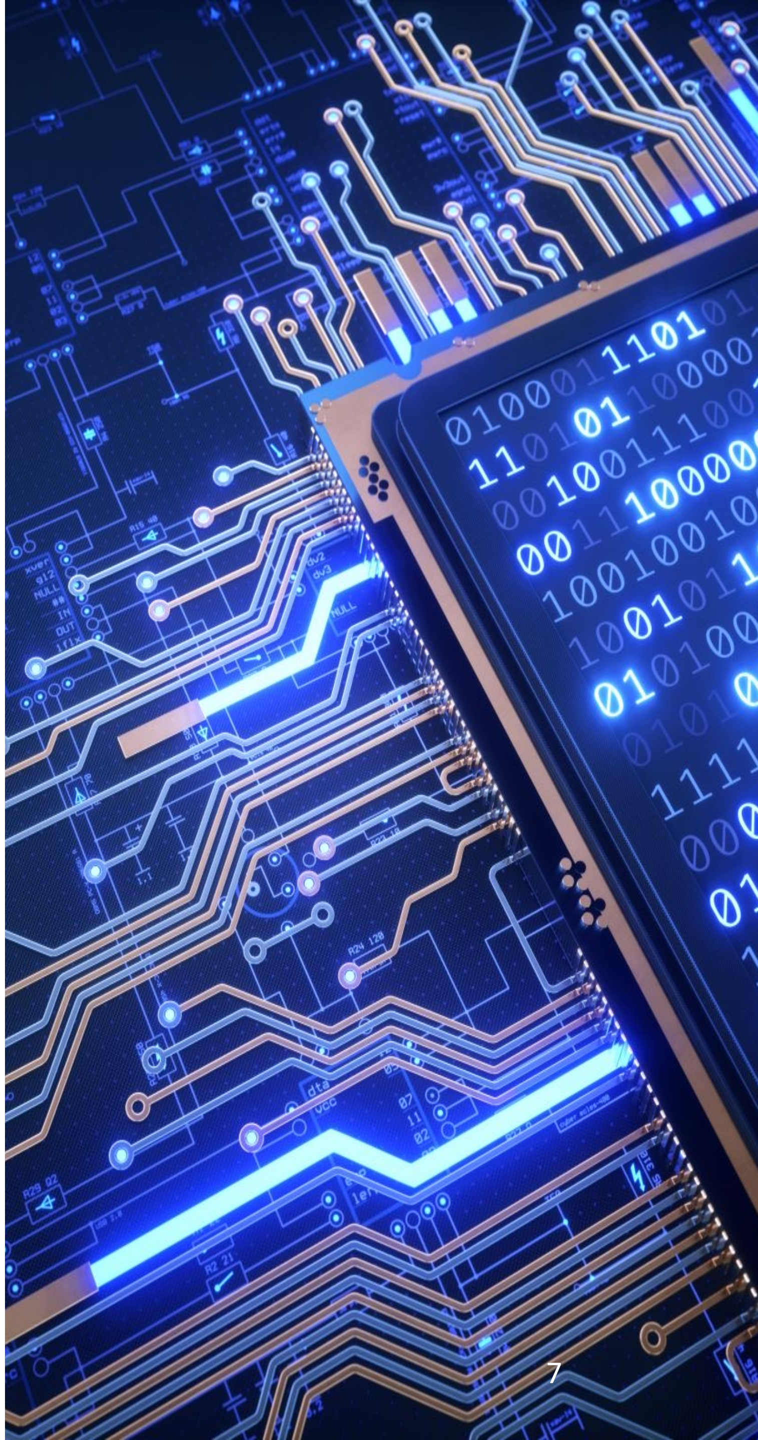
Answer:

When $n = 1$, $f(n+1) = f(1+1) = f(2) = f(1-1)/f(1) = f(0)/f(1) = -1/2$

When $n = 2$, $f(n+1) = f(2+1) = f(3) = f(2-1)/f(2) = f(1)/f(2) = 2/(-1/2) = -4$

When $n = 3$, $f(n+1) = f(3+1) = f(4) = f(3-1)/f(3) = f(2)/f(3) = (-1/2)/(-4) = 1/8$

When $n = 4$, $f(n+1) = f(4+1) = f(5) = f(4-1)/f(4) = f(3)/f(4) = -4/(1/8) = -32$



Defining functions recursively

- Worked example 1:

  Write a recursive definition of $\sum_{k=0}^{n} a_k$

  Answer:

$$\sum_{k=0}^{n} a_k = a_0 \quad = \text{(\textit{Basis step})}$$

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k \right) + a_{n+1} \quad = \text{(\textit{Recursive step})}$$

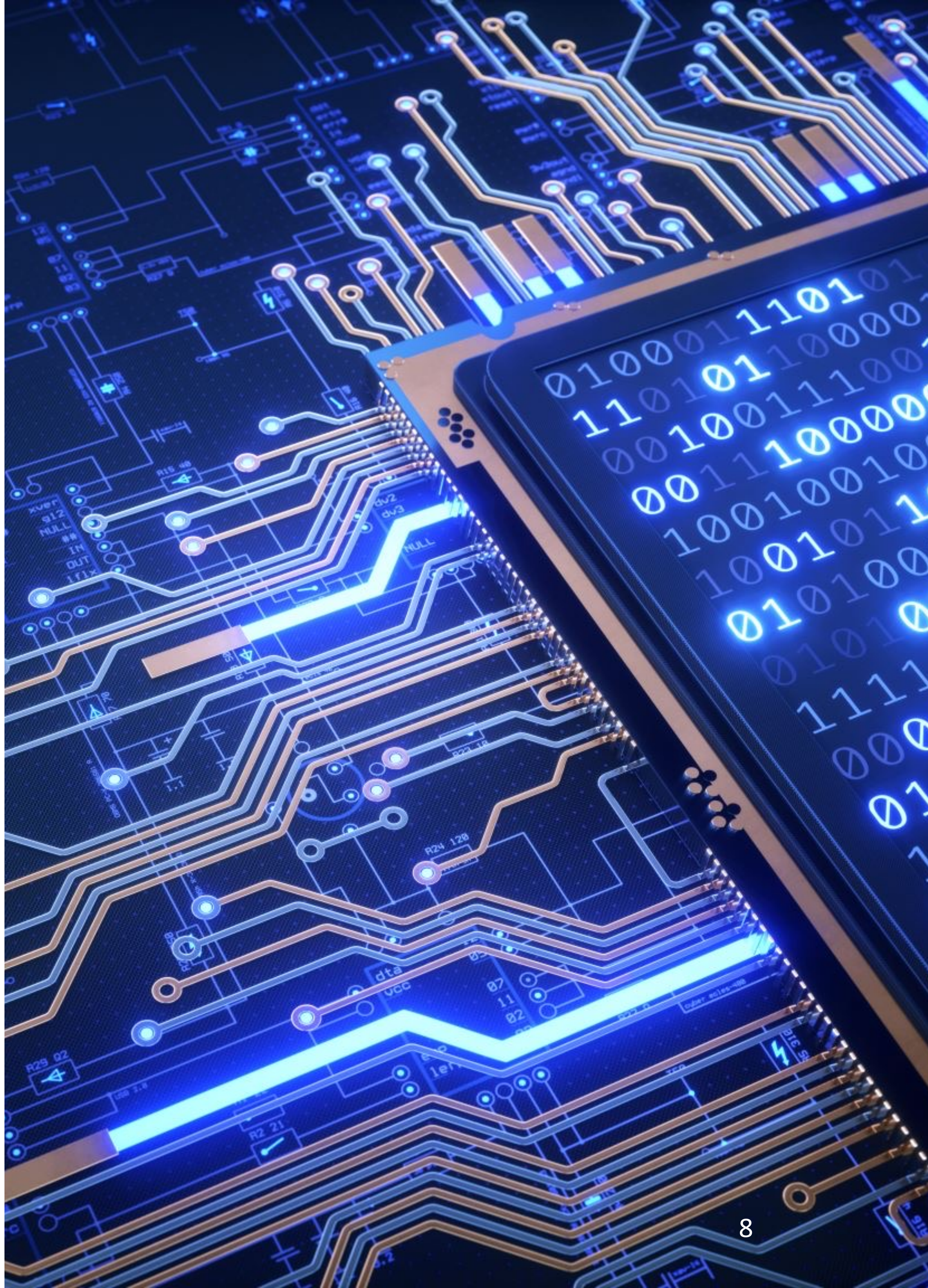
- Worked example 2:

  Give a recursive definition of Fibonacci sequence

  Answer: The recursive definition of **Fibonacci** number sequence is

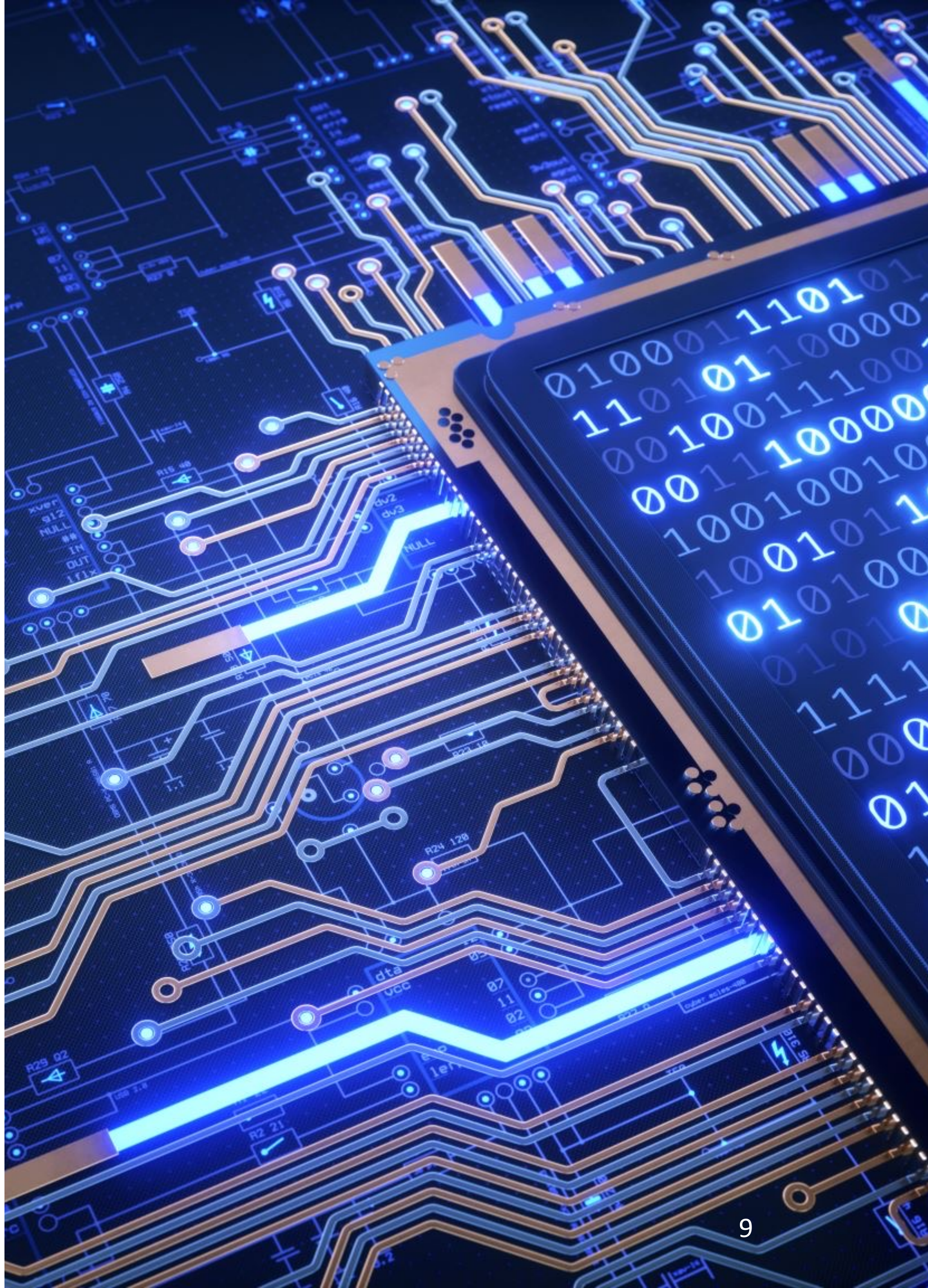
  $f_0 = 0, f_1 = 1$ (*Basis step*)

  $f_n = f_{n-1} + f_{n-2}$ (*Recursive step*)



Defining Sets

- Sets can be defined recursively, and the recursive definition of sets also contain the two steps: **basis step** and **recursive step**
- The *basis step* specifies an initial collection of elements
- The *recursive step* specifies the rules for forming new elements in the set from those already known to be in the set
- Recursive step may also include an **exclusion rule**, which specifies that, a recursively defined set contains *nothing other than* those elements specified in the basis step or those generated by application of the recursive step.



Defining Sets

- Example 1:

  Set of all positive multiples of 3 can be recursively defined as:

  BASIS STEP : $3 \in S$

  RECURSIVE STEP : If $x \in S$ and $y \in S$, then $x + y \in S$

- Example 2:

  Set of all even numbers can be recursively defined as:

  BASIS STEP : $0 \in S$

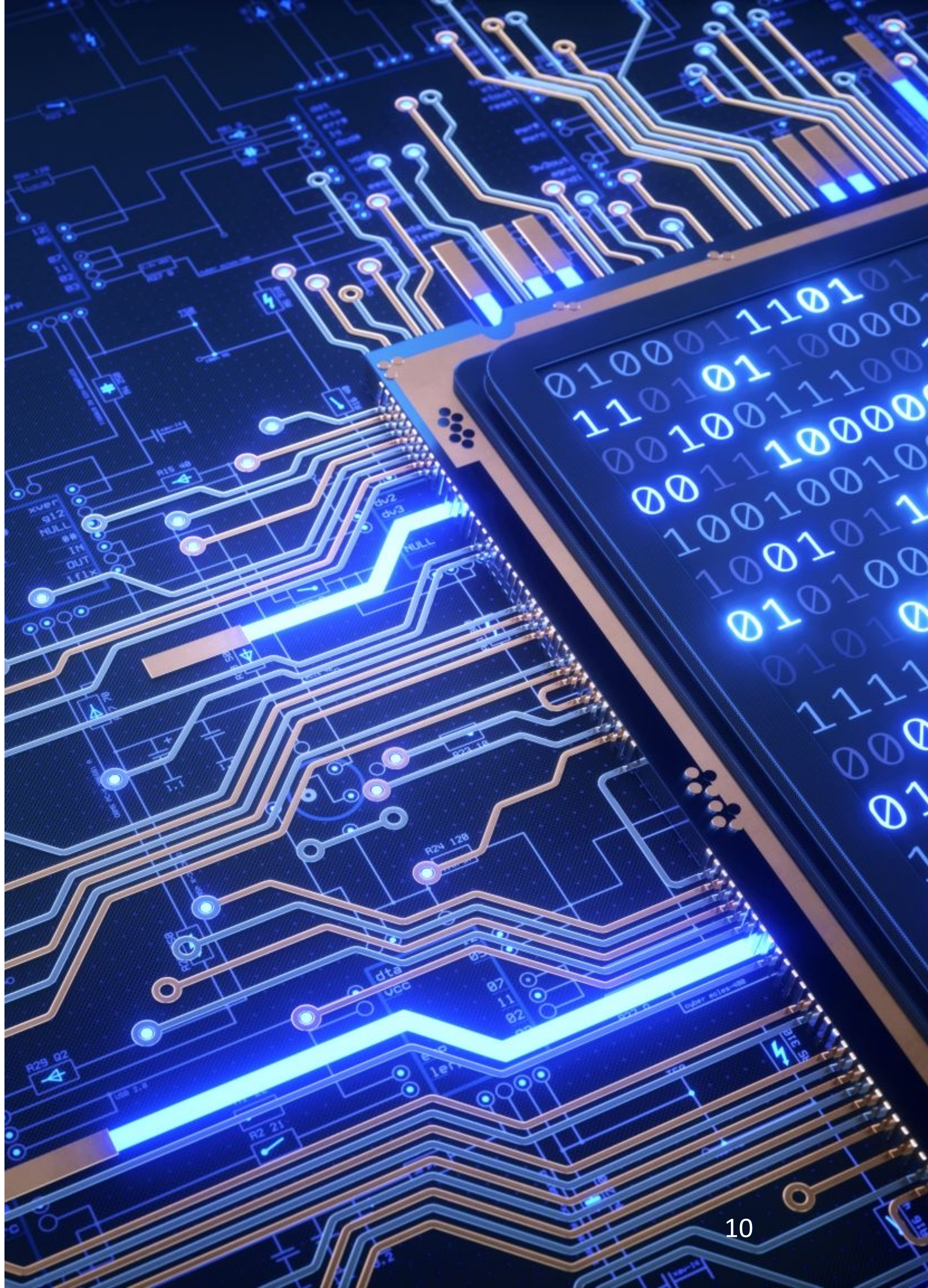
  RECURSIVE STEP : If $x \in S$ then $x + 2 \in S$

Example 3:

Set Σ_* of all *strings* over the alphabet Σ is defined recursively as:

BASIS STEP : $\varepsilon \in \Sigma_*$ (ε is the empty string) Appending at the end

RECURSIVE STEP: If $w \in \Sigma_*$ and $x \in \mathbf{\Sigma}$, then $wx \in \Sigma_*$



Relations

- Relationships between elements of sets are represented using the structure called a **relation**, which is a subset of the Cartesian product of the sets.

- Relations can be used to solve problems such as;

  – determining which pairs of cities are linked by airline flights

  – finding a viable order for the different phases of a complicated project

  – producing a useful way to store information in computer databases.

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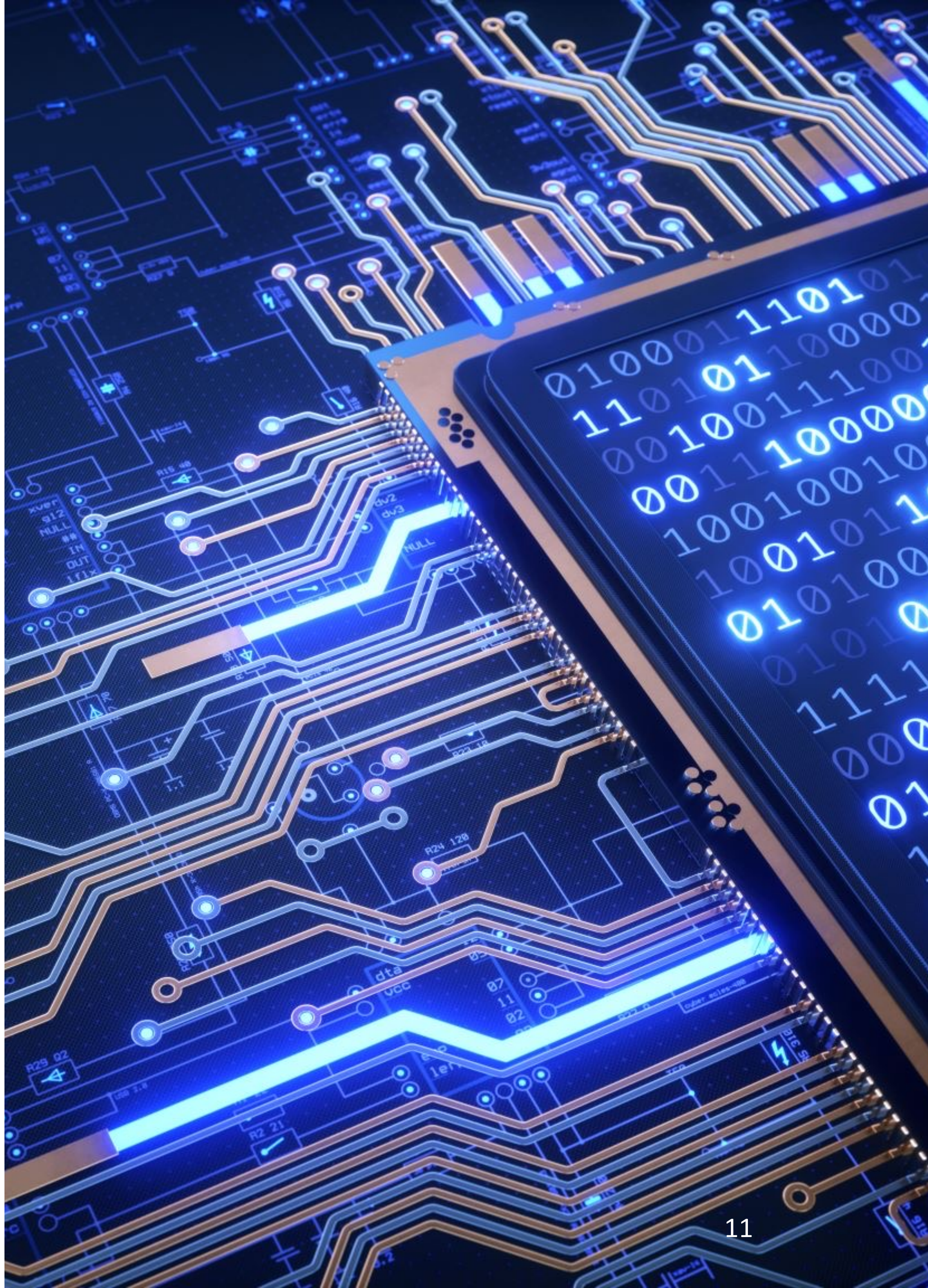

Terminology

- *Sets of ordered pairs* represent the relationship between elements of **two** sets, and hence called binary relations.
- A ***binary relation*** from a set A to a set B is a subset of $A \times B$, denoted by R
- The notation ***a*** *R* ***b*** to denotes that, $(a, b) \in R$ and *a R/ b* denotes that $(a, b) \notin R$
- When $(a, b) \in R$, ***a*** is said to be **related** to ***b*** by R
- The relationships among elements of more than two sets is expressed by ***n*-ary** relations

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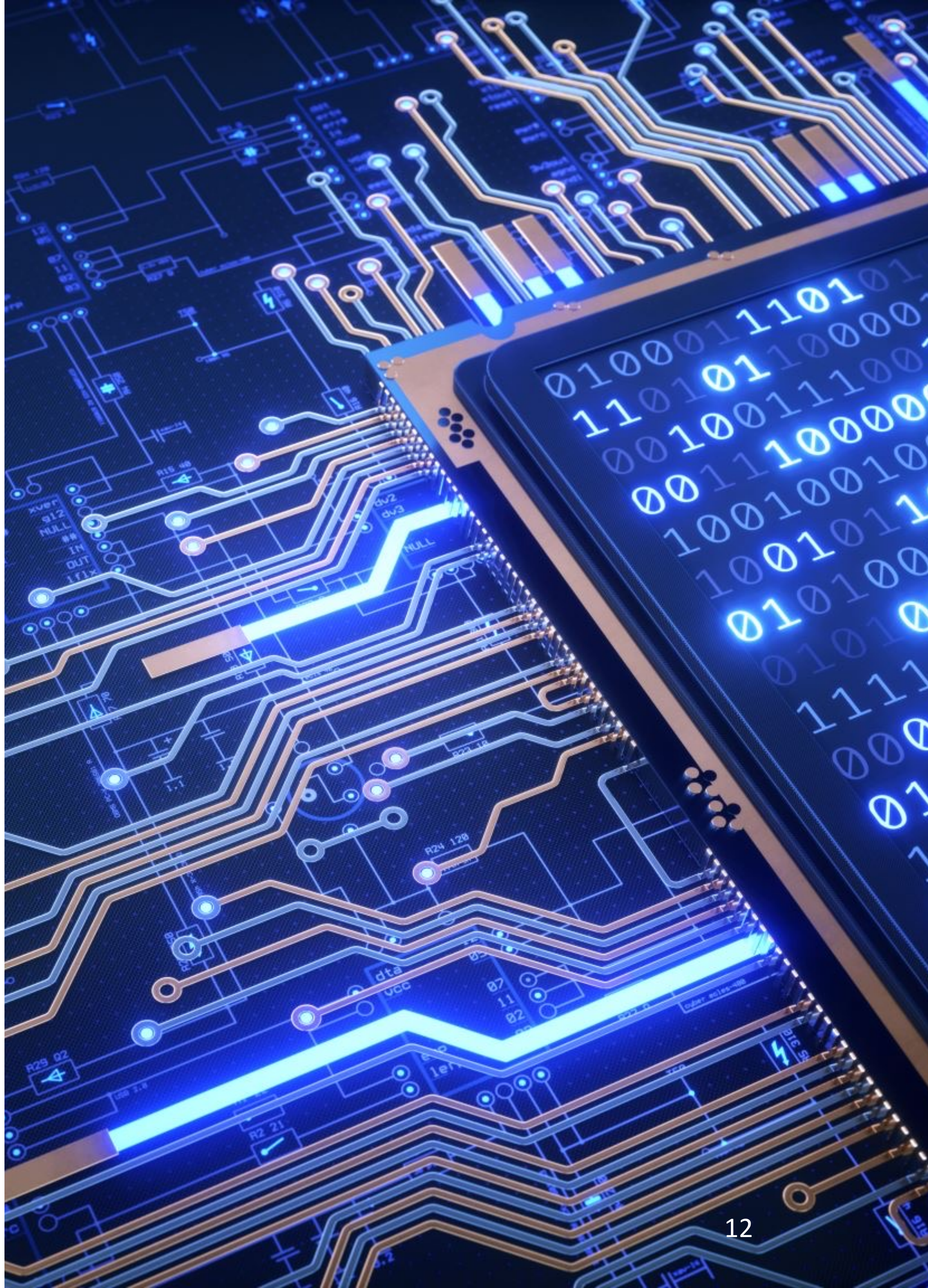


Binary Relations

- Example:

  Let *A* be the set of students, *B* be the set of courses

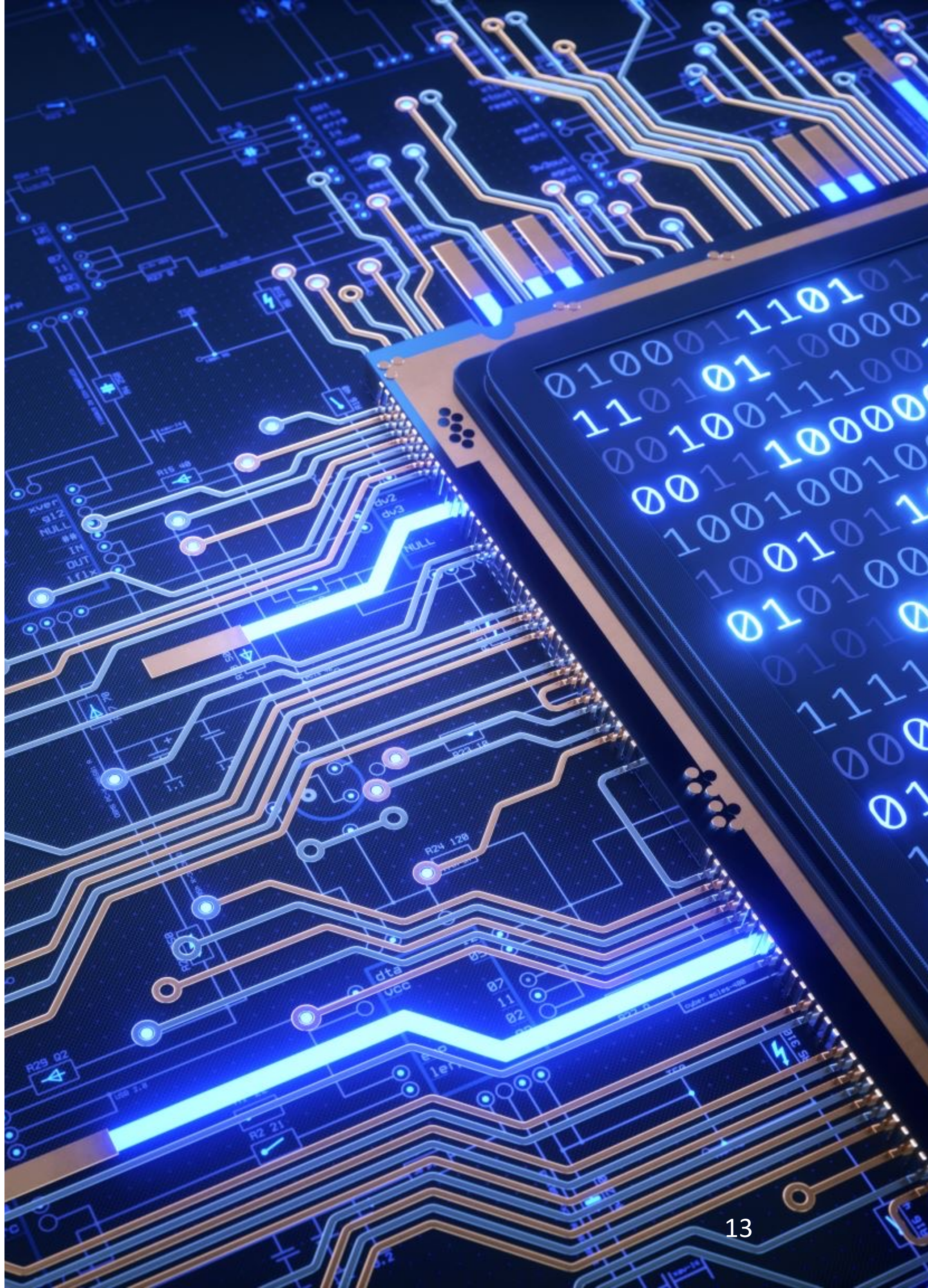
  The relation *R* that consists of the pairs (*a, b*) may represent that *a* is a student enrolled in course *b*

- Example:

  Let *A* be the set of cities in USA and *B* be the set of 50 states in the USA

  Let *R* be the relation with ordered pairs such that an ordered pair (***a, b***) belongs to *R* if a city ***a*** is in the state ***b***

  Thus, the ordered pairs (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in *R*

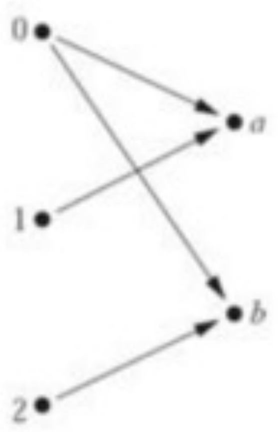


Binary Relations

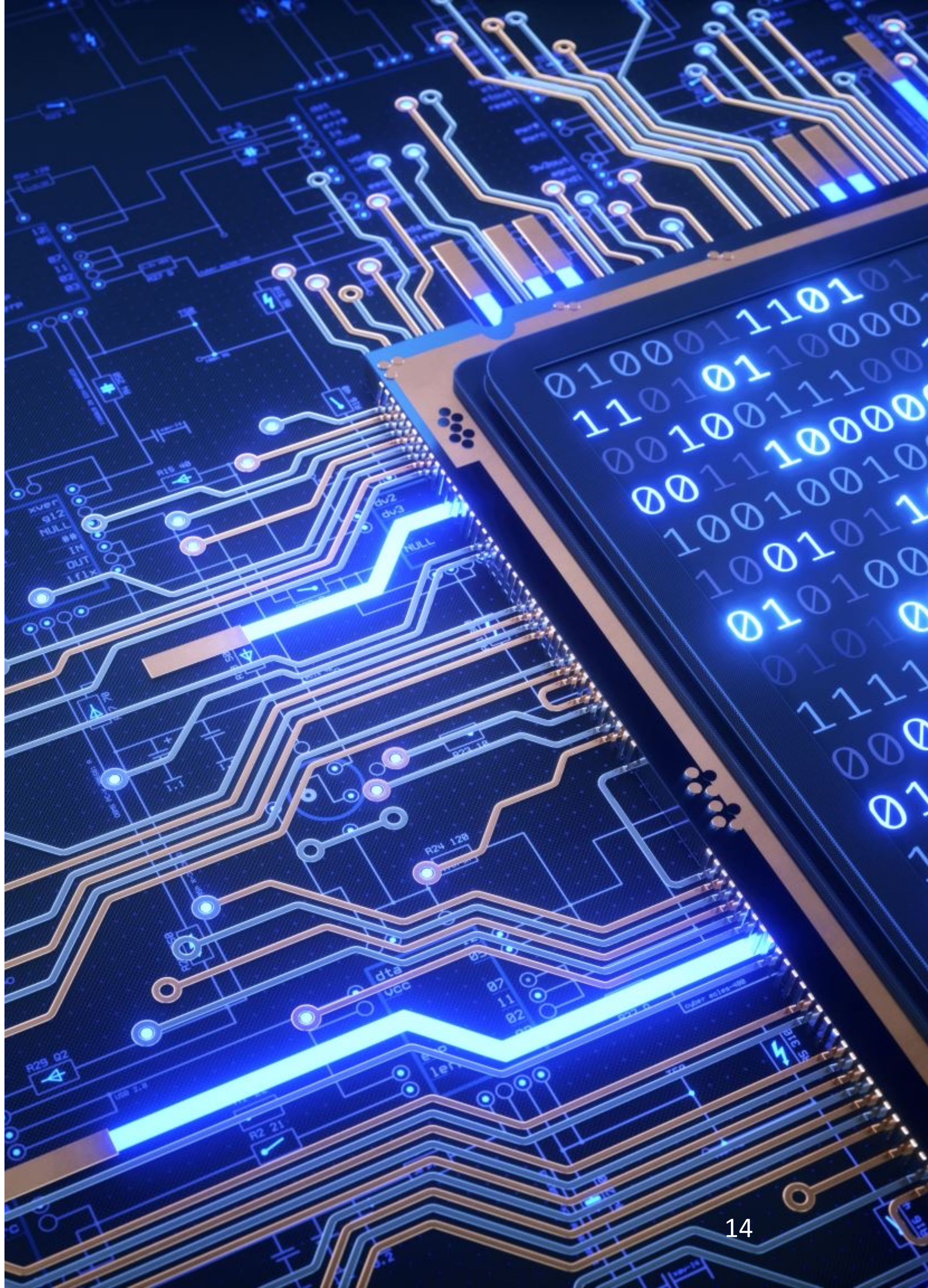
Relations can be represented graphically or by using a table

- Example
  - Let *A* = {0, 1, 2} and *B* = {*a*, *b*} and *R* = {(0, *a*), (0, *b*), (1, *a*), (2, *b*)} is a relation from *A* to *B*
  - We can say that 0 *R a*, but *1 R/ b*

    R can be represented as follows



R	a	b
0	×	×
1	×	
2		×



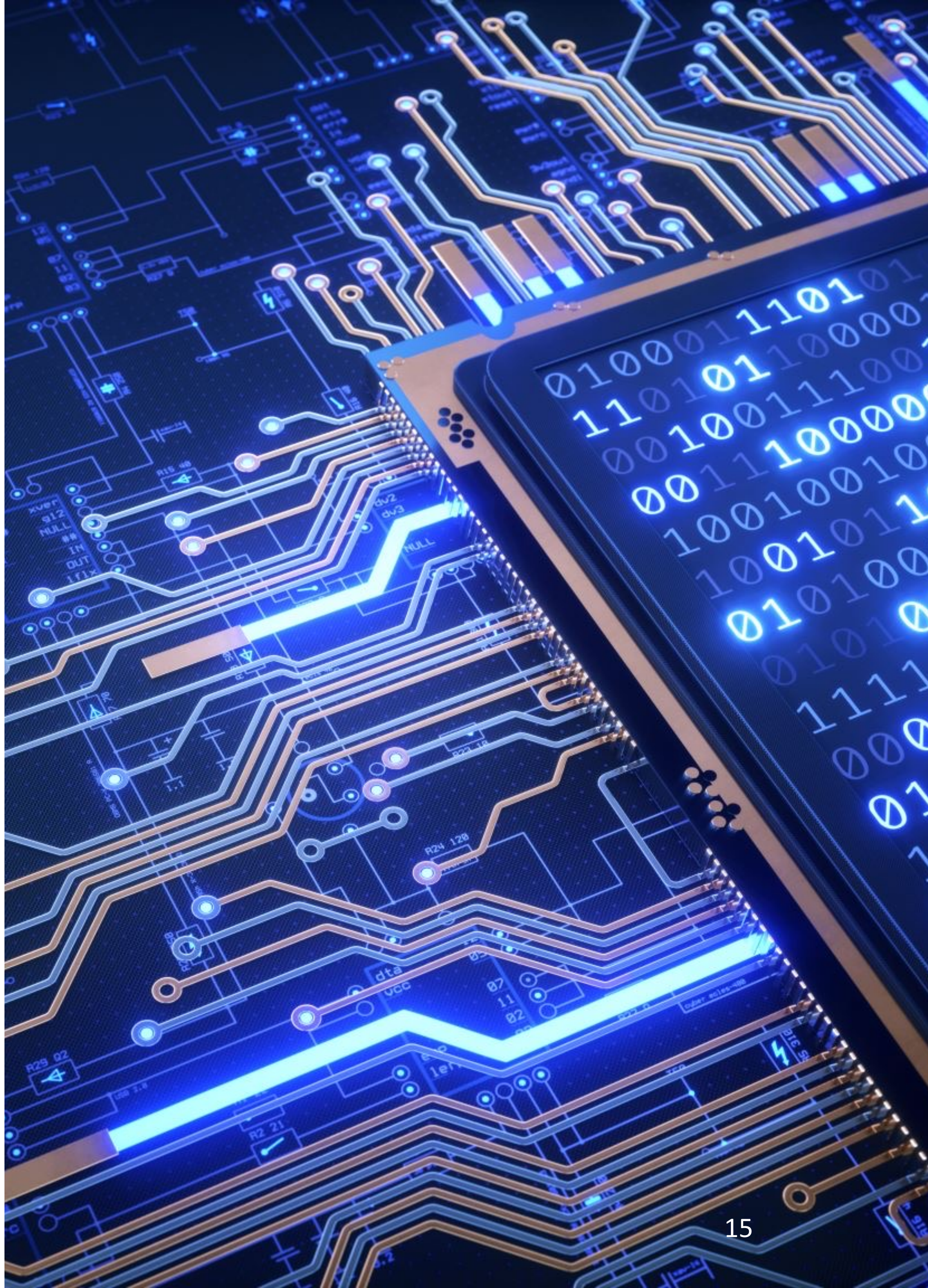
Reflexivity Property

- A relation *R* on a set *A* is called ***reflexive*** if (*a, a*) ∈ *R* for every element *a* ∈ *A*

  That means, an element ***a*** is always related to itself.

- Example:
  - Let R be the relation on the set of all people consisting of pairs (*x, y*) where x and y have the same mother and the same father. Then *x* R *x* for every person x.

- *Note*: A relation *R* on the set *A* is ***irreflexive*** if for every *a* ∈ *A*, (*a, a*) ∉ *R*. That is, *R* is irreflexive if no element in *A* is related to itself.



Reflexivity Property

- Example:

Consider the following relations on {1, 2, 3, 4}

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

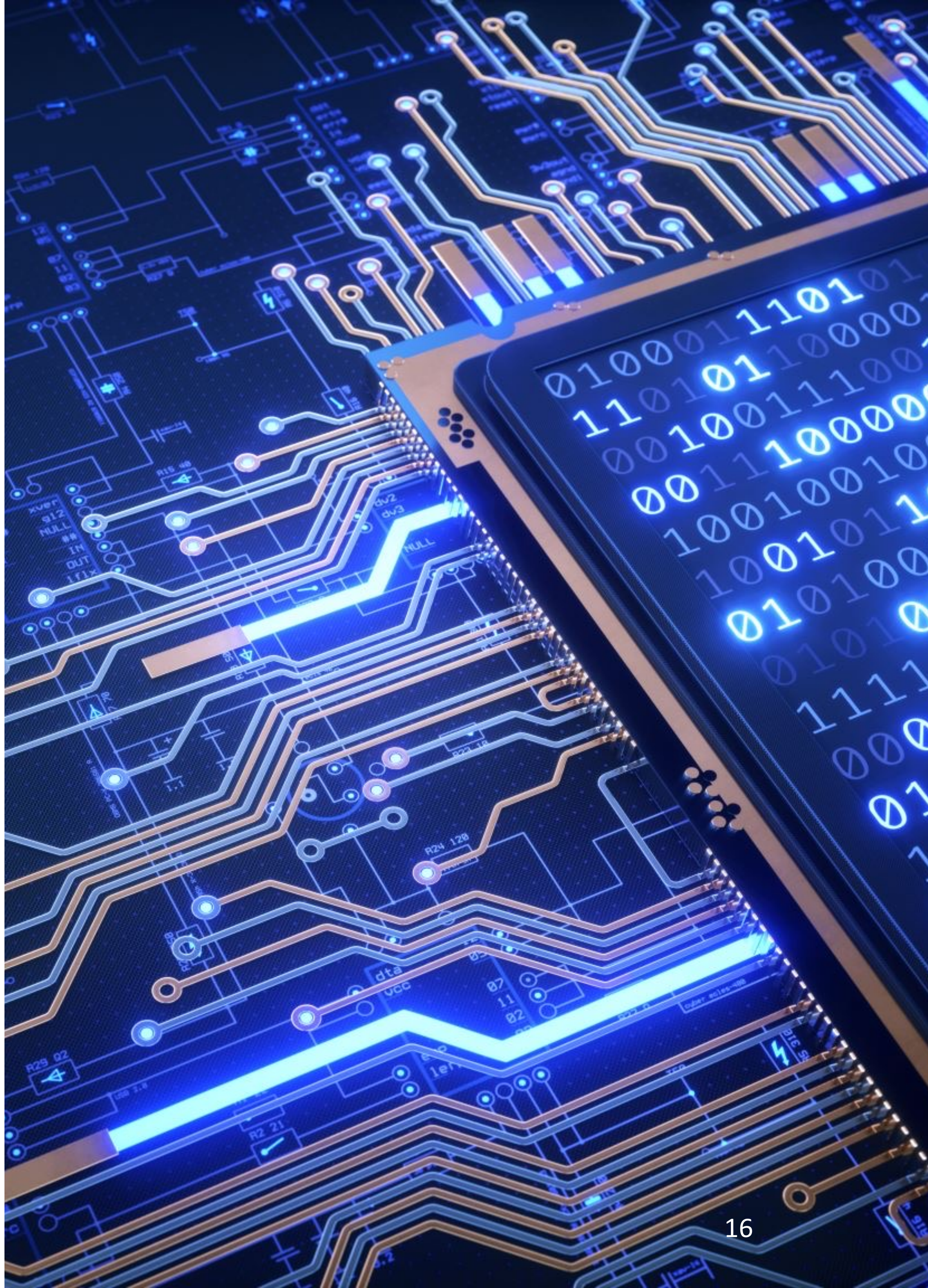
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$

Which of these relations are reflexive?

- ***Answer***: The relations R_3 and R_5 are reflexive



Reflexivity Property

- Example:

  Consider these relations on the set of **integers**:

  $R_1 = \{(a, b) \mid a \leq b\}$,
  $R_2 = \{(a, b) \mid a > b\}$,

  $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

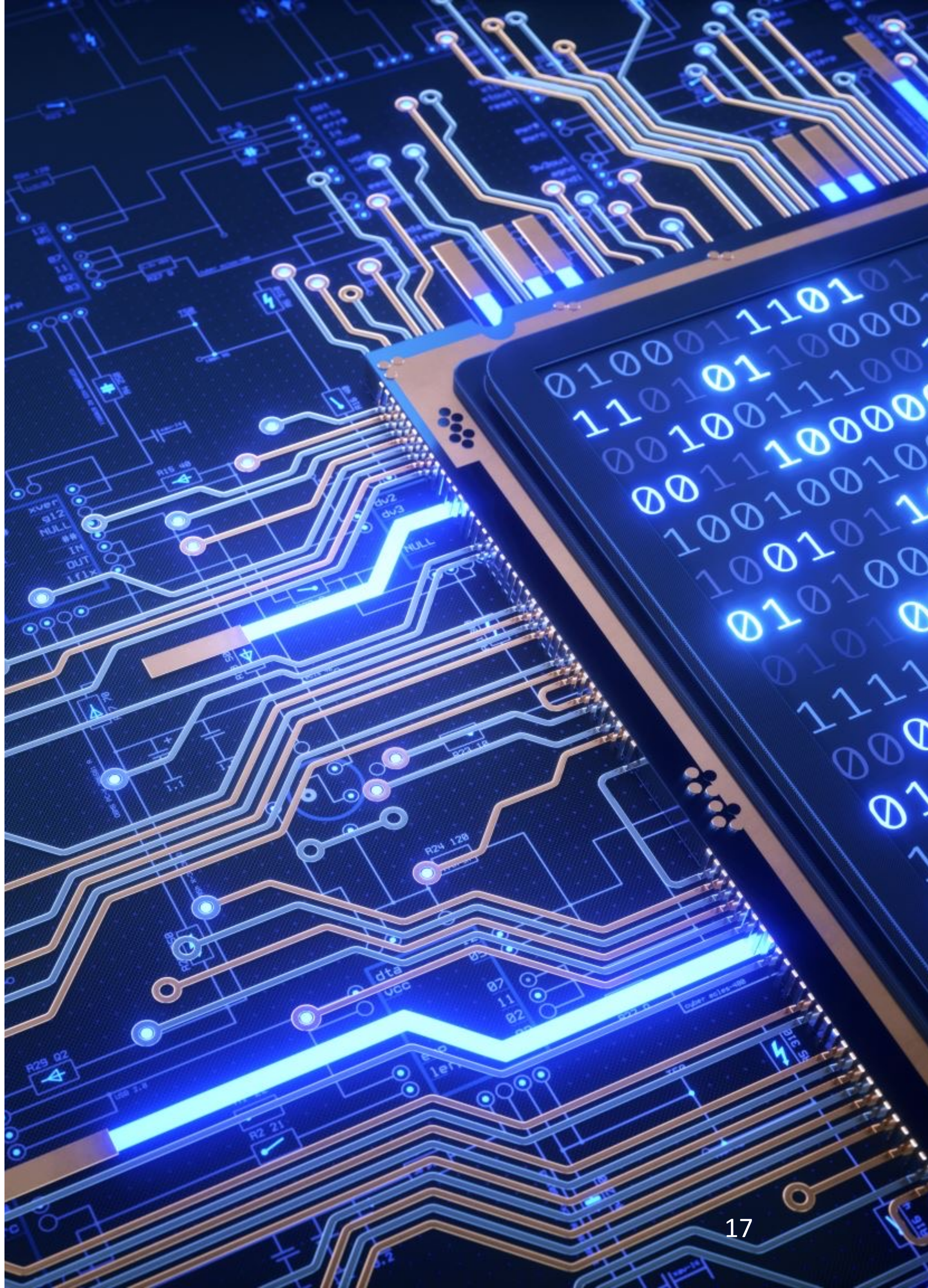
  $R_4 = \{(a, b) \mid a = b\}$,

  $R_5 = \{(a, b) \mid a = b + 1\}$,

  $R_6 = \{(a, b) \mid a + b \leq 3\}$.

  Which of these relations are reflexive?

Answer: The relations R_1, R_3, and R_4 are reflexive

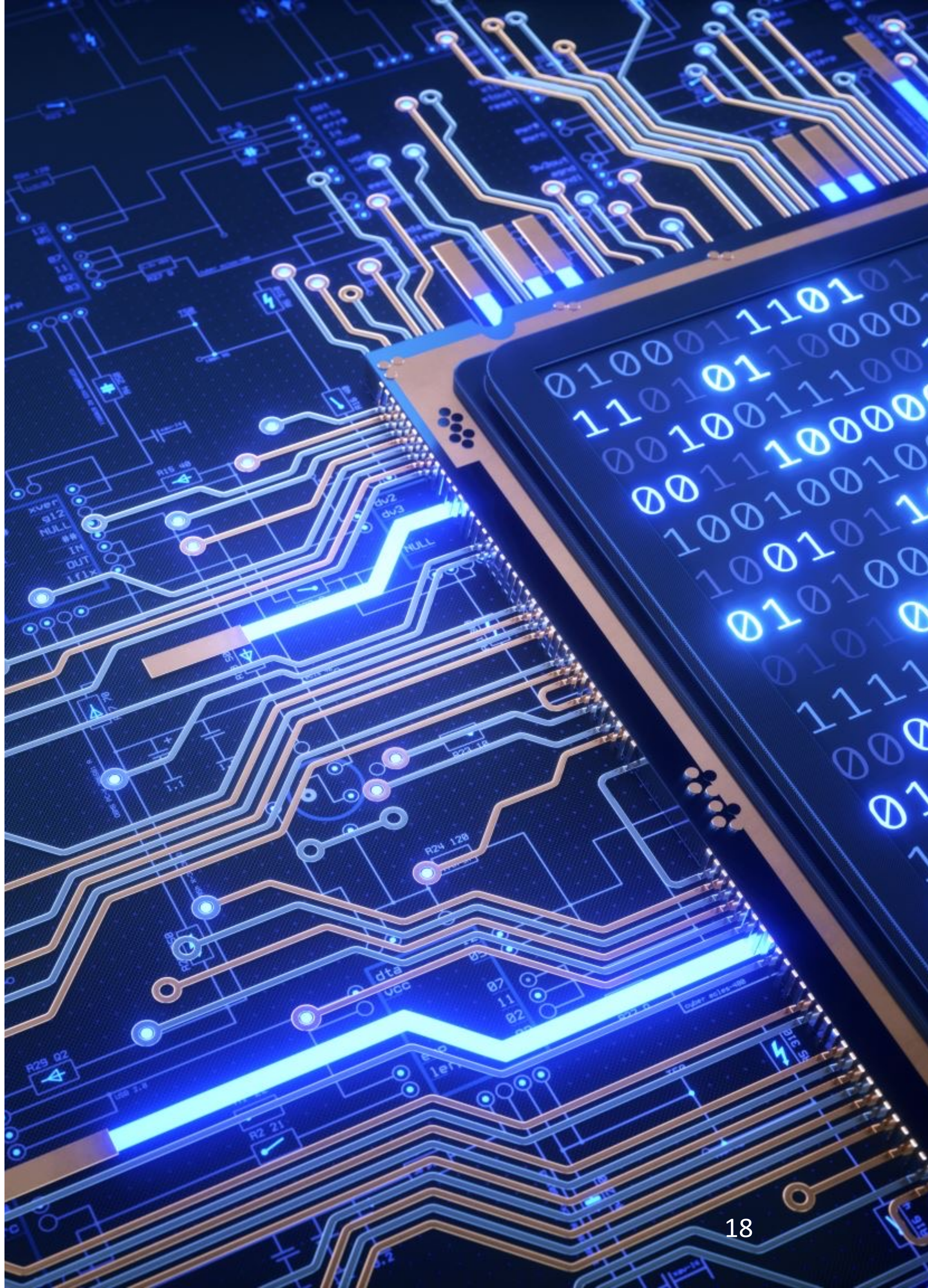


Symmetric Property

- A relation R on a set A is called ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$

- That means, an element is related to a second element if and only if the second element is also related to the first element

- If an element is related to a second element, but, the second element is not related to the first, then the relation is **not *symmetric***



Symmetric Property

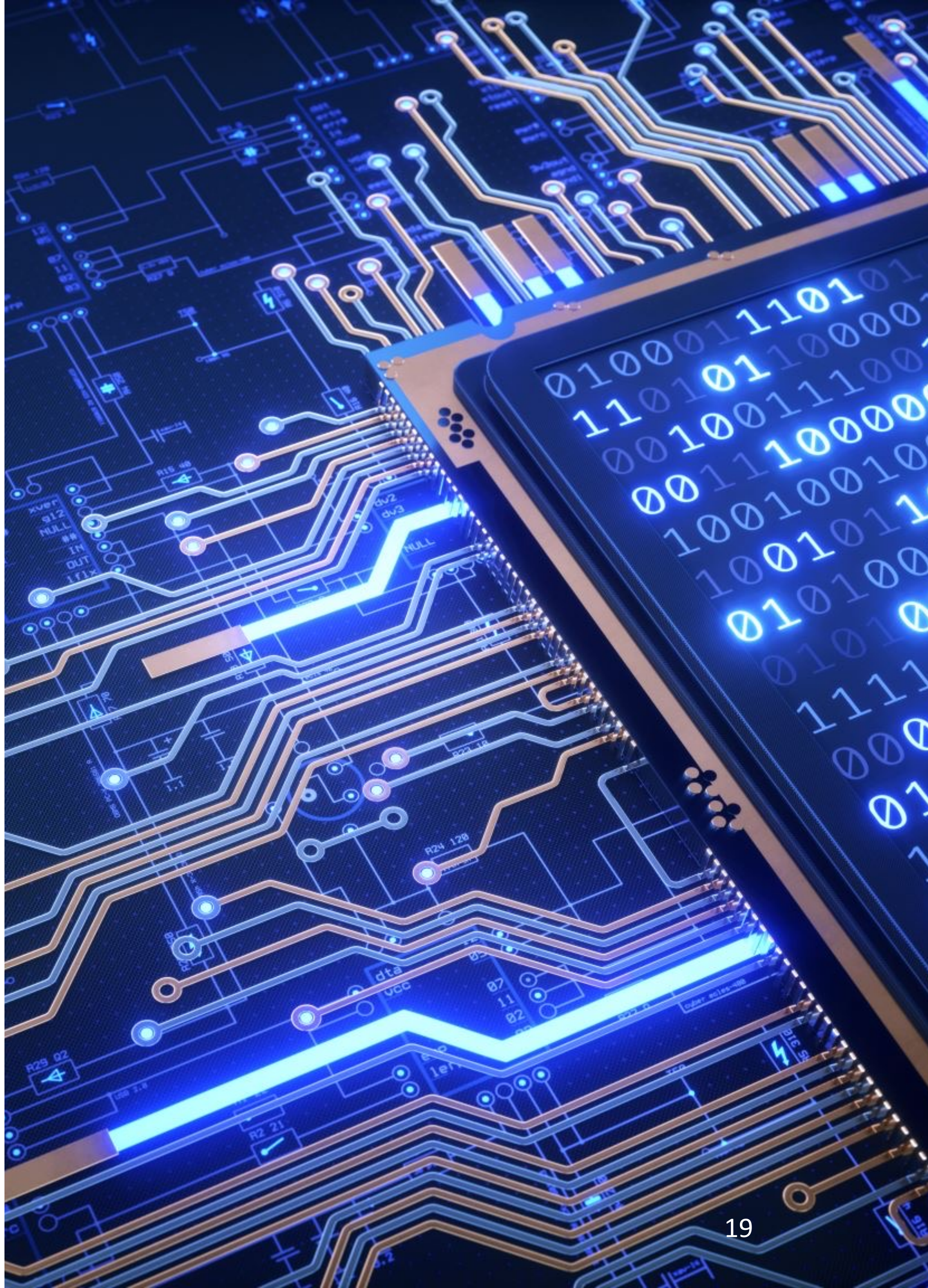
- ***Symmetry***

- Example:

  The relation consisting of pairs (*x*, *y*), where *x* and *y* are students at your school with at least one common class is ***symmetric.***

- Example:

  The relation consisting of the pairs (*x, y*), where *x* and *y* are students at your school, and *x* has a higher-grade point average than *y* is ***not symmetric***.



Symmetric Property

- Example:

Consider the following relations on {1, 2, 3, 4}

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

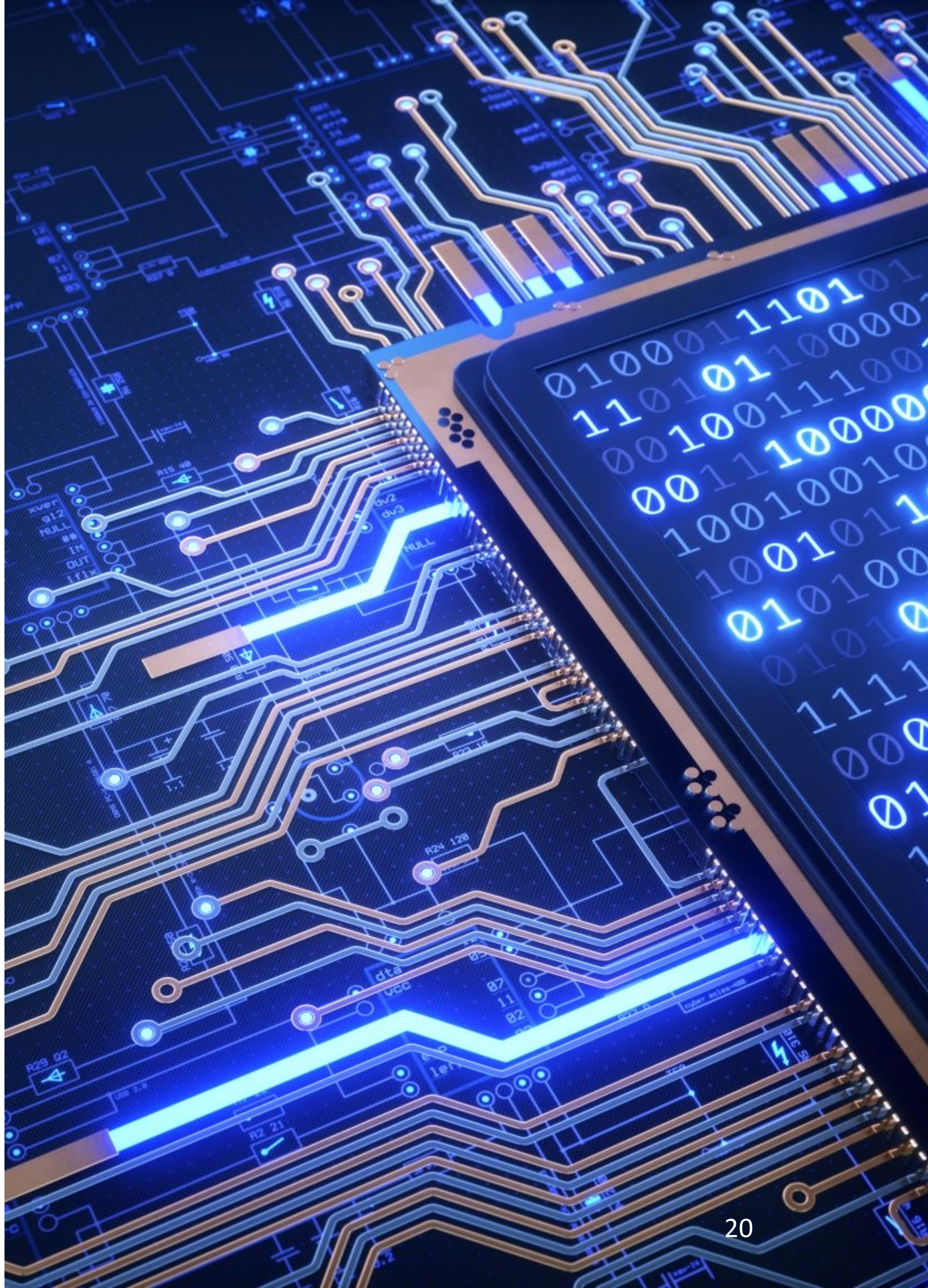
R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}

Which of these relations are symmetric?

Answer: The relations R_2 and R_3 are symmetric



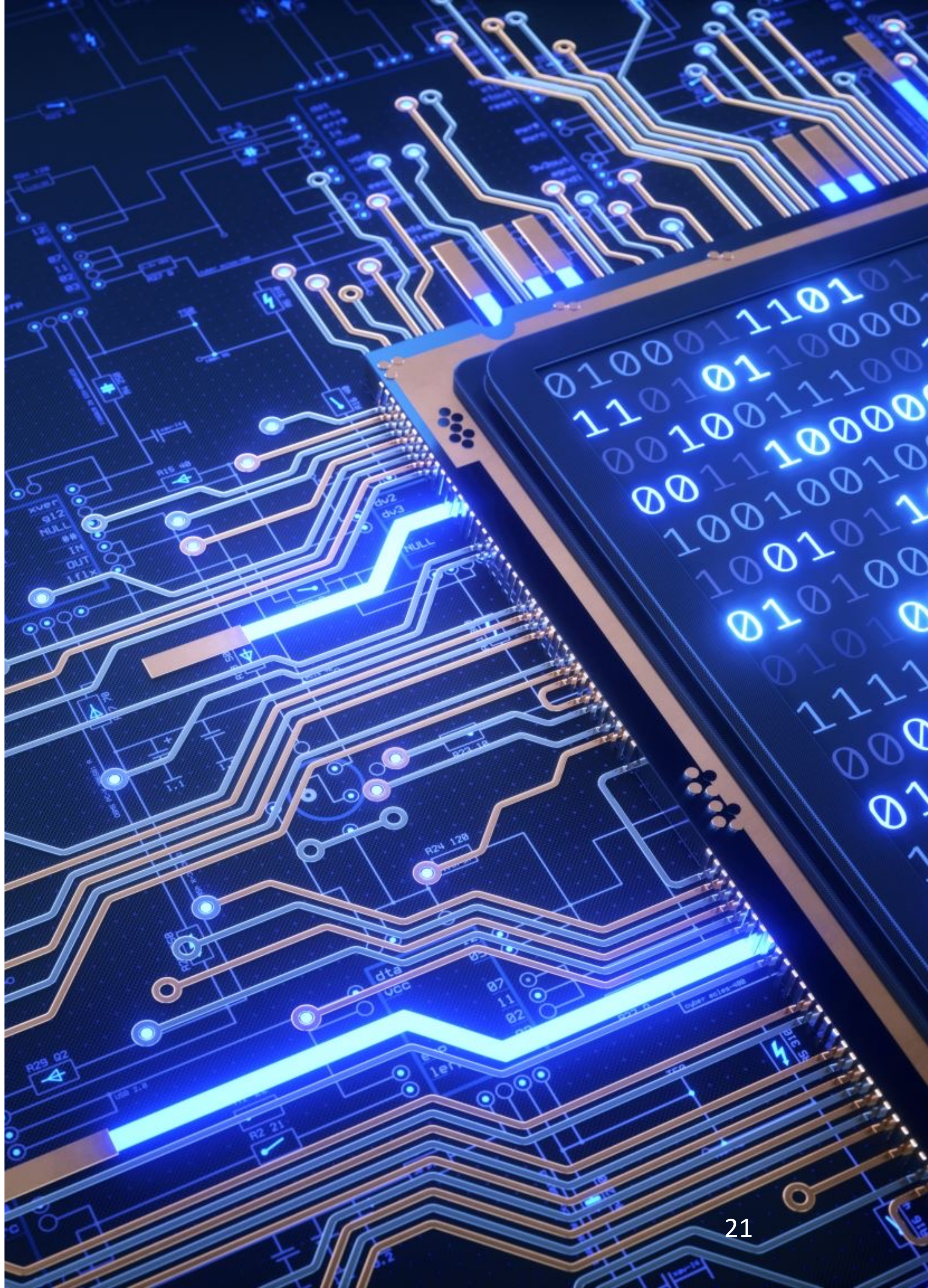
Antisymmetric Property

- A relation *R* on a set *A* such that for all *a*, *b* ∈ *A*, if (*a*, *b*) ∈ *R* then (*b*, *a*) ∉ *R* is called ***antisymmetric***

- That means, a relation is antisymmetric if and only if there are *no pairs* of distinct elements *a* and *b* such that *a* is related to *b* and *b* is related to *a*.

- The terms *symmetric* and *antisymmetric* **are not opposites**, because a relation can have both of these properties or may lack both of them.

- ***Remark***: A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (*a, b*), where *a* ≠ *b*.



Antisymmetric Property

Example: Consider the following relations on {1, 2, 3, 4}

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

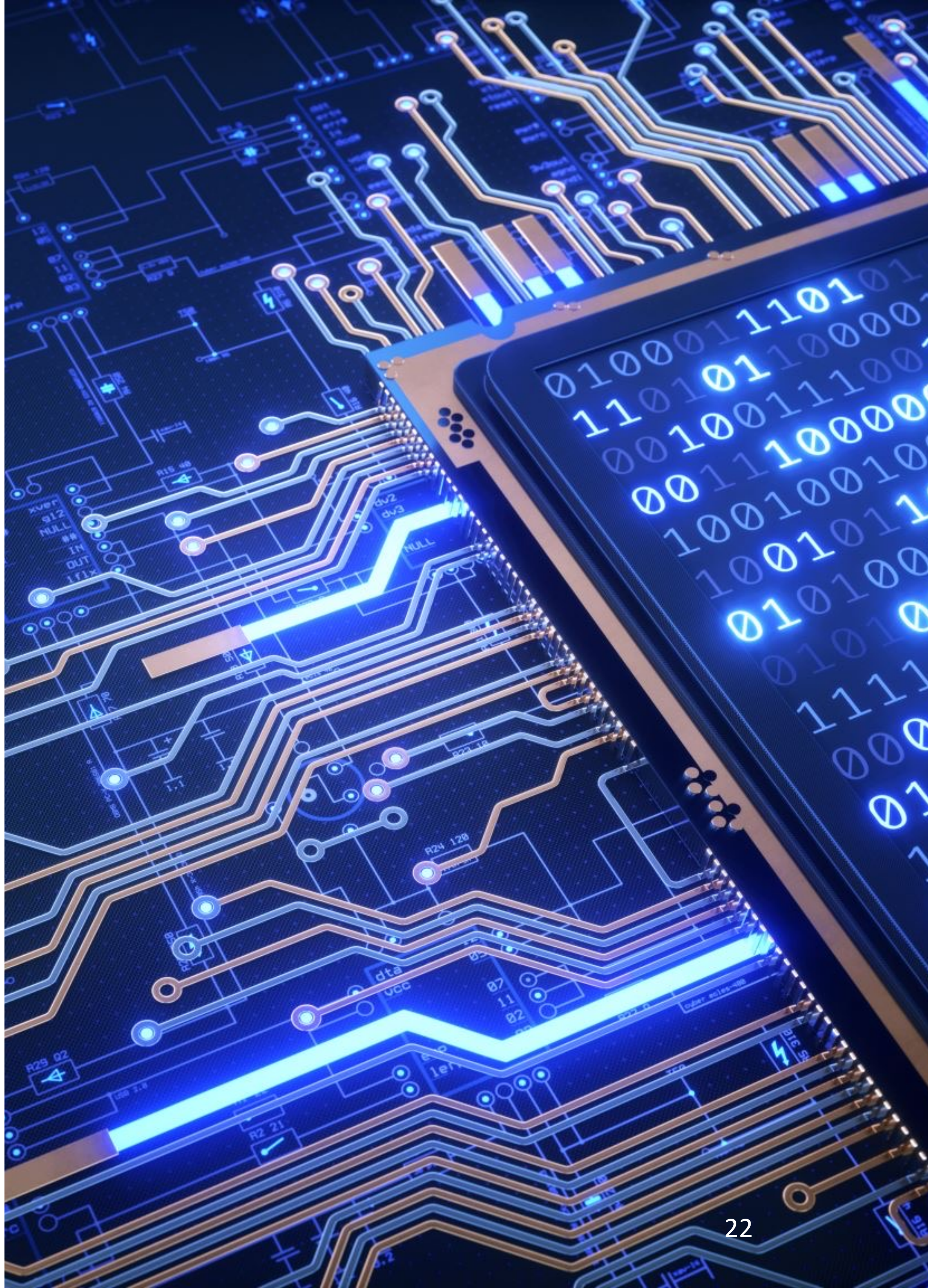
R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}

Which of these are antisymmetric?

Answer: R_4, R_5, and R_6 are all antisymmetric



In-Class Assignment

Write a relation *R* on the set {1, 2, 3, 4} such that,

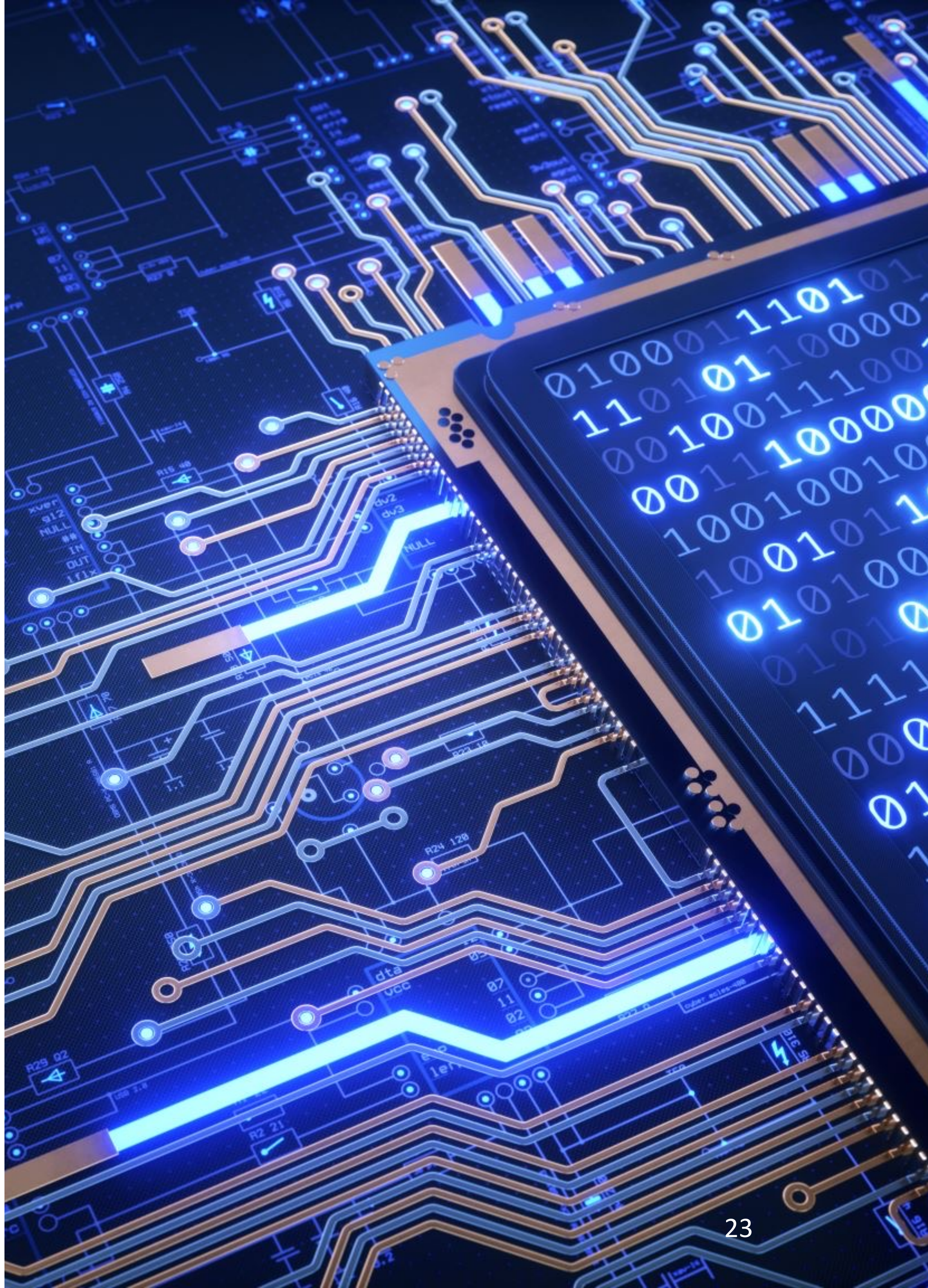
i) R is both symmetric as well as antisymmetric

ii) *R* is neither symmetric nor antisymmetric.

Answer:

i) R = {(1,1), (2,2), (3,3), (4,4)}

ii) R = {(1,2), (2,1), (3,4)}

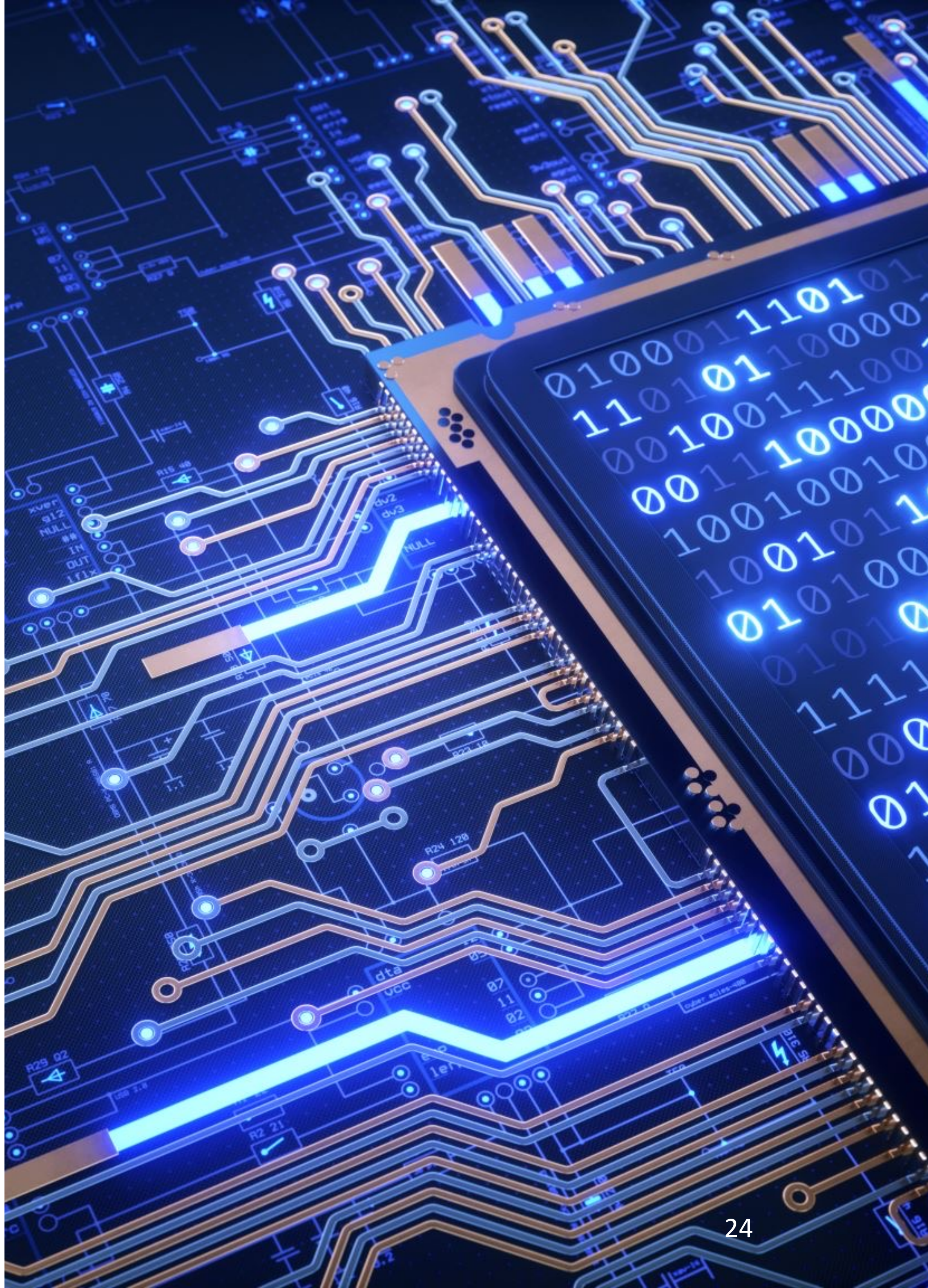


Transitive Property

- A relation *R* on a set *A* is called ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

- Example:

Let *R* be the relation consisting of all pairs (x, y) of students at your school, where *x* has taken more credits than *y*. Suppose that *x* is related to *y* and *y* is related to *z*. This means that *x* has taken more credits than *y* and *y* has taken more credits than *z*. We can conclude that *x* has taken more credits than *z*, so that *x* is related to *z*.



Transitive Property

- Example: Consider the following relations on {1, 2, 3, 4}

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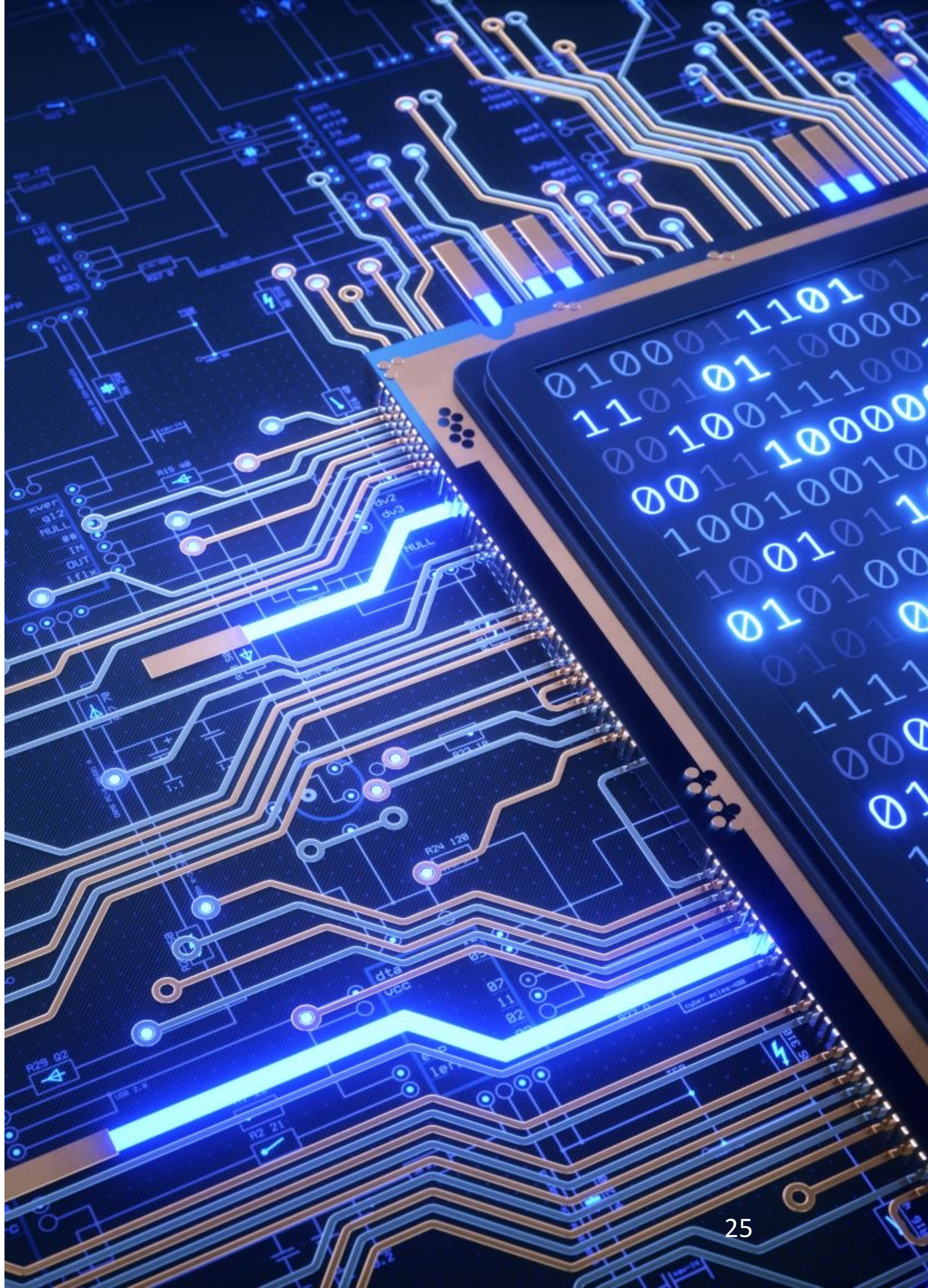
R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}

Which of these relations are transitive?

Answer: The relations R_2, R_3, R_4, and R_5 are transitive



Equivalence Relation

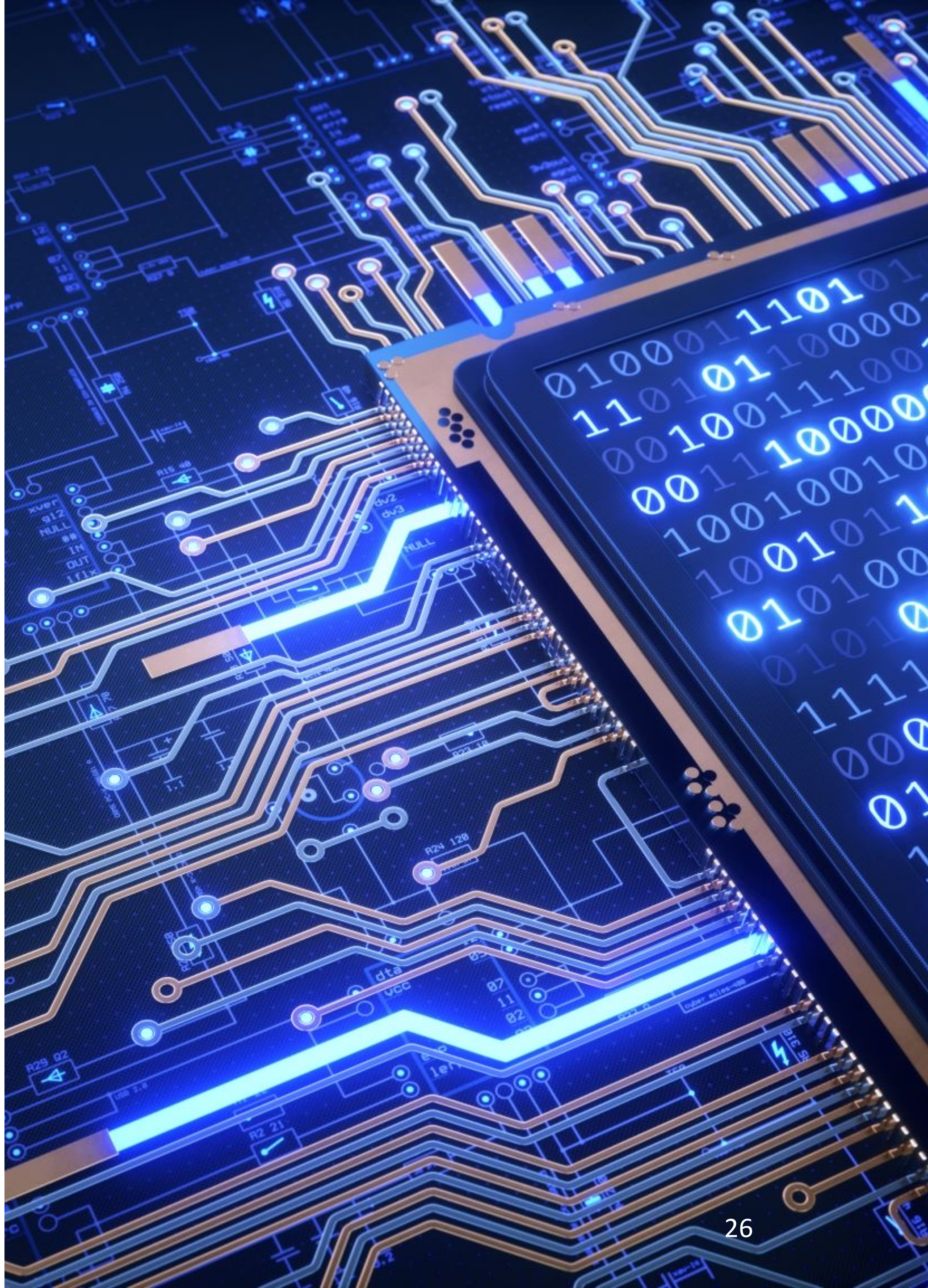
A relation on a set A is called an ***equivalence relation*** if it is reflexive, symmetric, and transitive

- Two elements a and b that are related by an equivalence relation are called *equivalent*.

- The notation $a \sim b$ is used to denote that a and b are equivalent elements with respect to a particular equivalence relation

- Example:

  Let R be the relation on the set of integers such that $a\ R\ b$ if and only if $a = b$ or $a = -b$. R is reflexive, symmetric, and transitive. Hence, R is an equivalence relation.



Equivalence Relation

- Example:

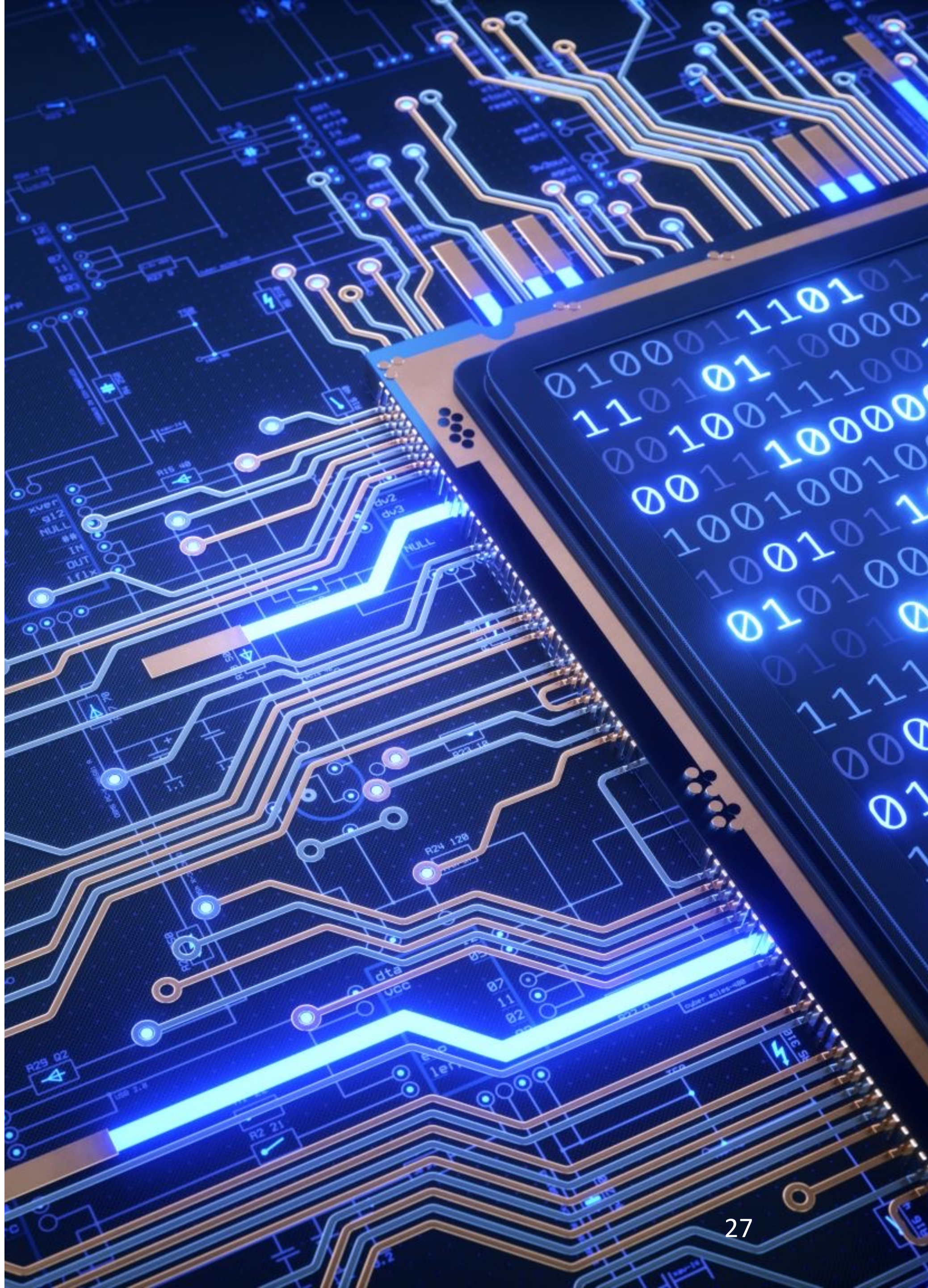
  Let R be the relation on the set of real numbers such that a R b if and only if $a - b$ is an integer. Is R an equivalence relation?

  Answer: R is reflexive, symmetric, and transitive. Hence, R is an equivalence relation.

- Example:

Suppose that R is the relation on the set of strings of English letters such that a R b if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Answer: R is reflexive, symmetric, and transitive. Hence, R is an equivalence relation.



In-class Exercise

Which of these relations on {0, 1, 2, 3} are equivalence relations?

Determine the properties of an equivalence relation that the others lack.

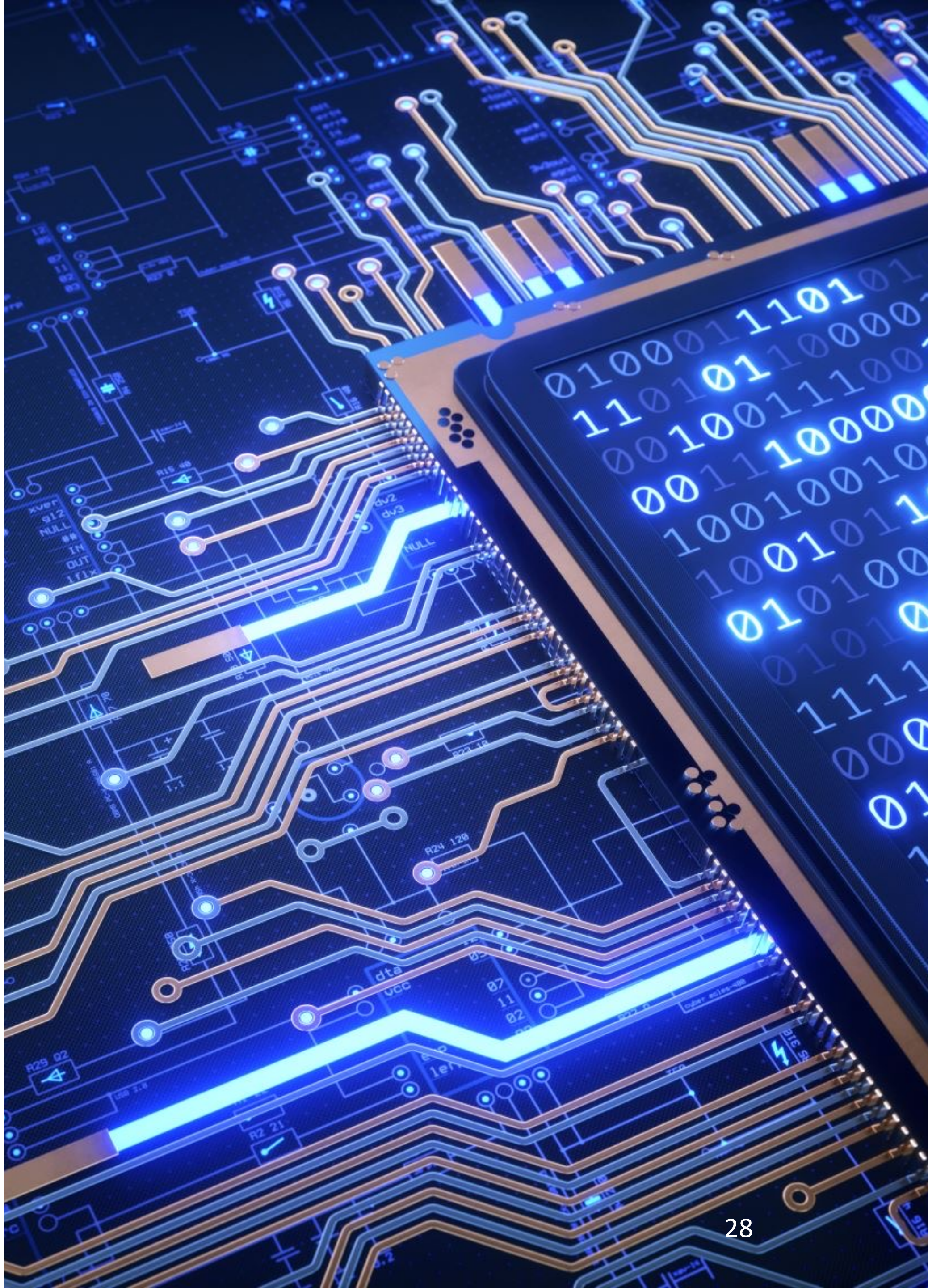
a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}



In-class Exercise

Solution:

a) The relation {(0, 0), (1, 1), (2, 2), (3, 3)} is an **equivalence relation**, because it is reflexive, symmetric and transitive.

It is **reflexive** because it contains all reflexive elements (i.e. (a, a)) formed from all the elements of the given set {0, 1, 2, 3}

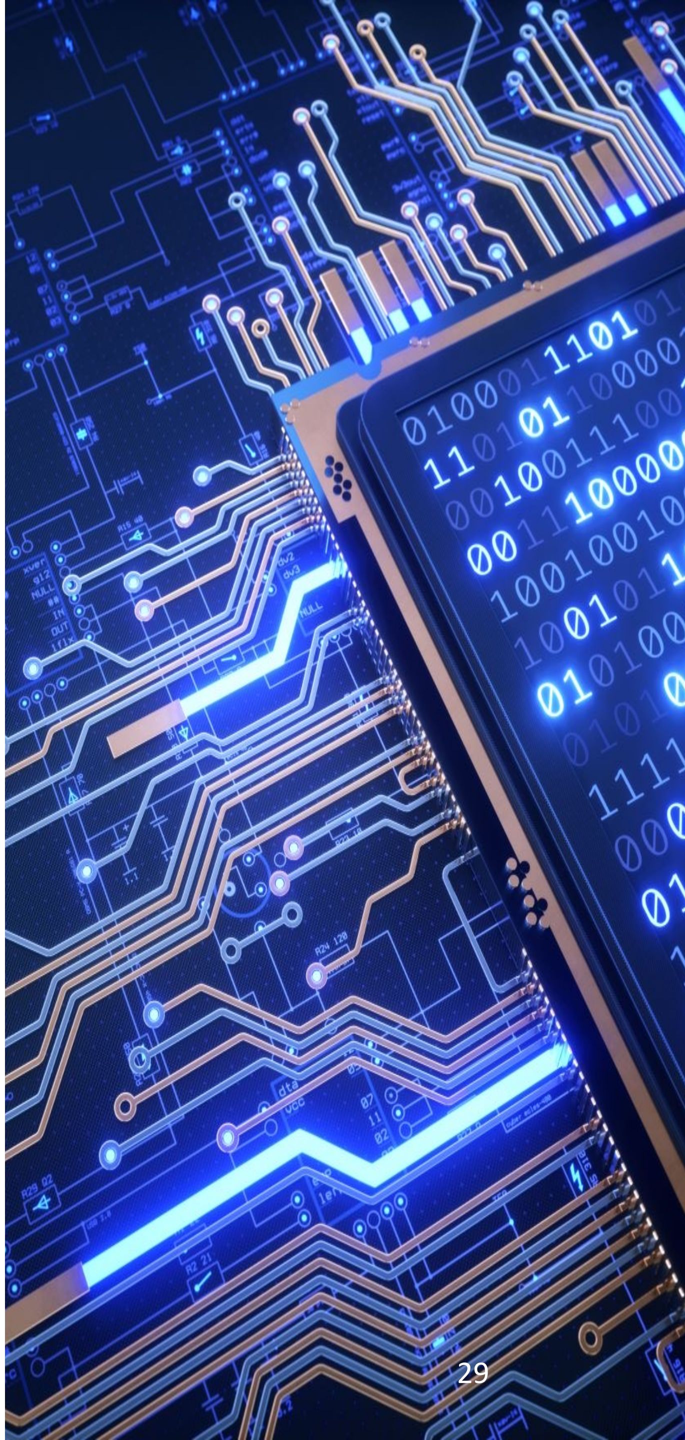
It is **symmetric** because, for each pair in the relation, say (2, 2), the reverse ordered pair which is again (2, 2) is present.

It is **transitive** (see the solution to the problem 8 (e) above), because, each pair of the form (a, a) can be treated as (a, b), (b, c) as well as (a, c) *all three*.

b) The relation {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)} is **not an equivalence relation**, because it **lacks the properties**; **reflexive, transitive**.

It is not reflexive because, (1, 1) is not in the relation.

It is not transitive because, for the set of ordered pairs (3, 2), (2, 0), the pair (3, 0) is not in the relation



In-class Exercise

Solution:

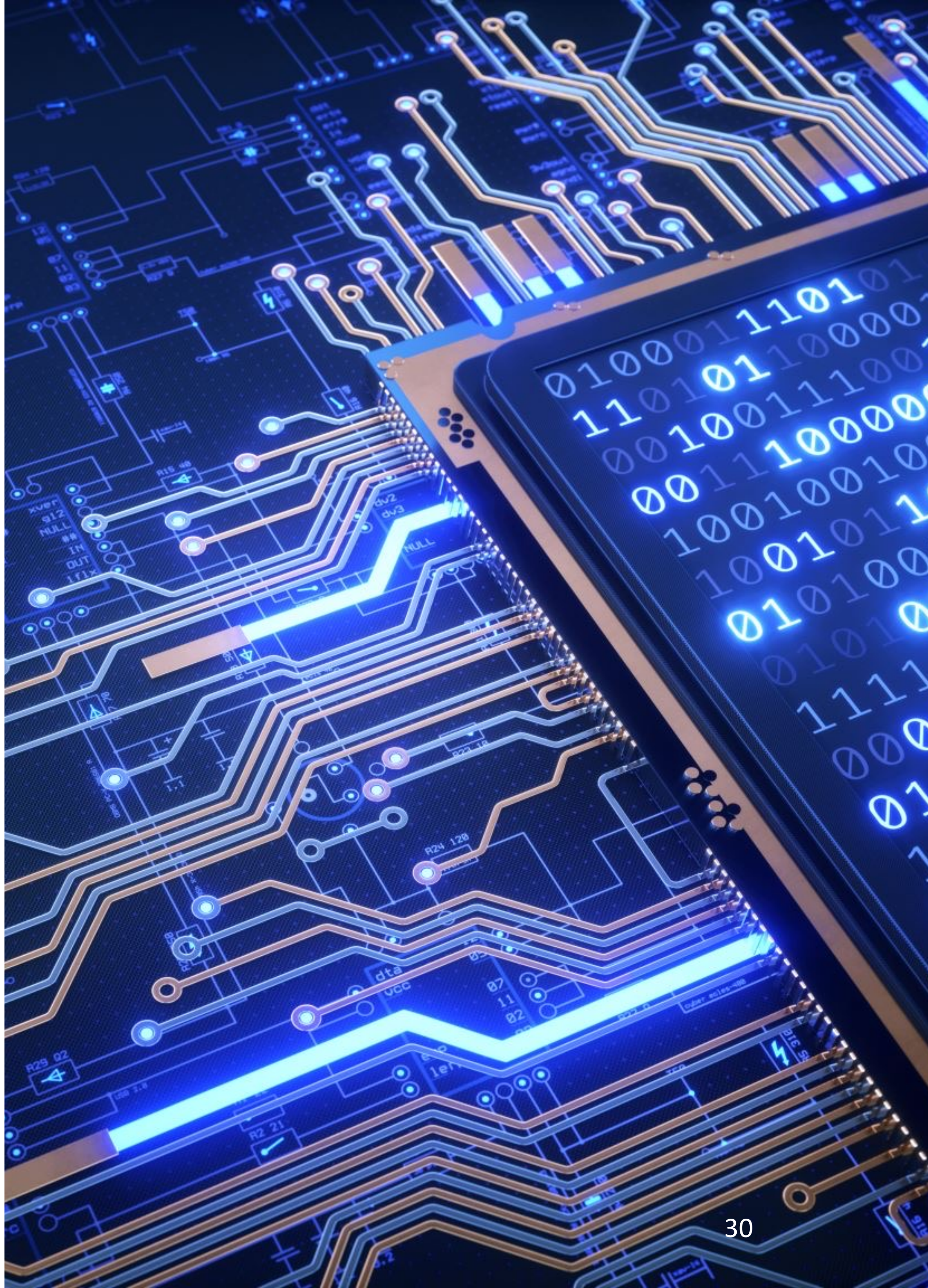
c) The relation {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)} is an **equivalence relation**, because it is reflexive, symmetric and transitive.

It is **reflexive** because it contains all reflexive elements (i.e. (*a*, *a*)) formed from all the elements of the given set {0, 1, 2, 3}

It is **symmetric** because, for each pair in the relation, say for example (1, 1), the reverse ordered pair which is again (1, 1) is present. Also, for (1, 2) we have (2, 1).

It is **transitive** because, for each set of pairs of the form (*a*, *b*), (*b*, *c*) we have (*a*, *c*). For the case of (3, 3) there is no pair with 3 as first element and hence can be ignored.

The rule is, whenever (*a*, *b*), (*b*, *c*) are present, we must have (*a*, *c*). For some (*a*, *b*), if there is no (*b*, *c*) to decide upon transitive property, such an (*a*, *b*) can be ignored and we continue treating the relation as transitive.



In-class Exercise

Solution:

d) The relation {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} is **not an equivalence relation**, because it **lacks the transitive property**.

It is not transitive because, for the set of ordered pairs (2, 3), (3, 1), the pair (2, 1) is not in the relation.

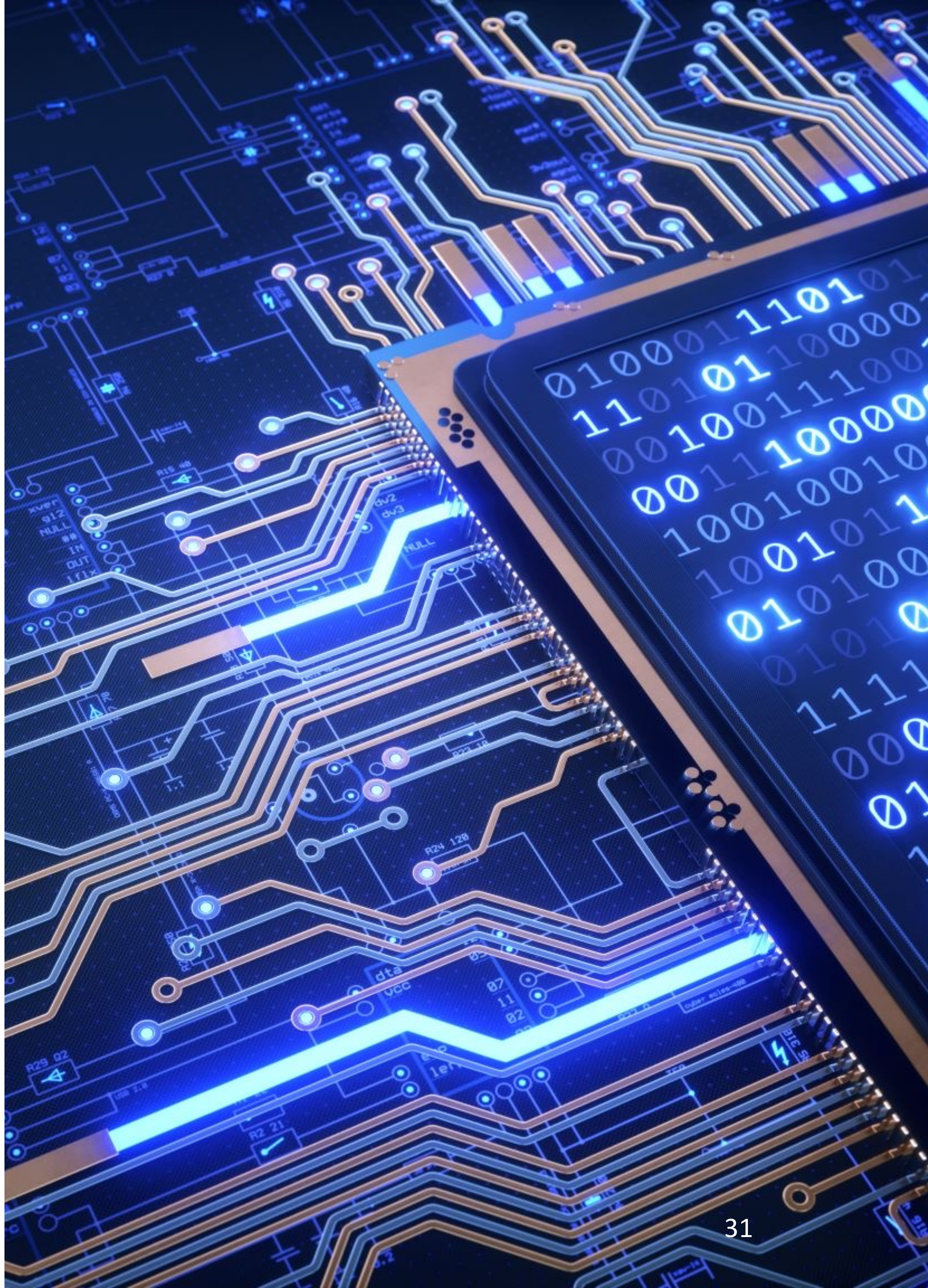
The reflexive and symmetric properties can be easily verified.

e) The relation {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)} is **not an equivalence relation**, because it **lacks the properties**; **symmetric**, **transitive**.

   It is not symmetric because, for the pair (1, 2) corresponding reverse ordered pair (2, 1) is not in the relation.

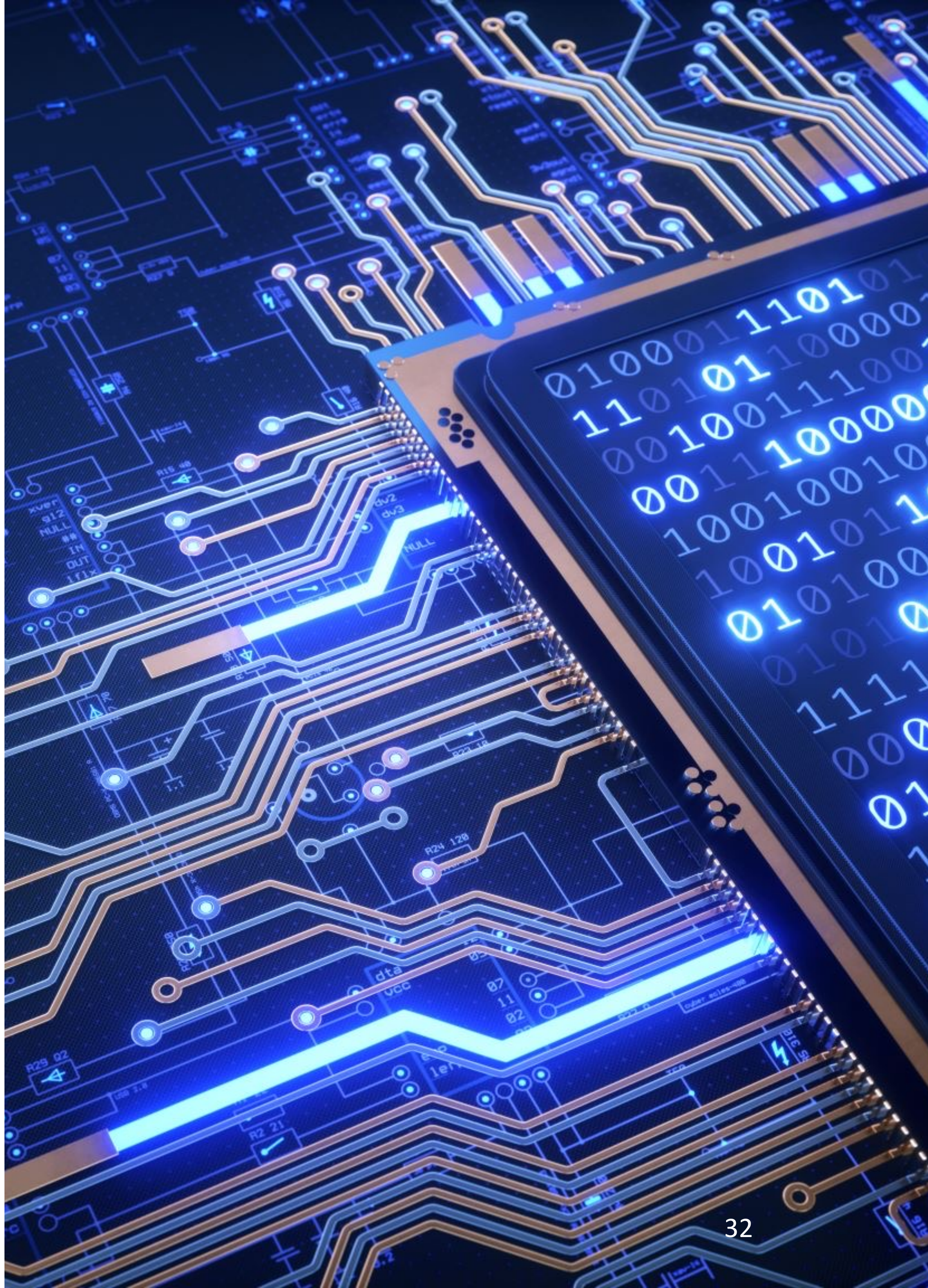
   It is not transitive because, for the set of ordered pairs (2, 0), (0, 1), the pair (2, 1) is not in the relation.

   The reflexive property can be easily verified.



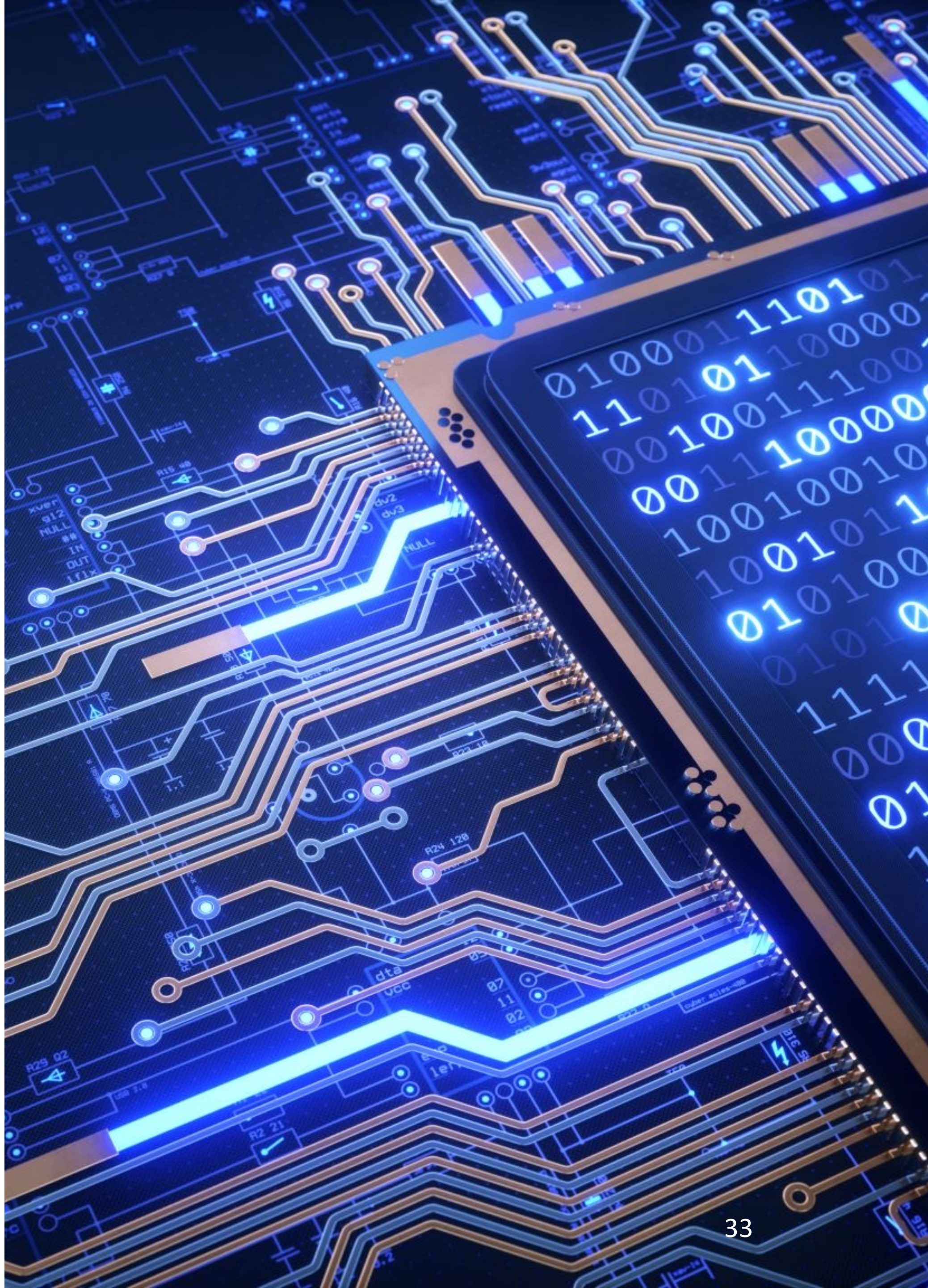
Applications of Recursion

- Algorithm design
- Data structures
- Dynamic Programming
- Fractals and Graphics
- Backtracking algorithms
- Parsing and expression evaluation
- Function calls and call stacks



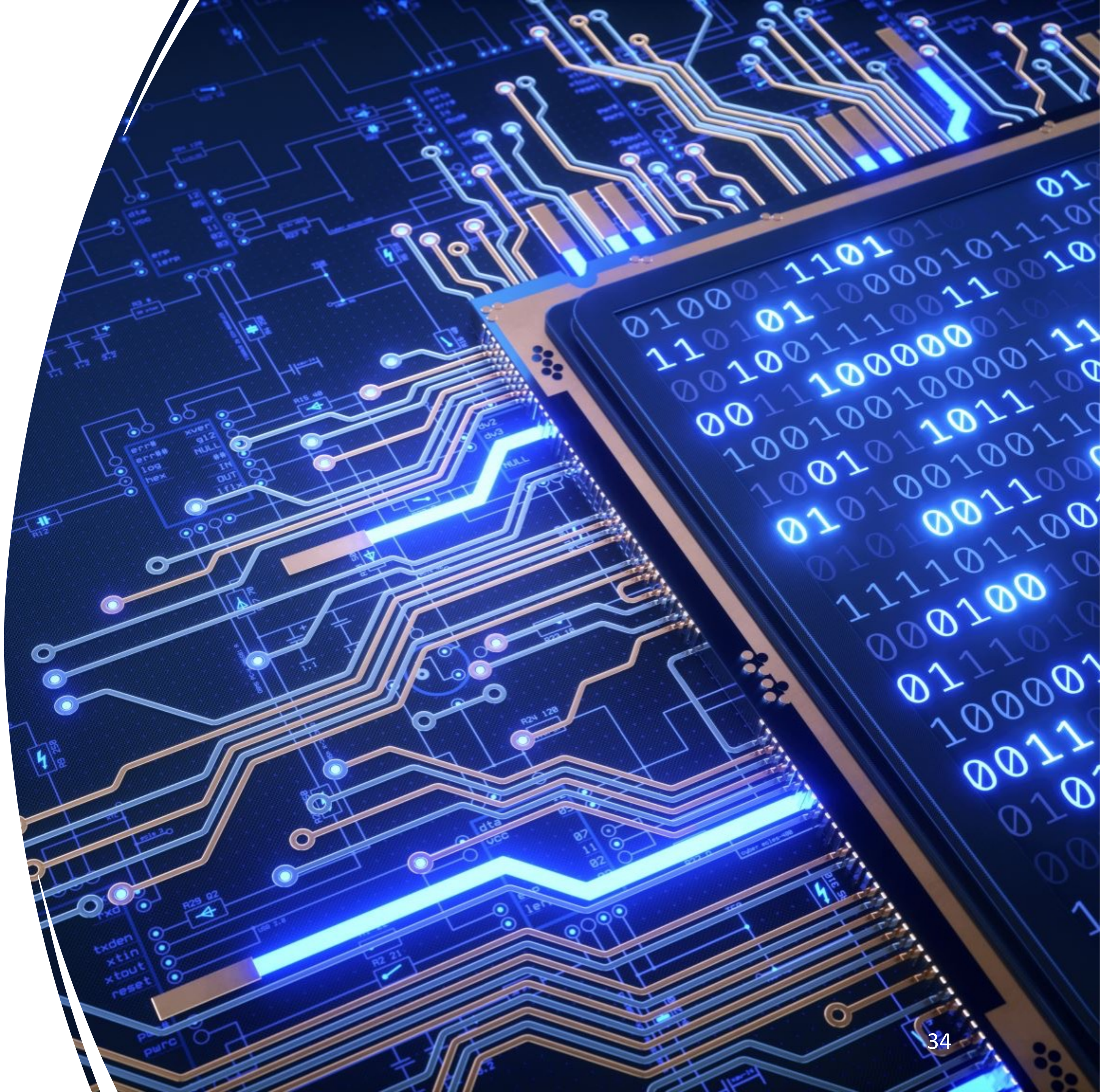
Applications of Relation

- Database Management
- Set theory
- Graph theory
- Order theory
- Equivalence relations
- Fuzzy logic
- Formal languages and automata theory



Summary

- Recursion and relations are powerful concepts with diverse applications in computer science, mathematics, and related fields,
  - enabling the design of efficient algorithms,
  - data structures,
  - and mathematical models.
- Recursive definition
- Binary relations
- Properties of relations
  - Reflexive
  - Symmetric
  - Transitive



Reference

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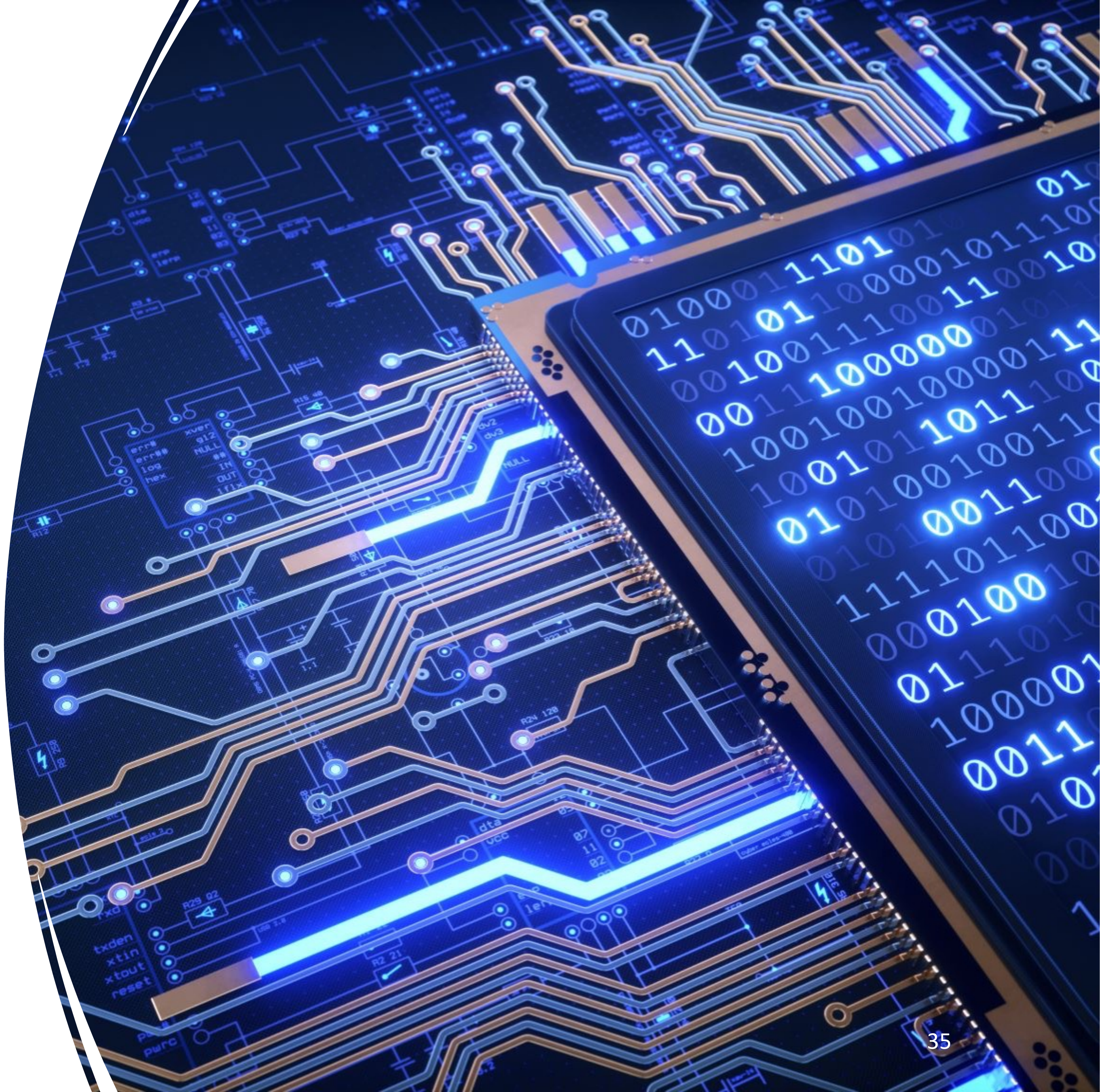
Chapters 5 & 9



See you next time!