Course: Mathematics for IT Professionals

Lecture 11

Differentiation

By

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Outline

The topics to be treated in this lecture are:

- The Limit concept
- Properties of Limits
- The mean value theorem
- Computation of Derivatives
- Leibniz Notation
- Product Rule
- Quotient Rule
- Chain Rule

Lecture Learning Outcomes

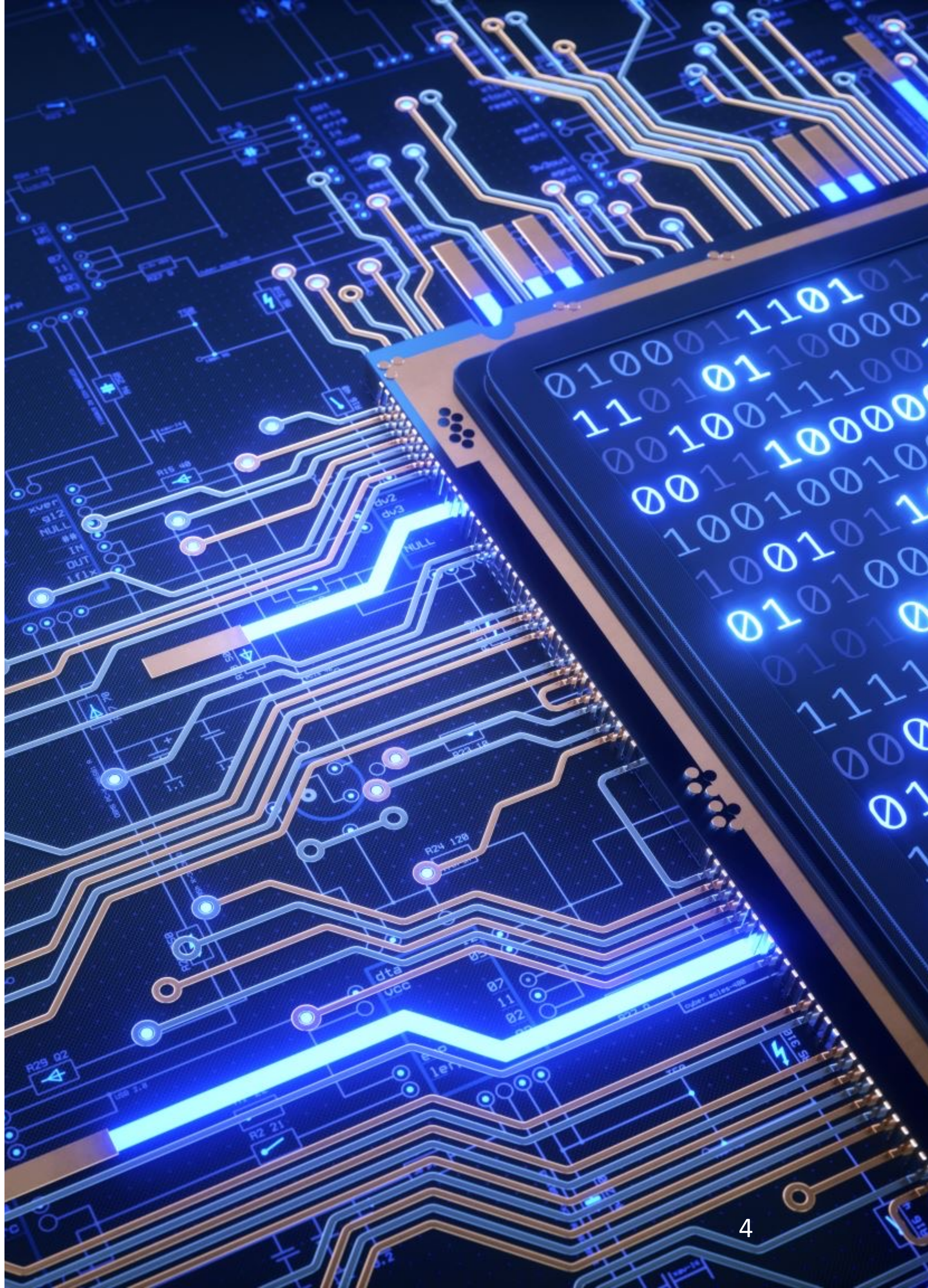
At the end of the session, you will be able to

- use the definition of limit to estimate limits
- determine whether limits of functions exist
- know how to compute simple derivatives from the definition of a derivative
- know the product rule, quotient rule, and chain rule and be able to use them to compute sums, products, quotients, and compositions of these functions

Introduction

- Differentiation is the process of computing the derivative of a function.
- Differentiation is all about measuring change.
- The derivative is used to find
  - *the "slope" of a function at a point*
  - *the "slope of the tangent line" to the graph of a function at a point*
  - *the "instantaneous rate of change" of a function at a point*
  - *the limit of the difference quotient as Δx approaches 0 (Limit Definition)*

Brokate, M., Manchanda, P., & Siddiqi, A. H. (2019). *Calculus for scientists and engineers*. Springer Singapore.



Introduction

$$y = a + bx$$

a = intercept

b = constant slope i.e. the impact of a unit change in x on the level of y

$$\mathbf{b} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

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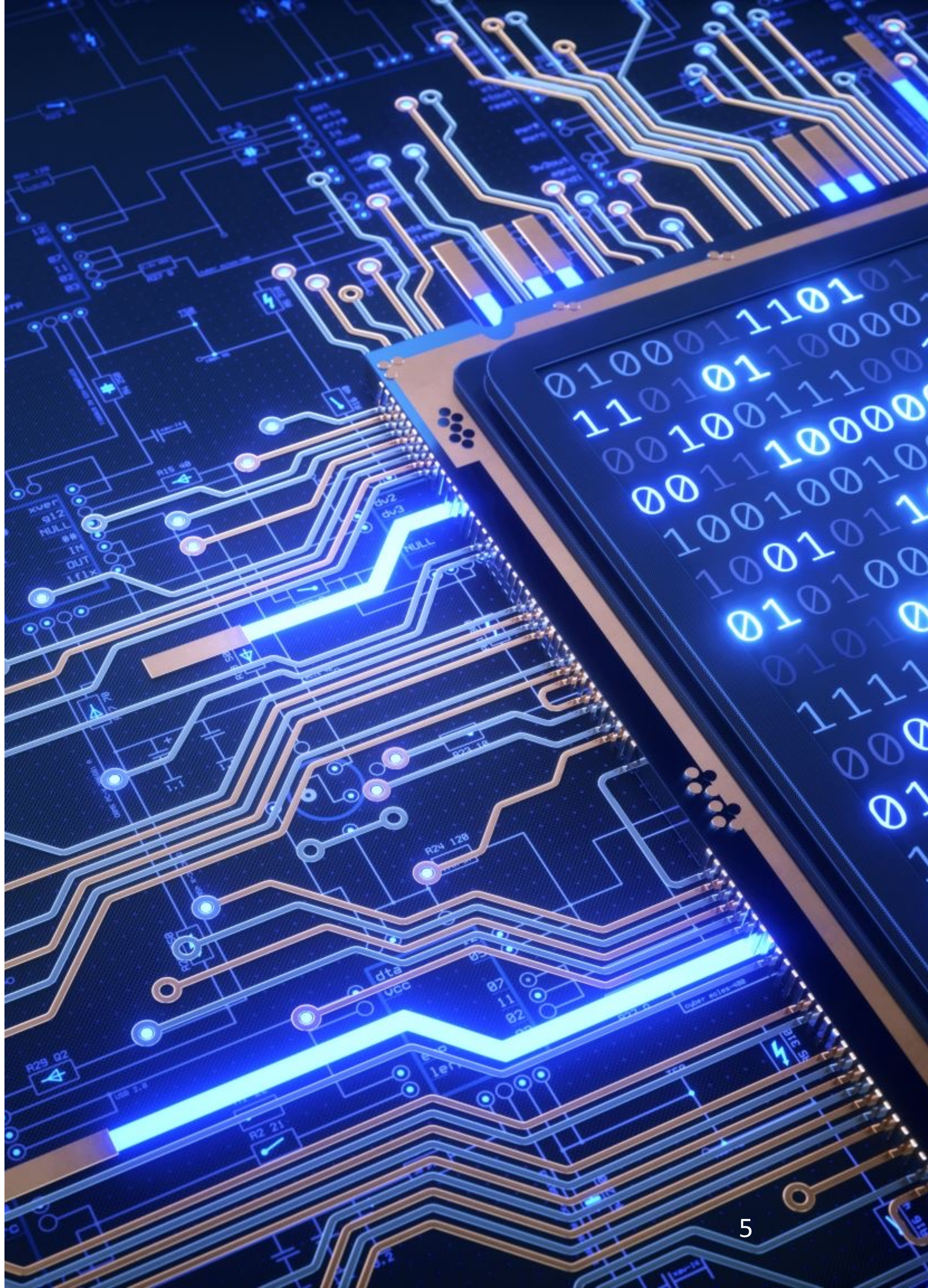


Worked Example I

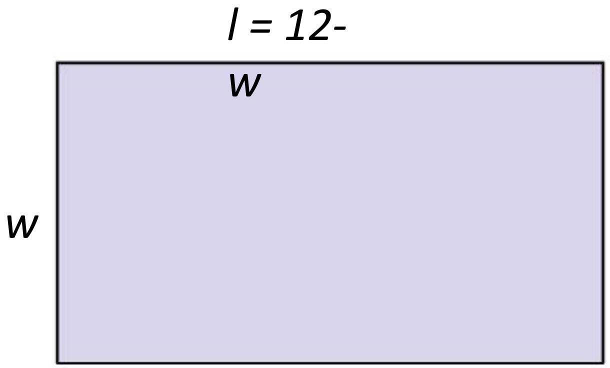
Find the dimensions of a rectangle that has a perimeter of 24 inches and a maximum area.

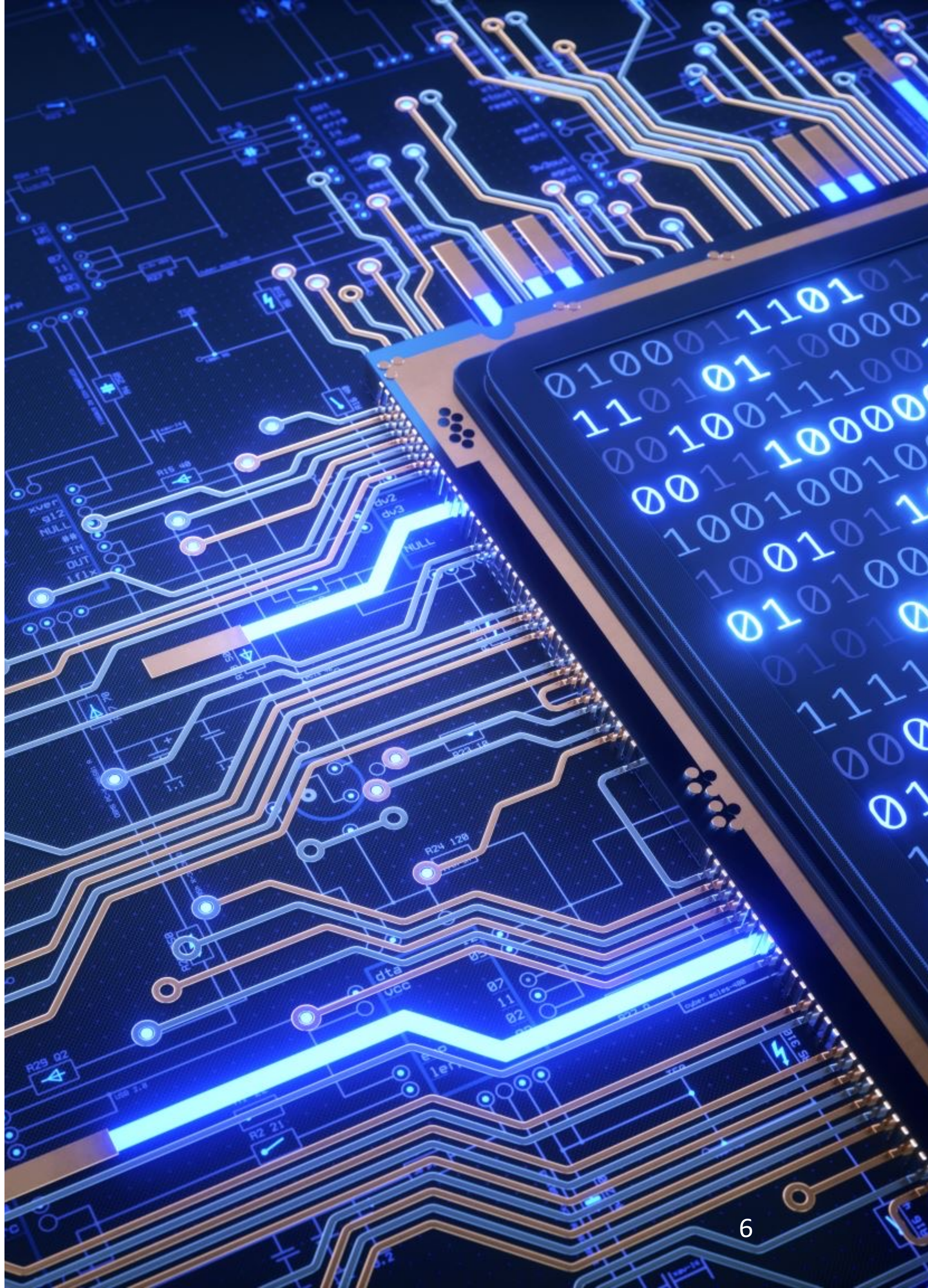
Solution:

Let *w* represents the width of the rectangle and let *l* represents the length of the rectangle. Because

Perimeter is 24

$2w + 2l = 24$





Worked Example I

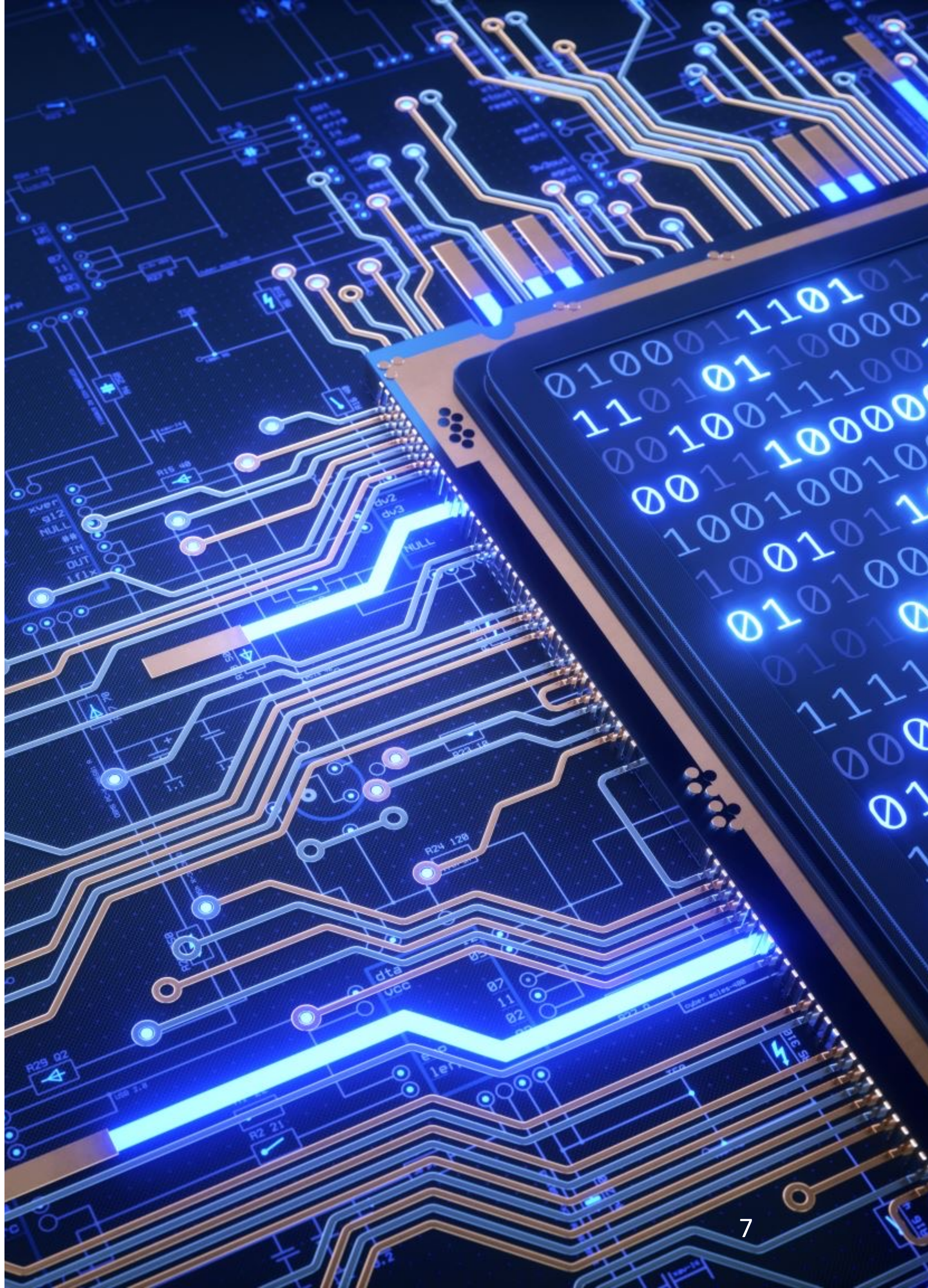
- So, the area of the rectangle is

$$A = lw$$

$$= (12 - w)w$$

$$= 12w - w^2.$$

- Using this model for area, experiment with different values of w to see how to obtain the maximum area.



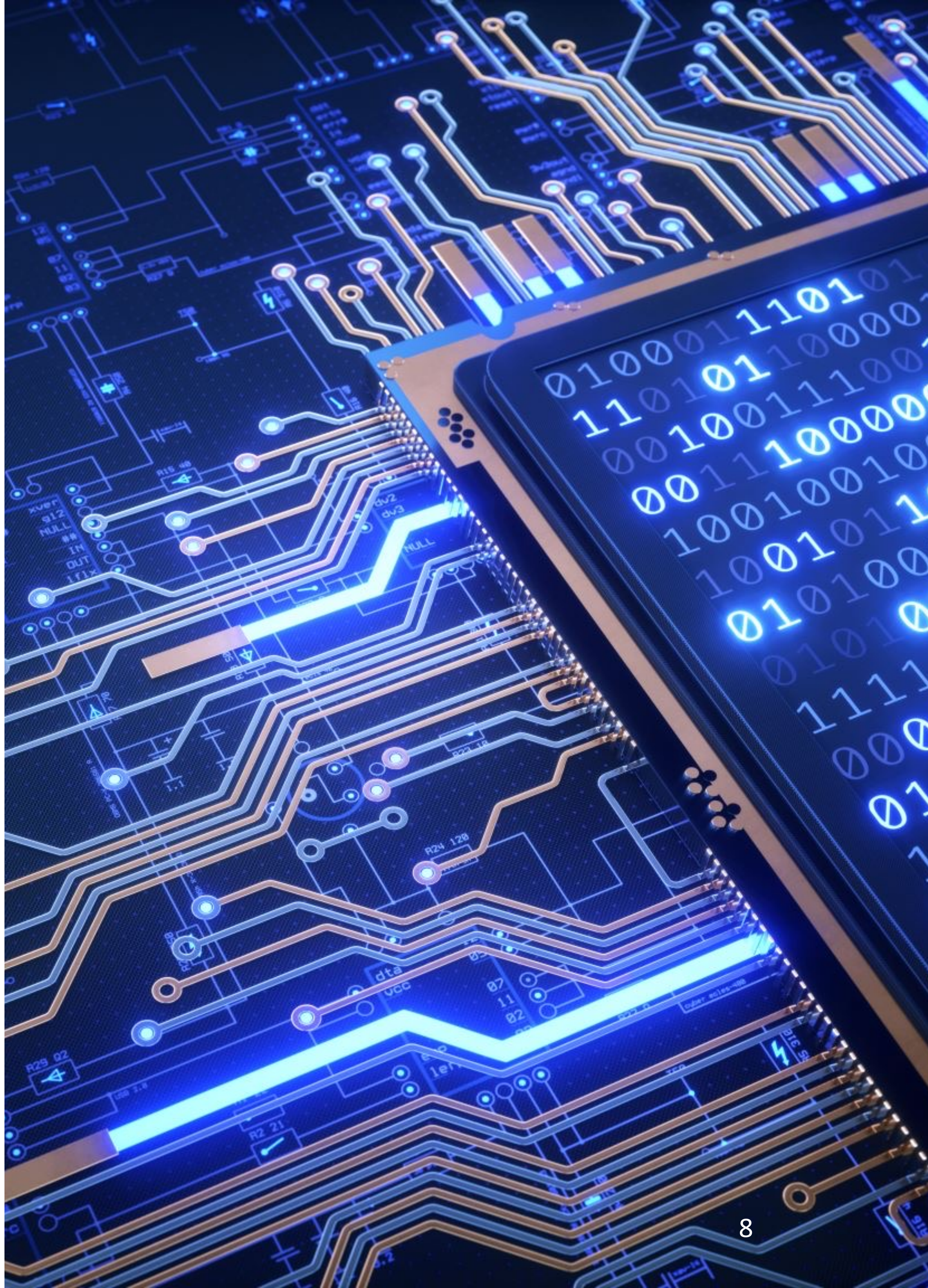
Worked Example I

- After trying several values, it appears that the maximum area occurs when $w = 6$, as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

- In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2)$$
$$= 36.$$



Definition of Limit

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

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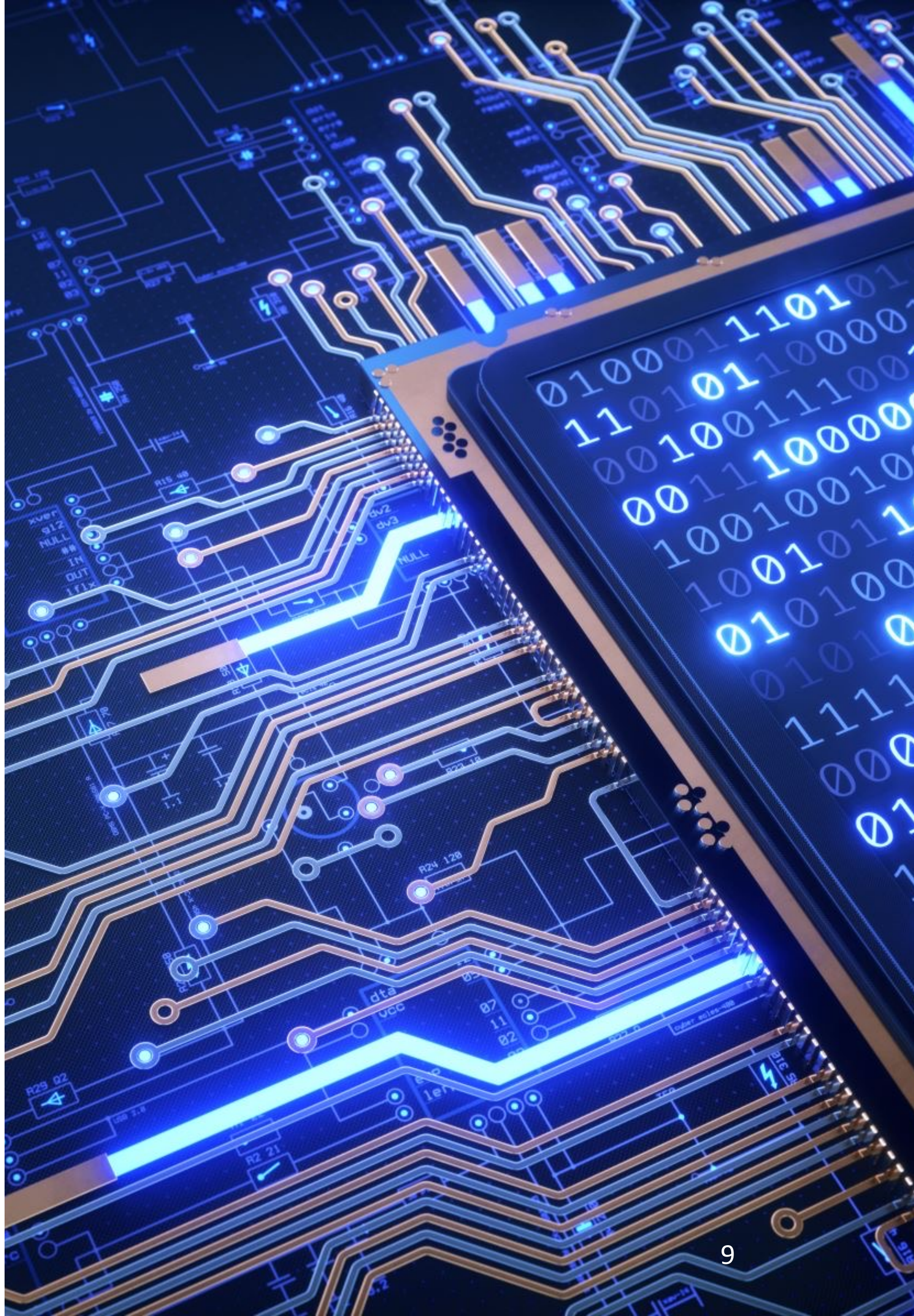


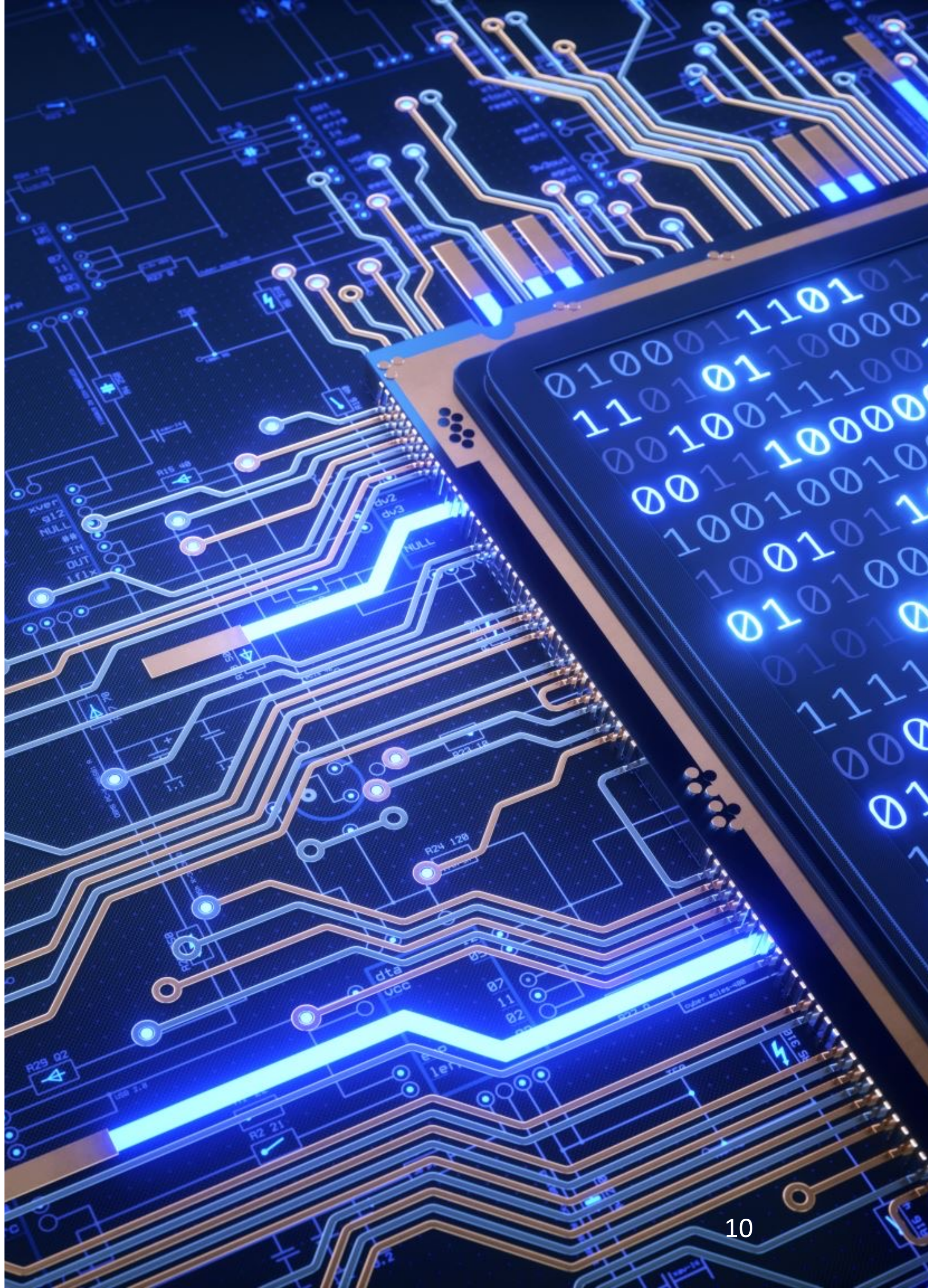
Worked Example II

- Use a table to estimate numerically the limit:

$$\lim_{x \to 2} (3x - 2)$$

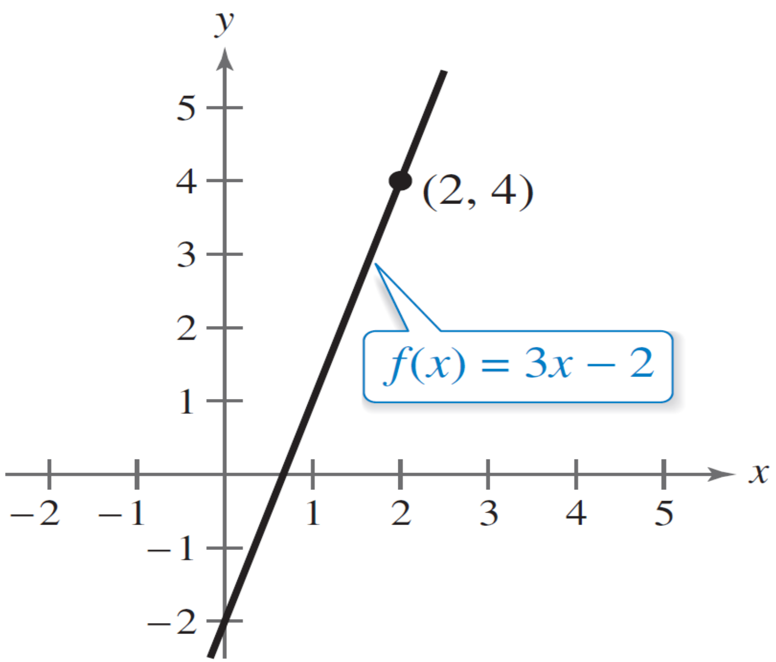
- Solution:
- Let $f(x) = 3x - 2$.
- Then construct a table that shows values of $f(x)$ for two sets of x-values
  - one set that approaches 2 from the left
  - and one that approaches 2 from the right

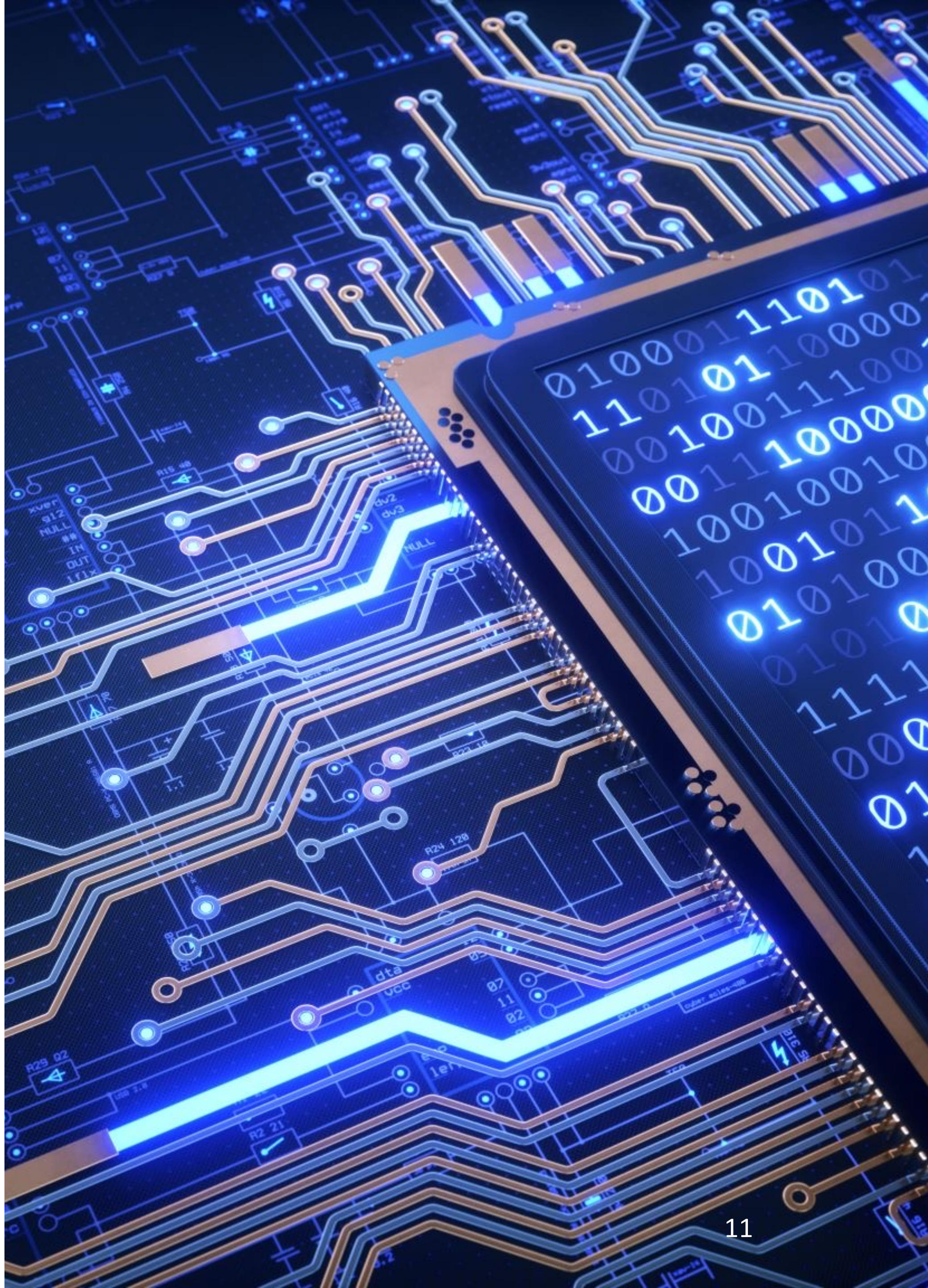
x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300



Worked Example II

- From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4.
- So, you can estimate the limit to be 4.



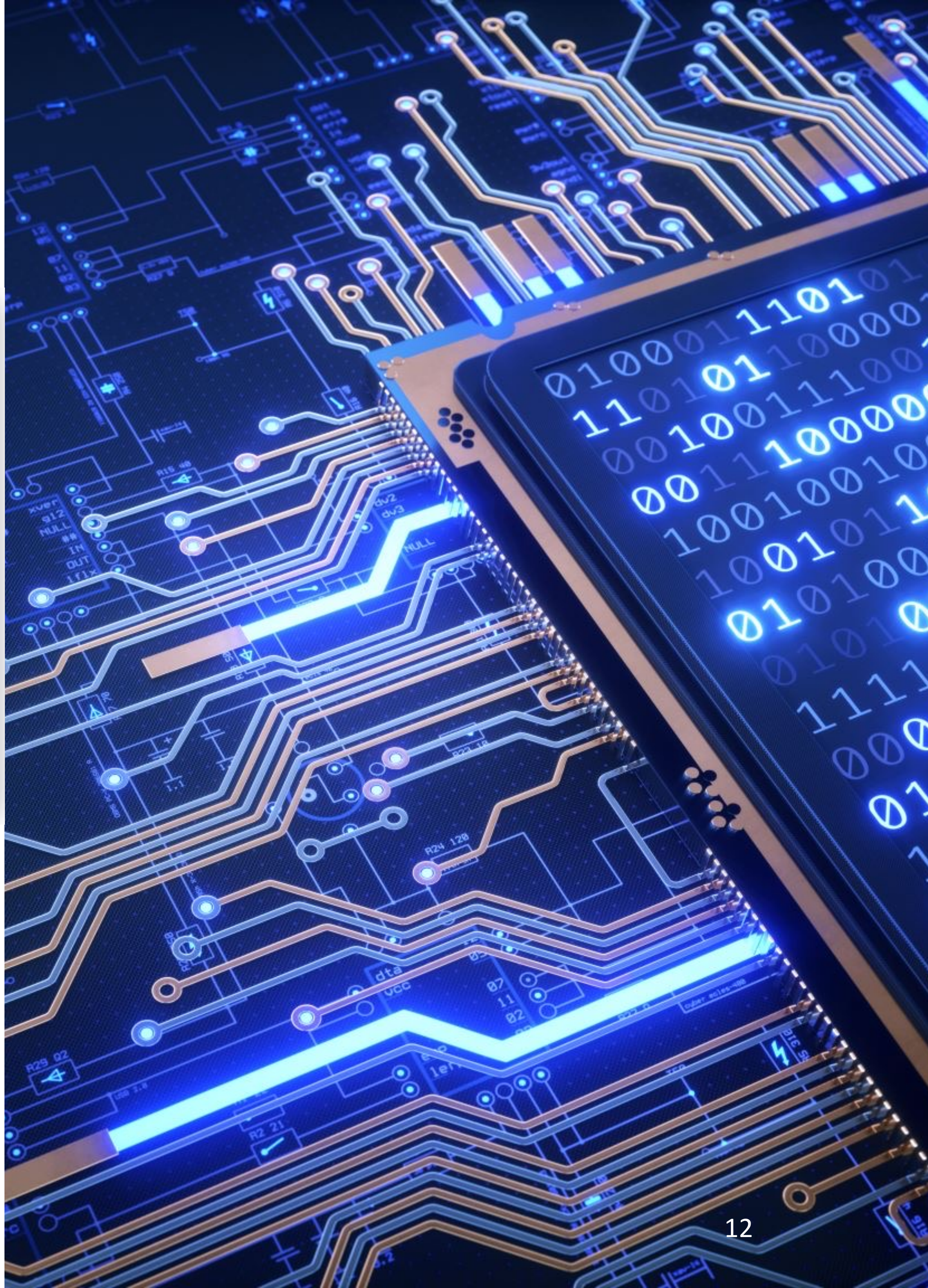


Limits That Fail to Exist

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the following conditions are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.



Worked Example III

- Show that the limit does not exist $\lim_{x \to 0} \frac{|x|}{x}$

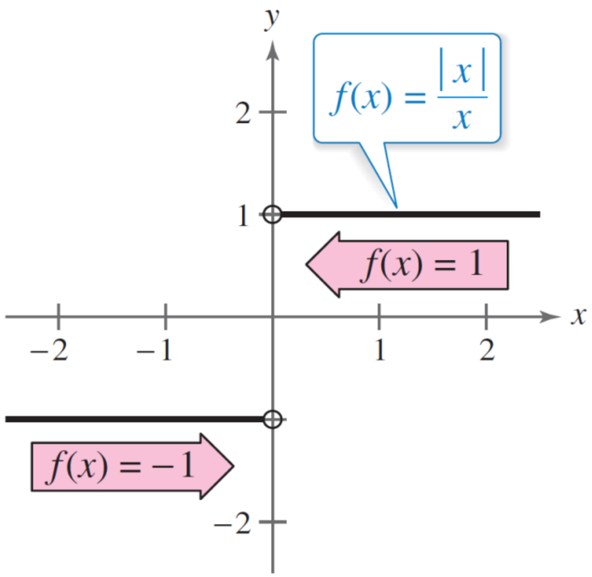
Solution:
Consider the graph of $f(x) = |x|/x$

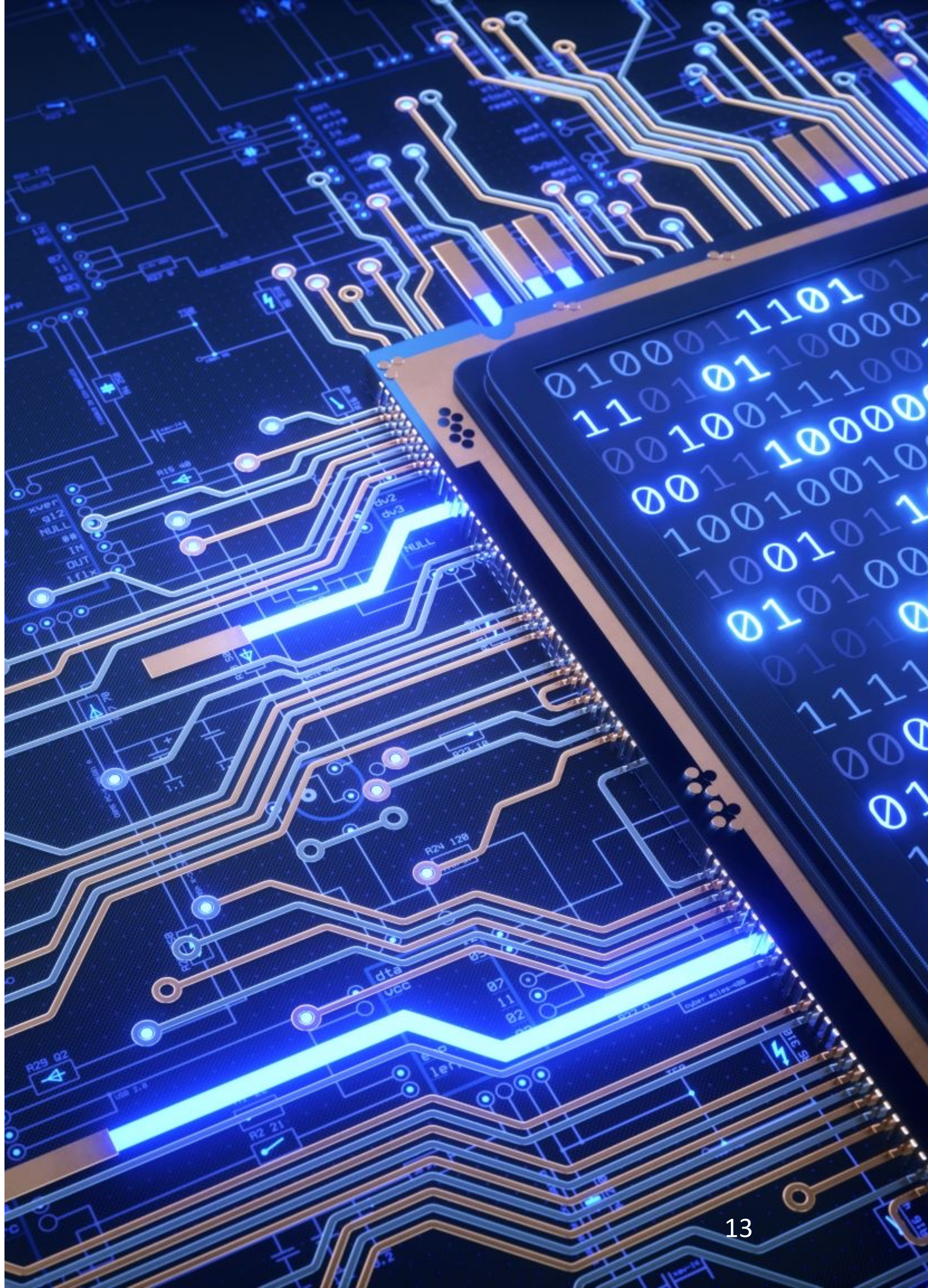
From the Figure, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative,

$$\frac{|x|}{x} = -1, \quad x < 0.$$

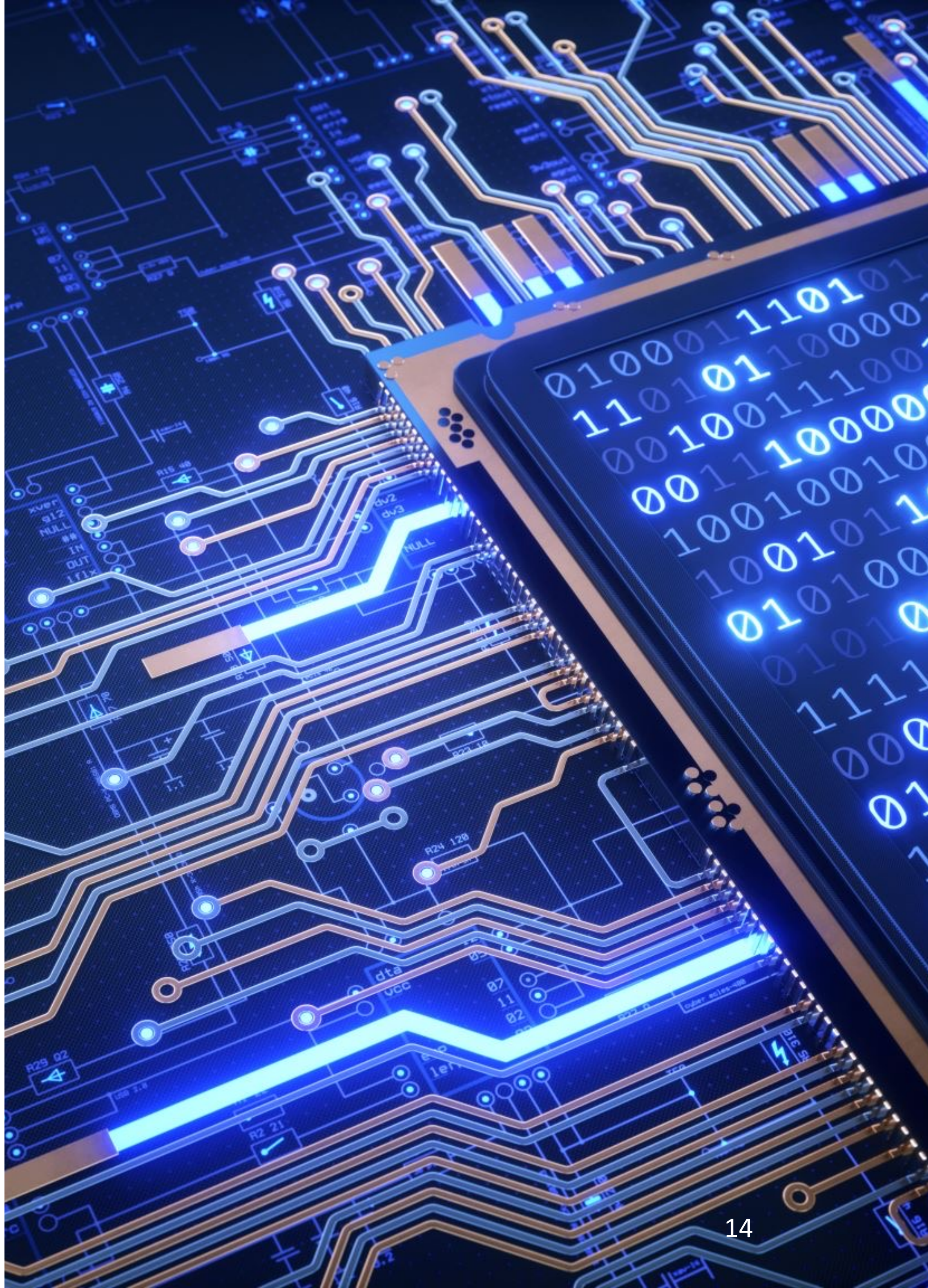




Worked Example III

- This means that no matter how close *x* gets to 0, there will be both positive and negative *x*-values that yield $f(x) = 1$ and $f(x) = -1$.

- This implies that the limit does not exist.



Properties of Limits and Direct Substitution

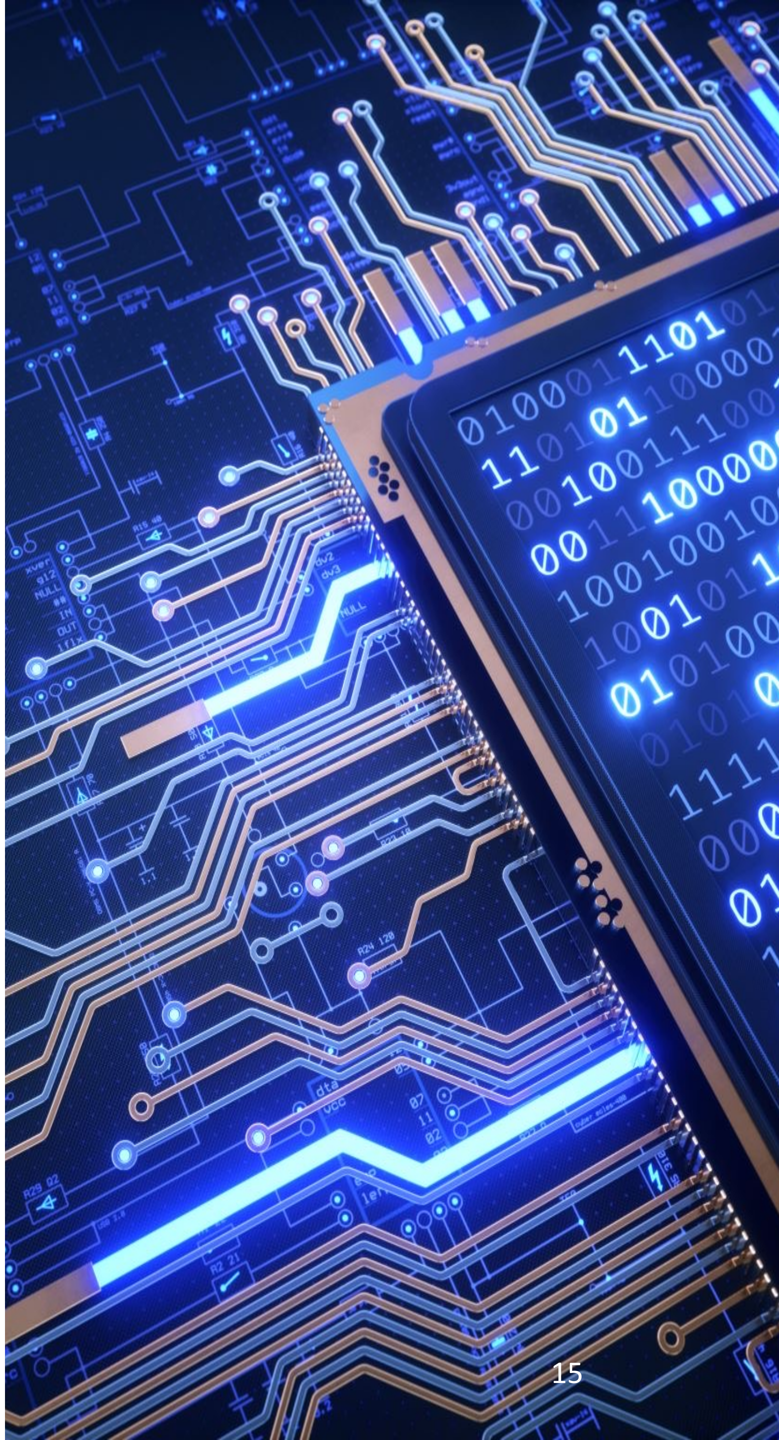
Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ — Limit of a constant function
2. $\lim_{x \to c} x = c$ — Limit of the identity function
3. $\lim_{x \to c} x^n = c^n$ — Limit of a power function
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$ — Limit of a radical function

- Trigonometric functions are also included
- E.g.:

$$\lim_{x \to \pi} \sin x = \sin \pi = 0 \qquad \lim_{x \to 0} \cos x = \cos 0 = 1.$$



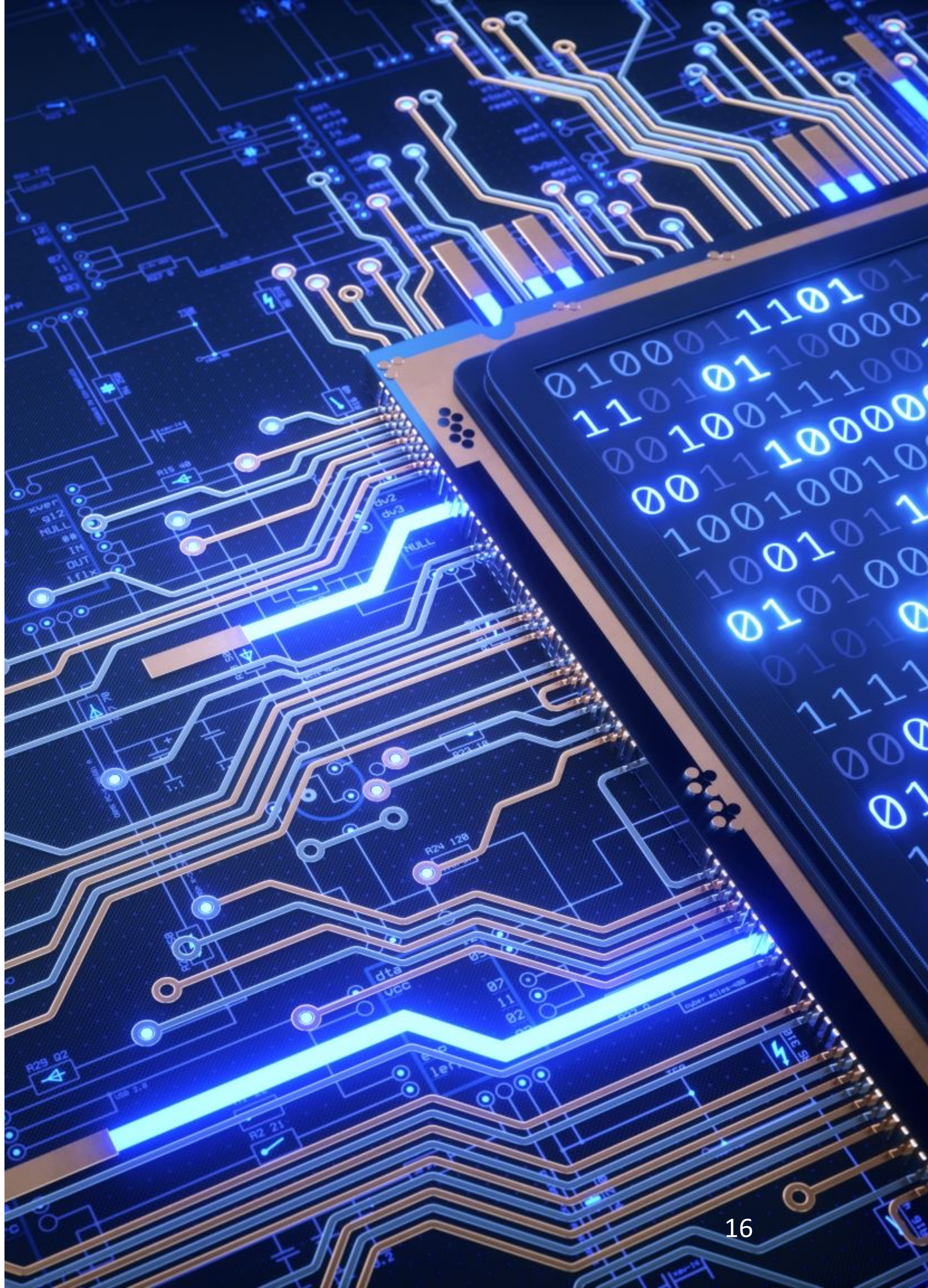
Properties of Limits and Direct Substitution

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$



Worked Example IV

Find each limit

a. $\lim_{x \to 4} x^2$

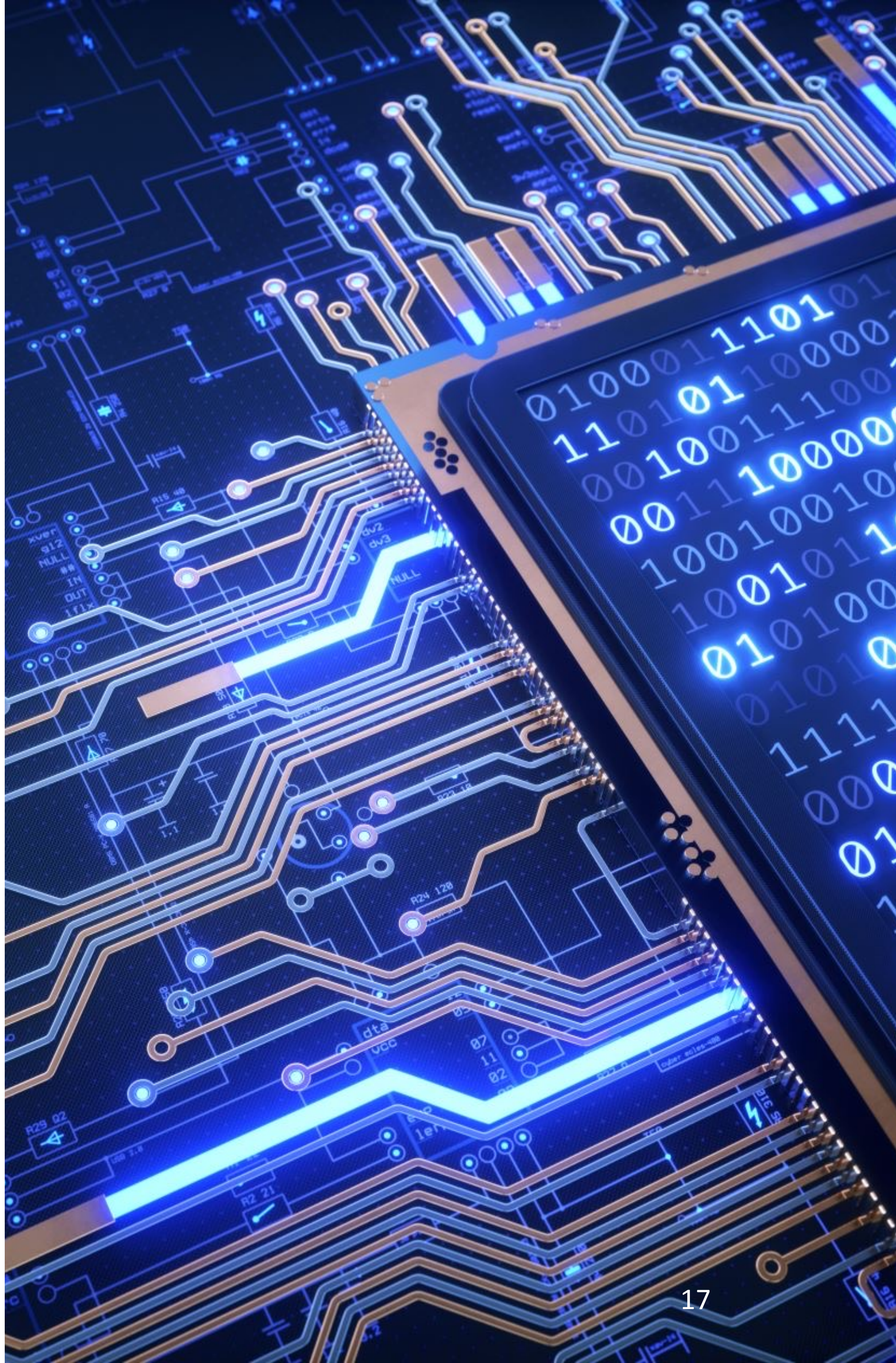
b. $\lim_{x \to 4} 5x$

c. $\lim_{x \to \pi} \frac{\tan x}{x}$

d. $\lim_{x \to 9} \sqrt{x}$

e. $\lim_{x \to \pi} (x \cos x)$

f. $\lim_{x \to 3} (x + 4)^2$



Solution

a. $\lim_{x\to 4} x^2 = (4)^2 = 16$

b. $\lim_{x\to 4} 5x = 5 \lim_{x\to 4} x$

$= 5(4) = 20$ Property 1

c. $\lim_{x\to\pi} \frac{\tan x}{x} = \frac{\lim_{x\to\pi} \tan x}{\lim_{x\to\pi} x} = \frac{0}{\pi} = 0$ Property 4

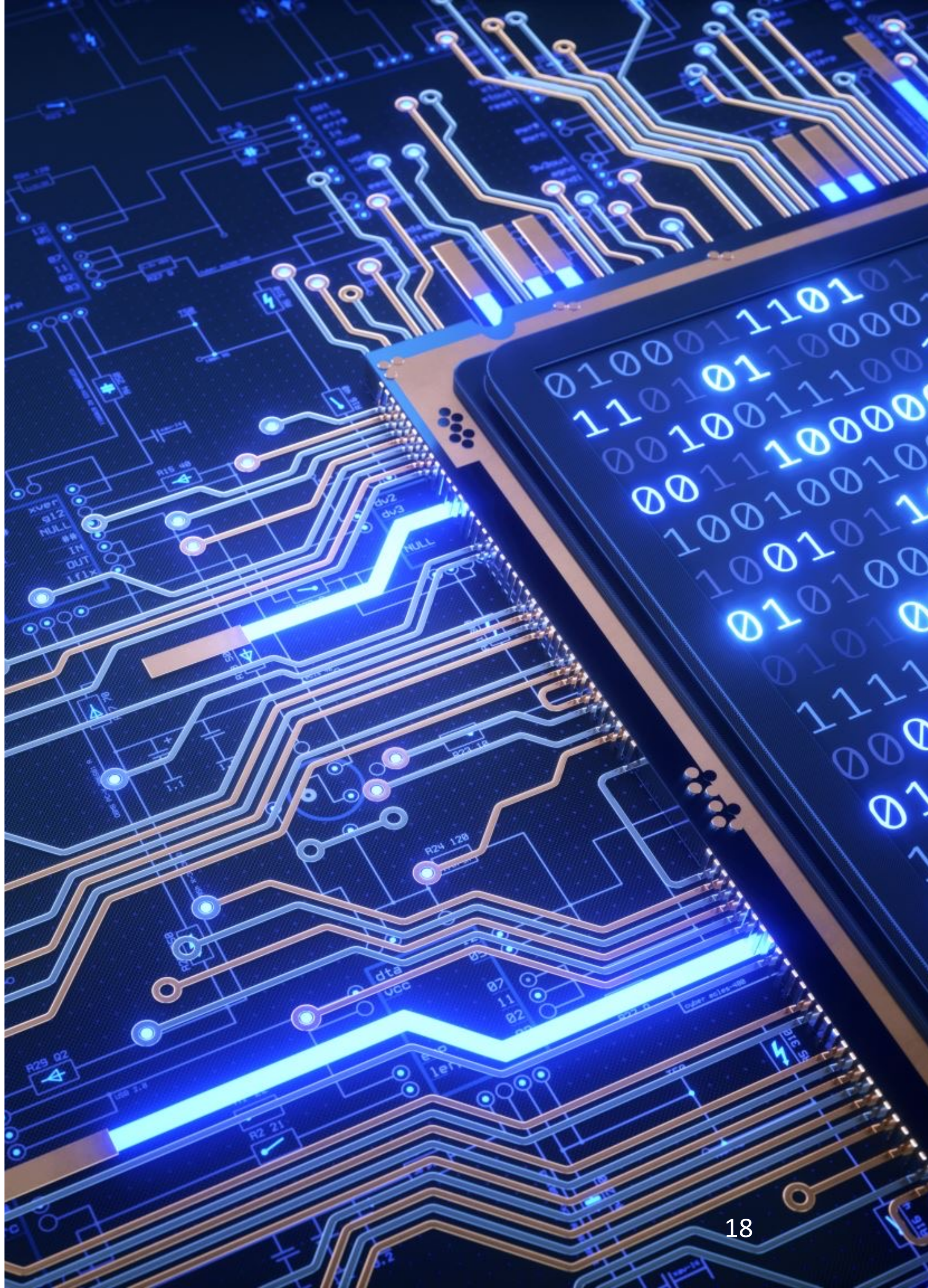
d. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3$

e. $\lim_{x\to\pi} (x \cos x) = \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi} \cos x\right)$ Property 3

$= \pi(\cos \pi) = -\pi$

f. $\lim_{x\to 3} (x + 4)^2 = \left[\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right)\right]^2$ Properties 2 and 5

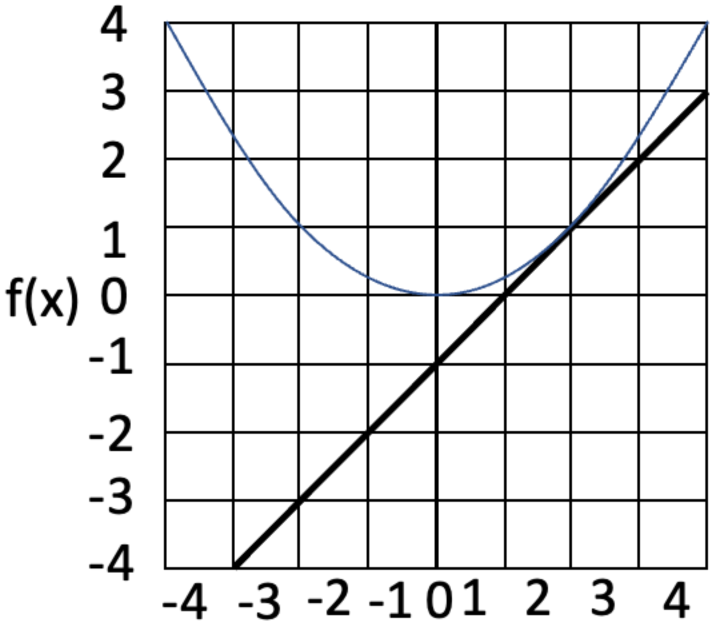
$= (3 + 4)^2 = 49$

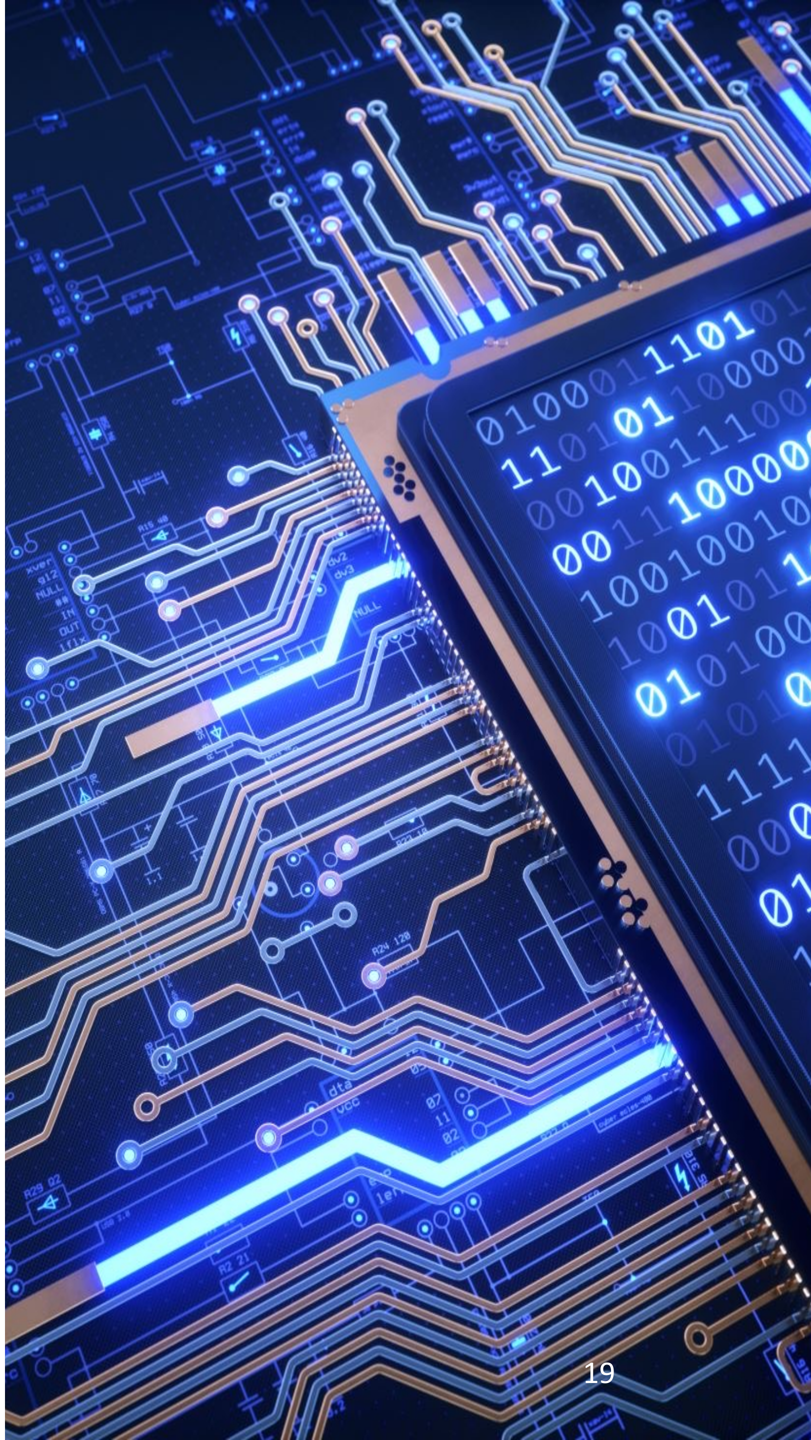


What is a derivative?

- The derivative $f'(x)$ of a function $f(x)$ says how fast $f(x)$ changes as x changes.
- Visually, $f'(x)$ is the slope of $f(x)$ at x.

Example: If $f(x) = \frac{1}{4}x^2$ then $f'(2) = 1$ because the slope of $f(x)$ at $x = 2$ is 1. We can see this by looking at the tangent line to $f(x)$ at $x = 2$.

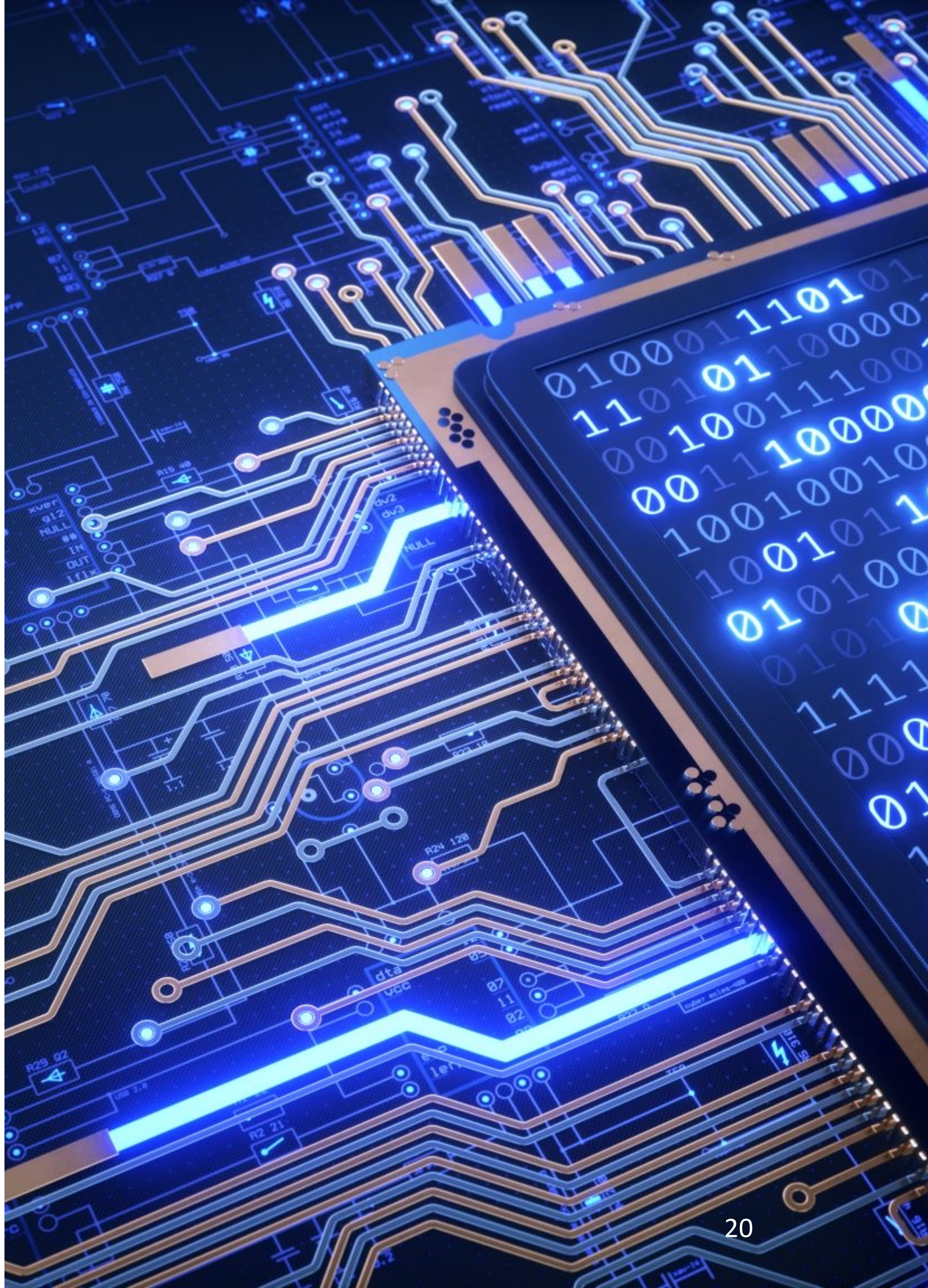




The Mean Value Theorem

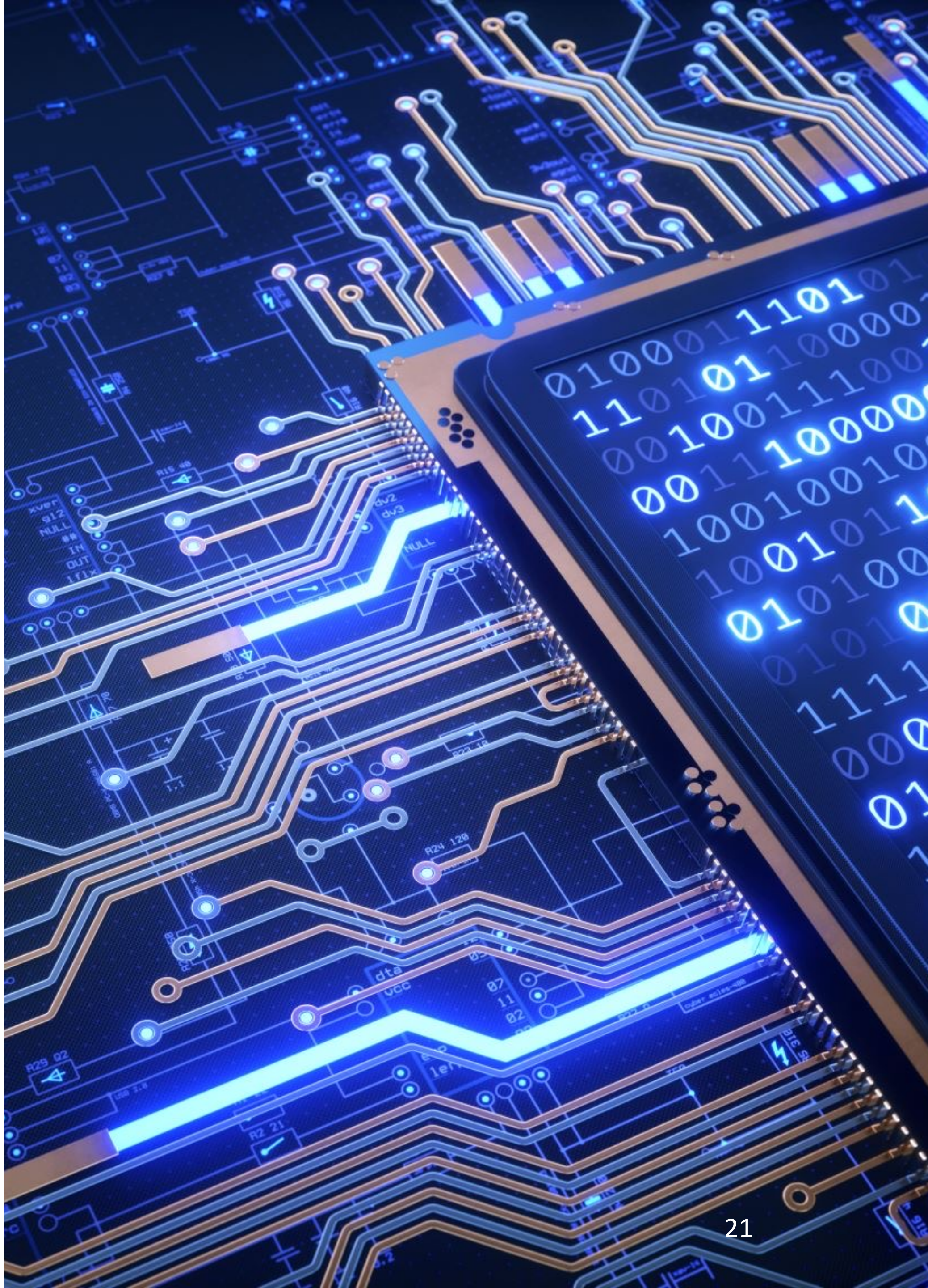
- Theorem: If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b) then there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
- Note that this is the slope of the line from $(a, f(a))$ to $(b, f(b))$

Example: If $f(x) = \frac{1}{4}x^2$ then if we take $a = 0$ and $b = 4$, the mean value theorem tells us that there is a $c \in (0, 4)$ such that $f'(c) = \frac{f(4)-f(0)}{4-0} = 1.$
This is indeed true as $f'(2) = 1$



The Mean Value Theorem

- If $f'(x) > 0$ on an interval then f is increasing on that interval (increasing means that $a < b$ implies that $f(a) < f(b)$).
- If $f'(x) < 0$ on an interval then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval then f is constant on that interval.



Leibniz Notation

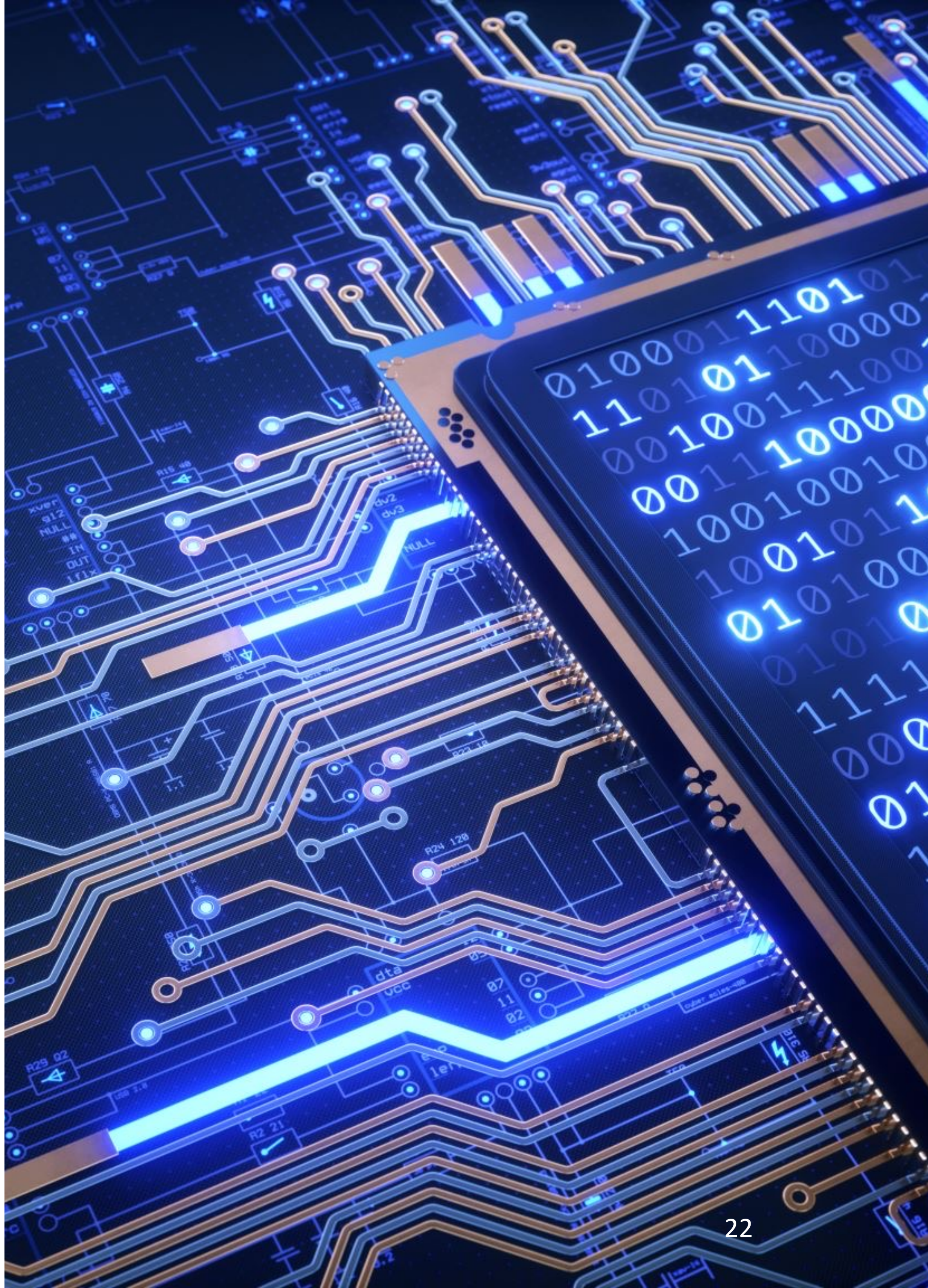
- We have written the derivative of a function f as f'.
- Another notation, devised by Leibniz, is $\frac{df}{dx}$
- NB: $\frac{df}{dx}$ is a single function. df and dx do not have values on their own.

Advantages of Leibniz notation:

- Emphasizes how the derivative is computed $\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$
- Makes it easier to express the product rule, quotient rule, and chain rule

Disadvantage of Leibniz notation:

- Need clumsy notation like $\left(\frac{df}{dx}\right)_{x=2}$ or $\left.\frac{df}{dx}\right|_{x=2}$ to write the derivative of a function at a particular point.



Differentiation Rules

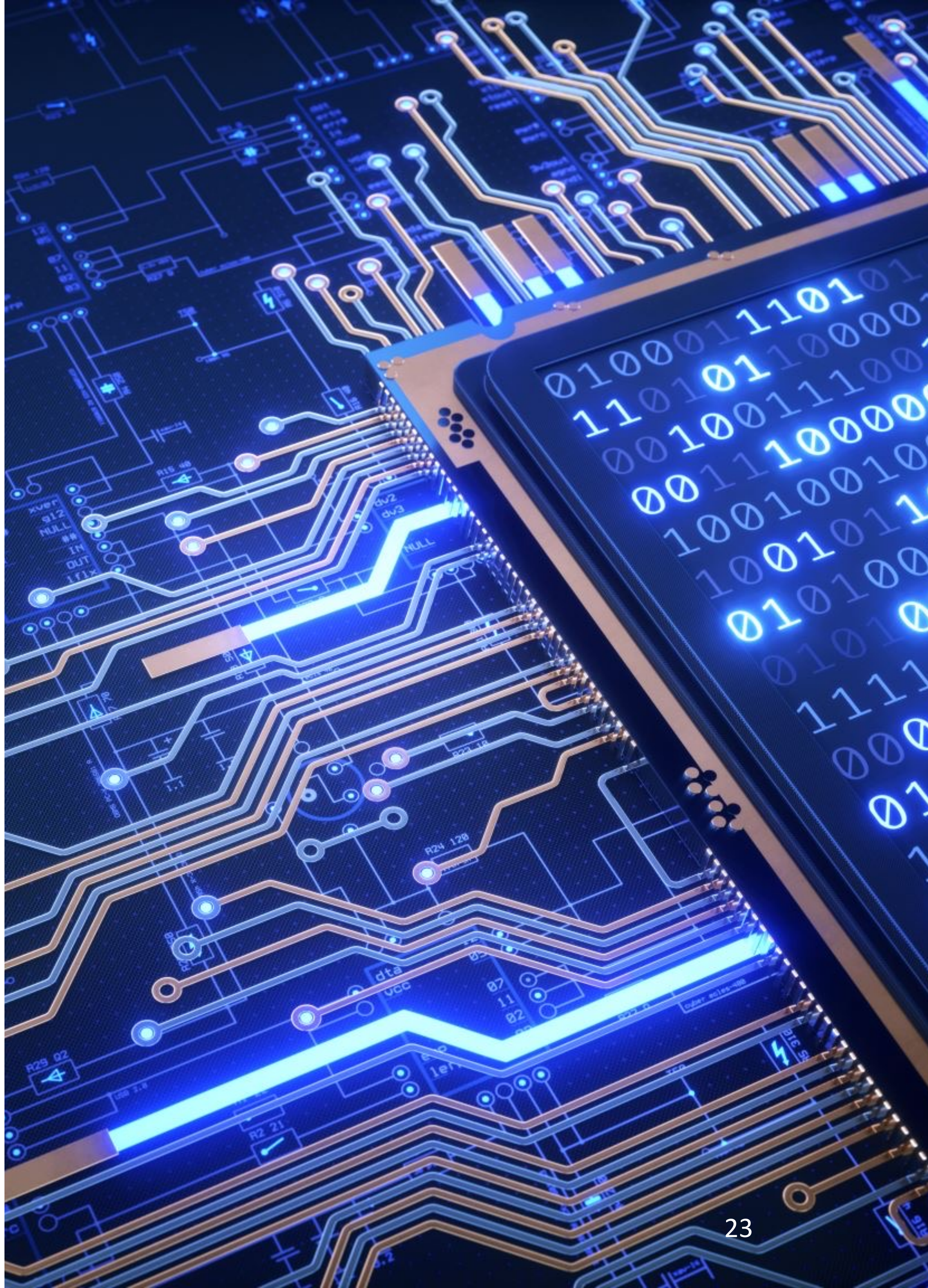
1. $$\frac{d}{dx}(c) = 0 \quad (c \text{ is a constant})$$

E.g. $$f(x) = 5$$
$$f'(x) = 0$$

2. $$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad (n \text{ is a real number})$$

Power Rule

E.g. $$f(x) = x^7$$
$$f'(x) = 7x^6$$



Differentiation Rules

3. $$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right) \quad (c \text{ is a constant})$$

Constant Multiple Rule

E.g. $f(x) = 3x^8$

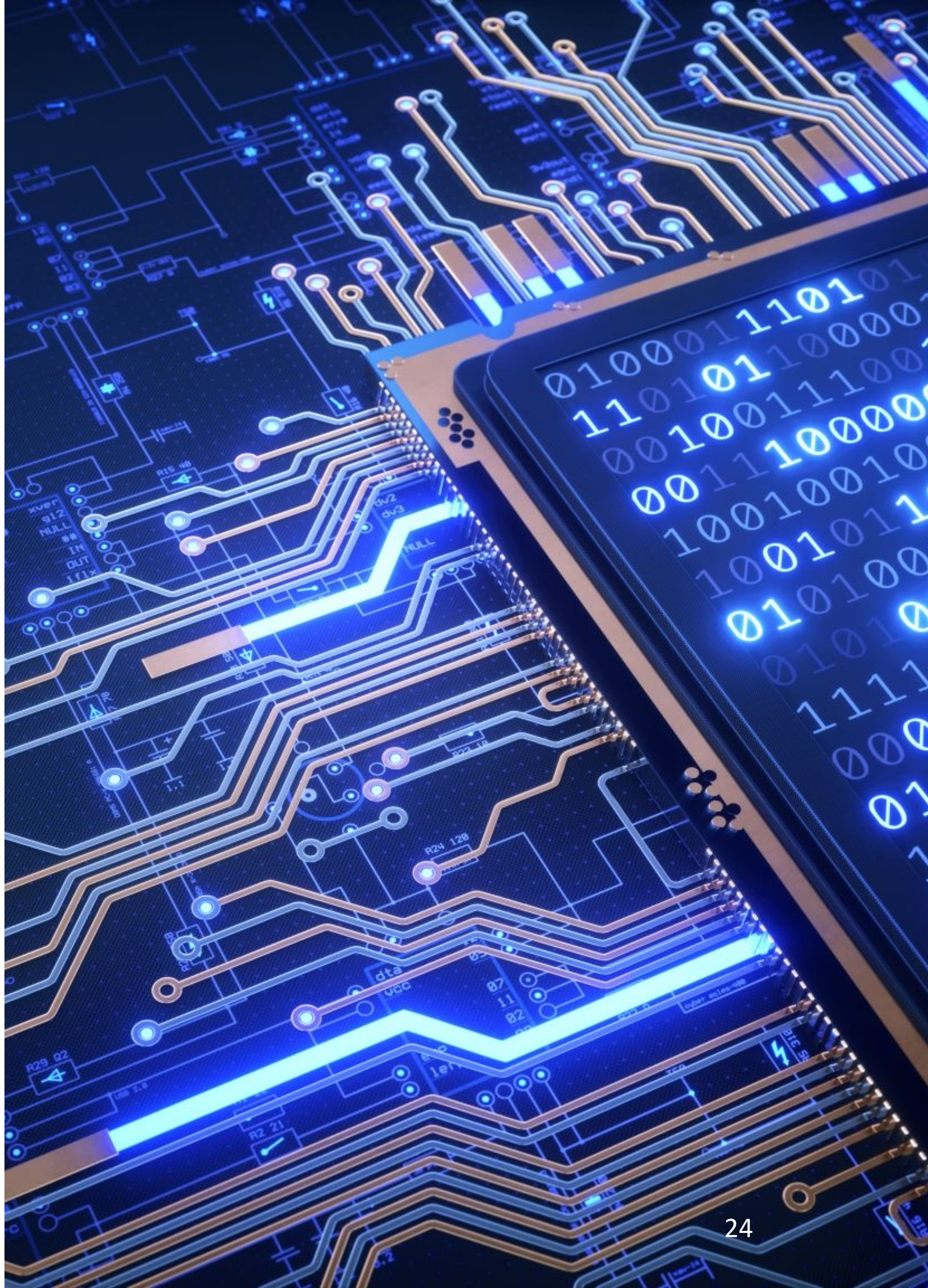
$$f'(x) = 3\left(8x^7\right) = 24x^7$$

4. $$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

E.g. $f(x) = 7 + x^{12}$

$$f'(x) = 0 + 12x^{11} = 12x^{11}$$

The Sum and Difference Rule



Differentiation Rules

5. Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=\frac{d}{dx}[f(x)]g(x)+\frac{d}{dx}[g(x)]f(x)$$

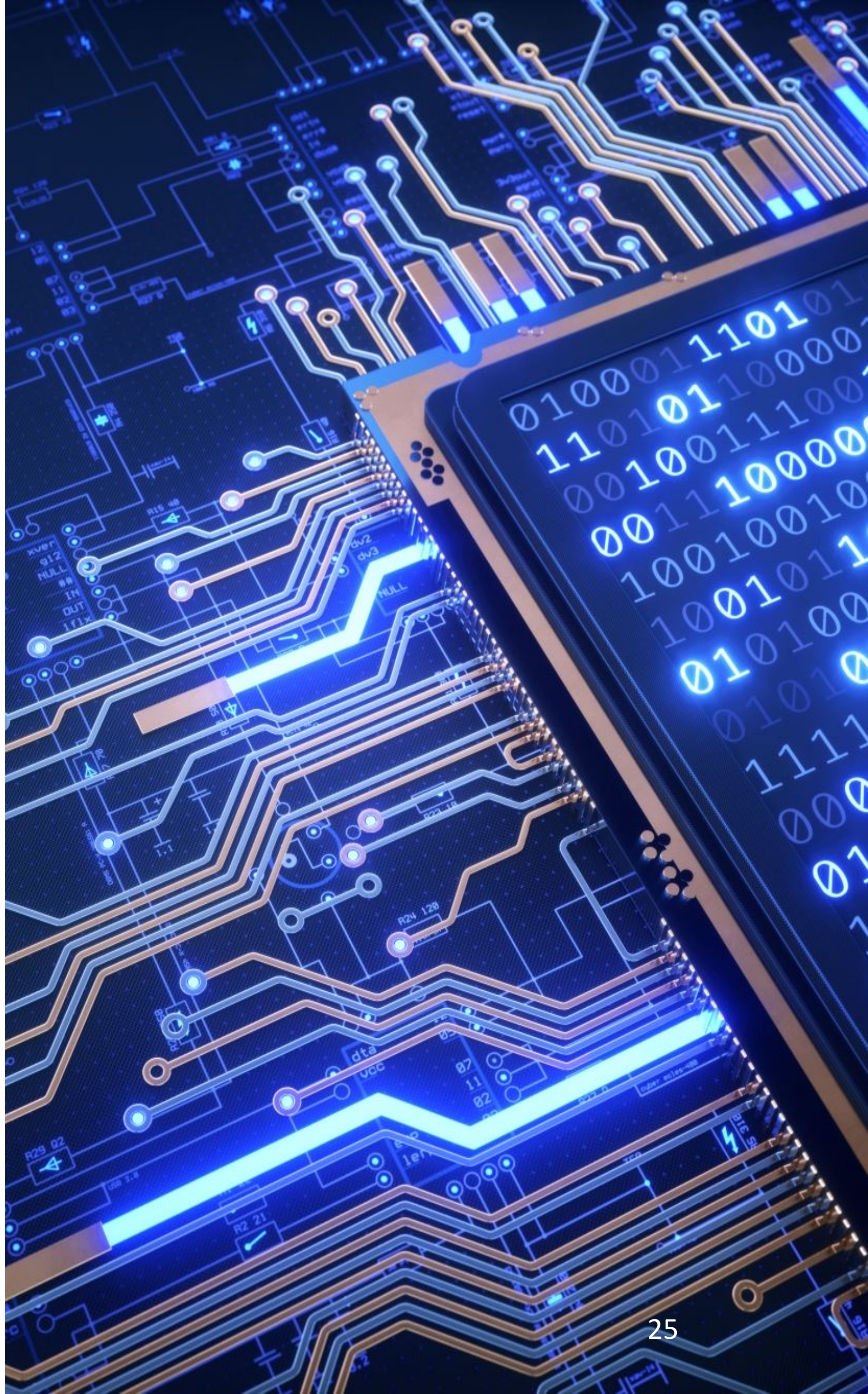
E.g. $f(x)=\left(x^3+2x+5\right)\left(3x^7-8x^2+1\right)$

$$f'(x)=\left(3x^2+2\right)\left(3x^7-8x^2+1\right)+\left(x^3+2x+5\right)\left(21x^6-16x\right)$$

Derivative of the first function

Derivative of the second function

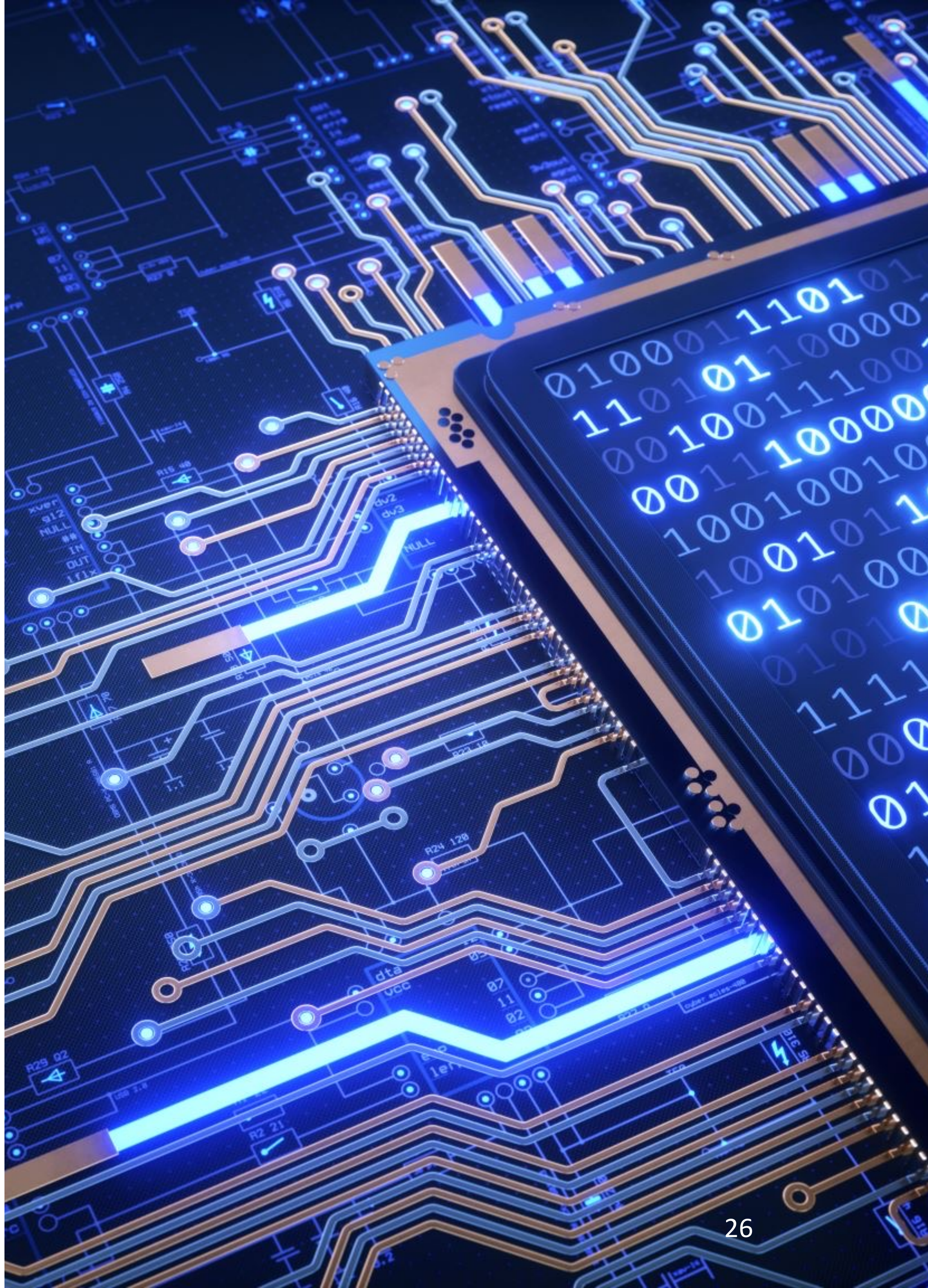
$$f'(x)=30x^9+48x^7+105x^6-40x^4-45x^2-80x+2$$



Differentiation Rules

6. Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$



Differentiation Rules

6. Quotient Rule (cont.)

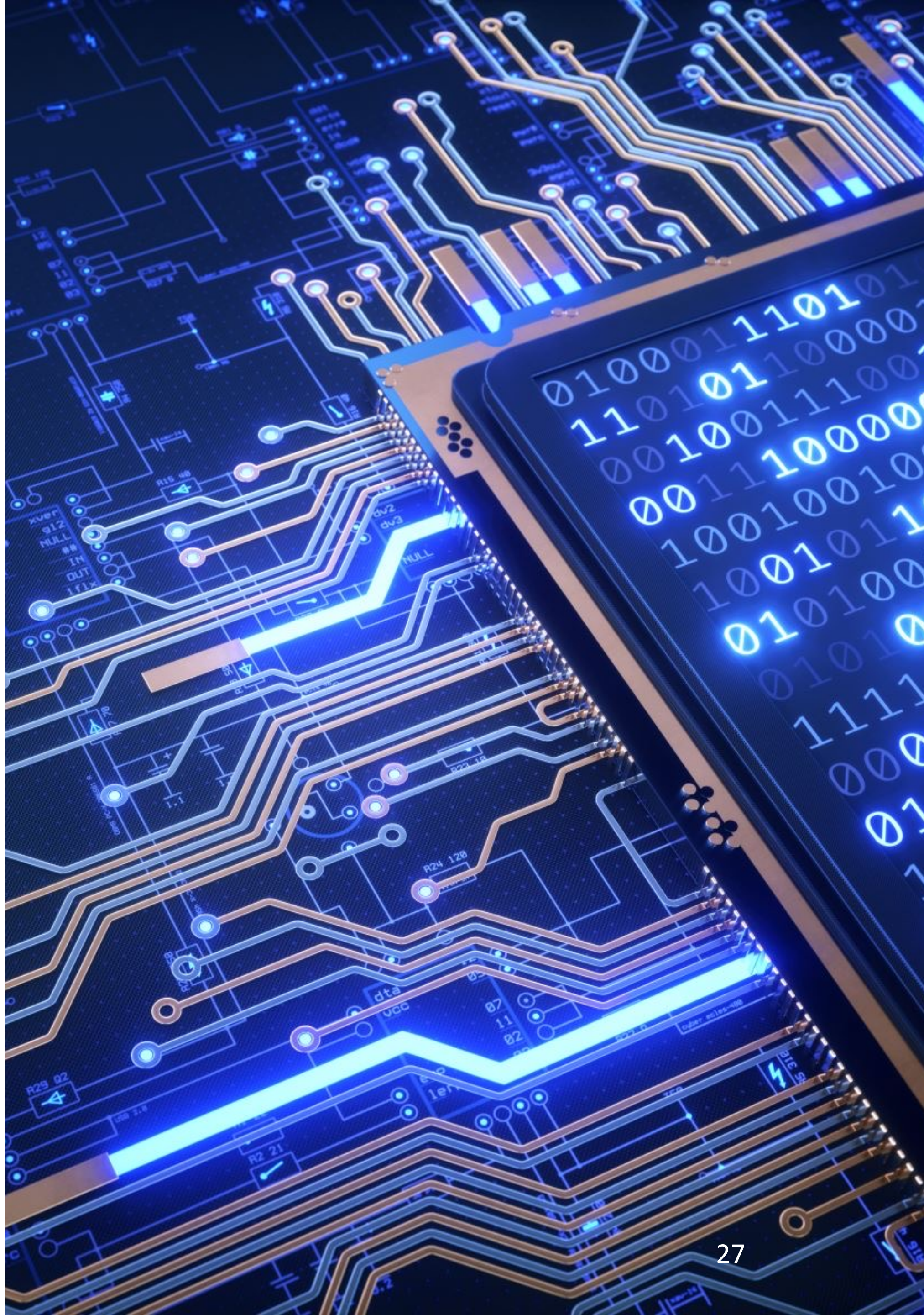
E.g. $$f(x)=\frac{3x+5}{x^2-2}$$

Derivative of the numerator

Derivative of the denominator

$$f'(x)=\frac{3\left(x^2-2\right)-2x\left(3x+5\right)}{\left(x^2-2\right)^2}$$

$$=\frac{-3x^2-10x-6}{\left(x^2-2\right)^2}$$



Differentiation Rules

7. The Chain Rule

If $h(x) = g\left(f(x)\right)$ then

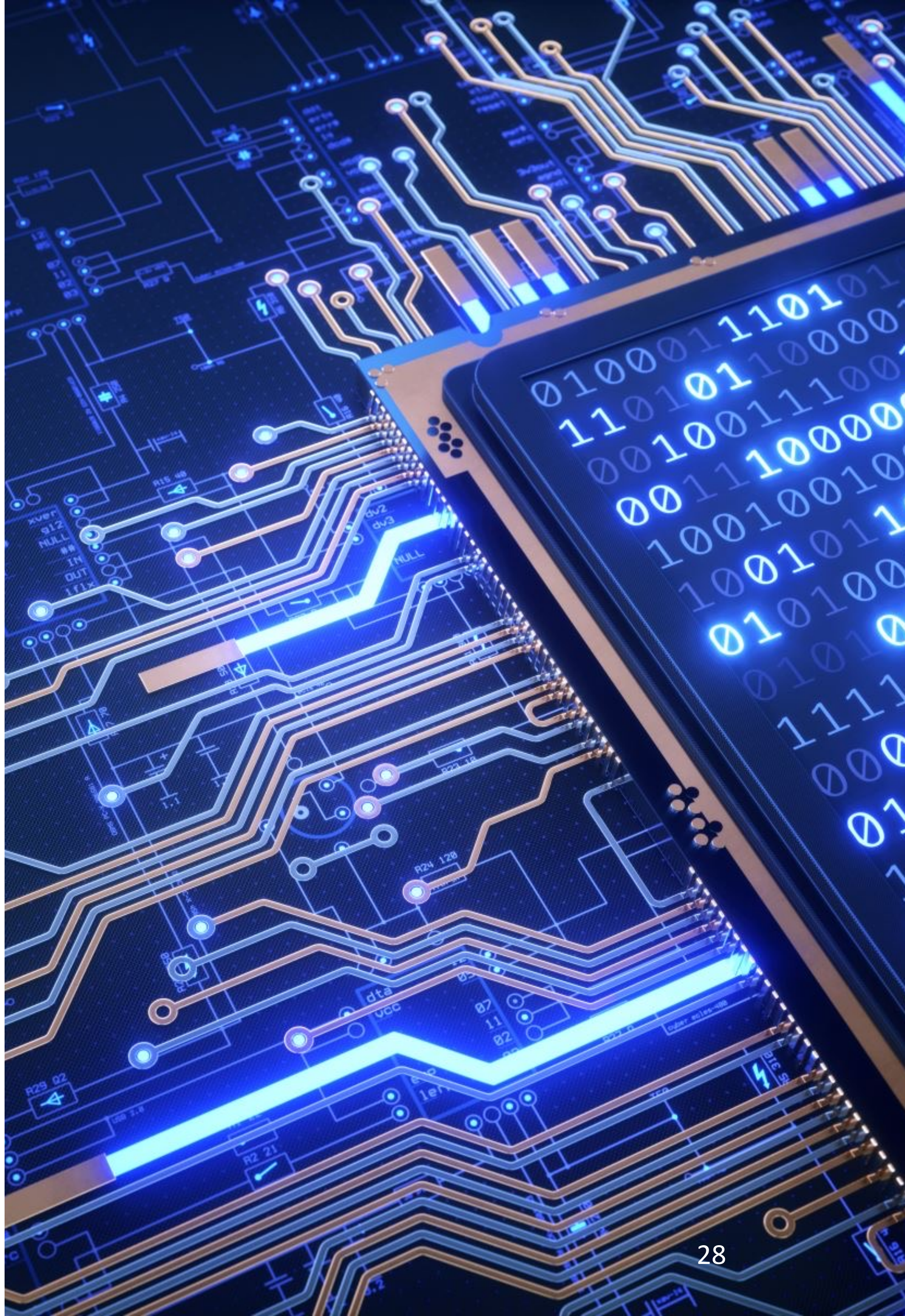
$$h'(x) = g'\left(f(x)\right) \cdot f'(x)$$

Note: $h(x)$ is a composite function.

Another Version:

If $y = h(x) = g\left(u\right)$, where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

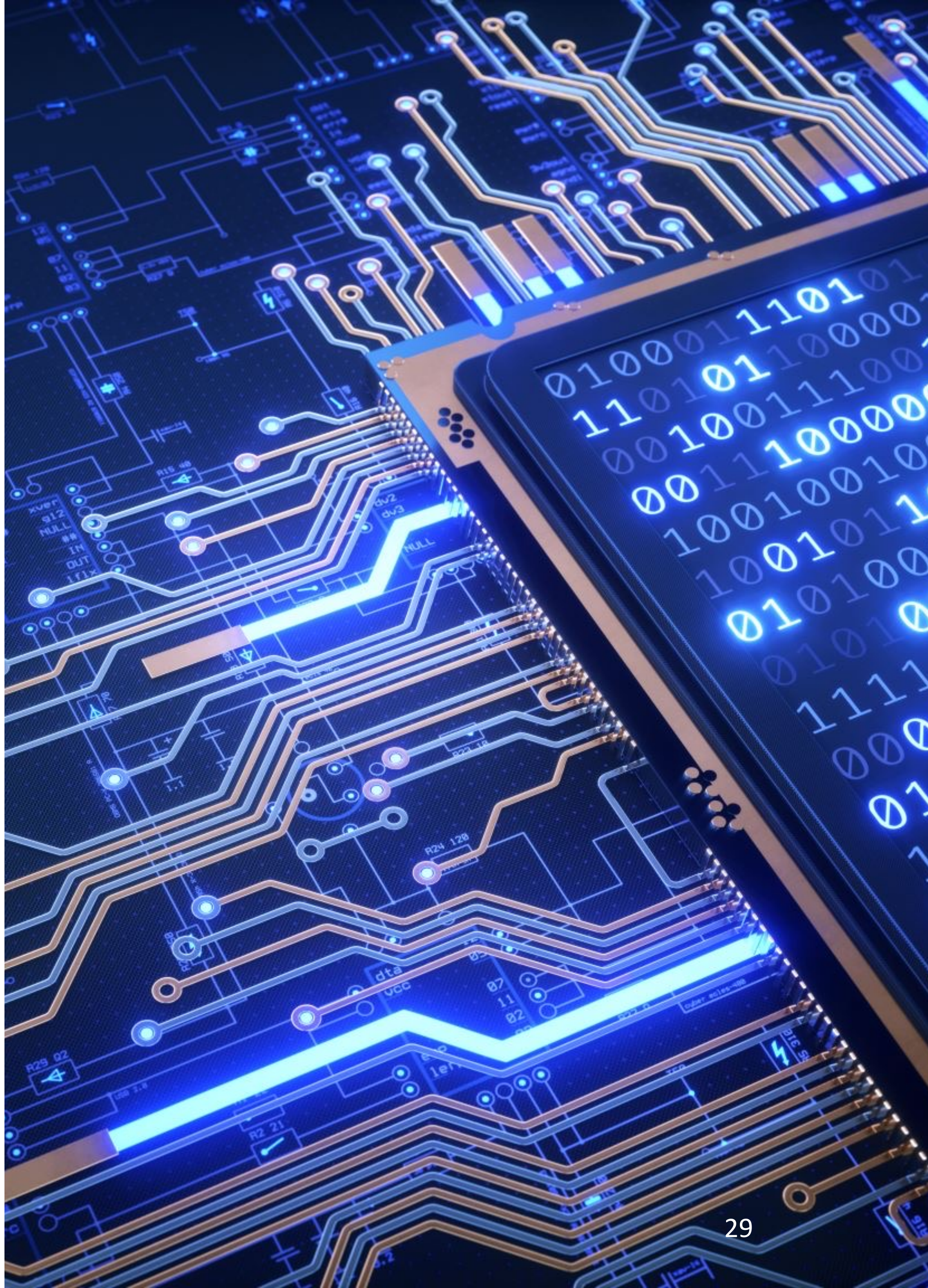


Differentiation Rules

E.g.: Differentiate $f(x) = \sqrt{1 + x^2}$

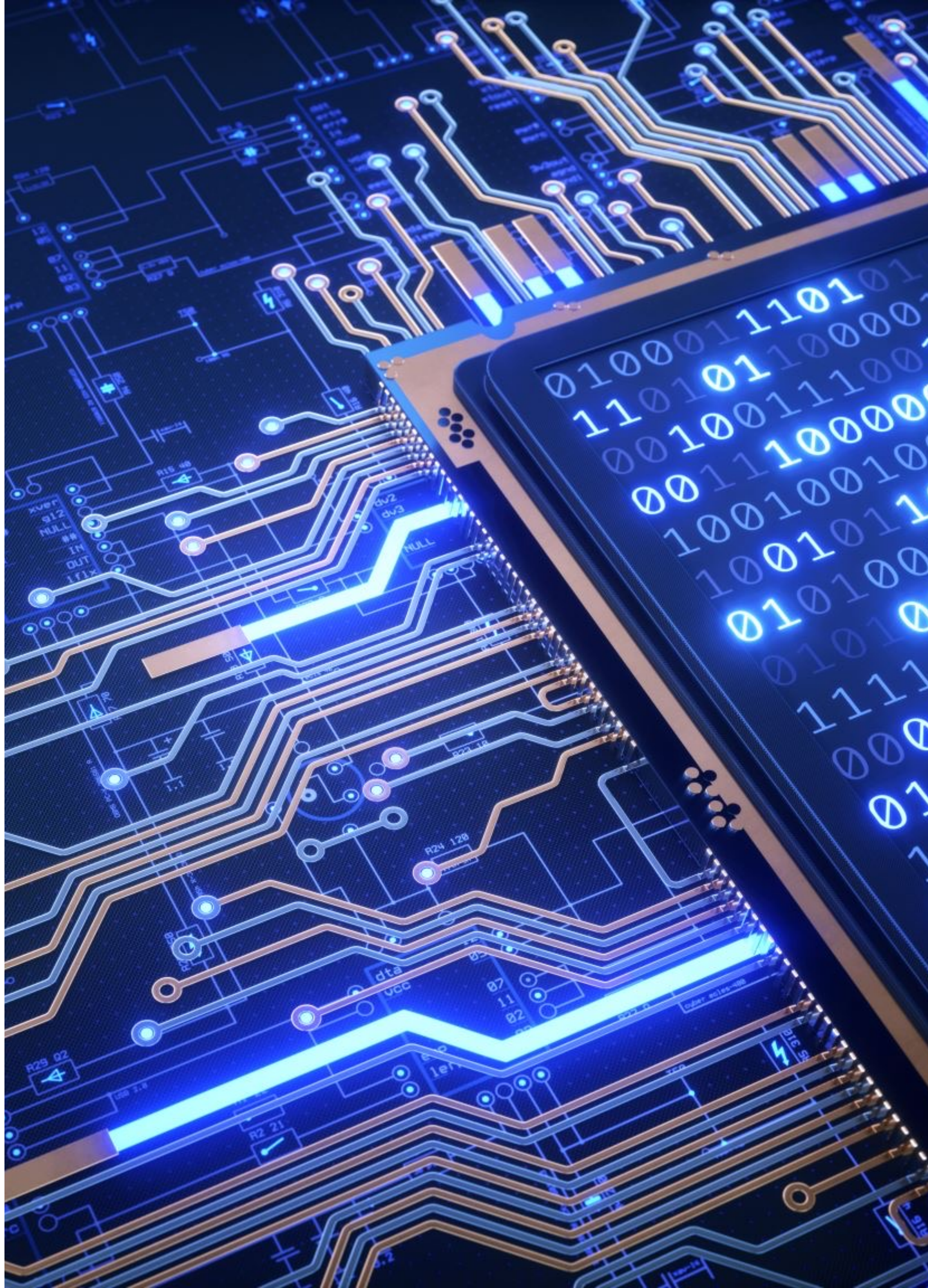
Solution: If $f(x) = \sqrt{1 + x^2}$ then taking
$u = 1 + x^2$ and $f(u) = \sqrt{u}$,

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$



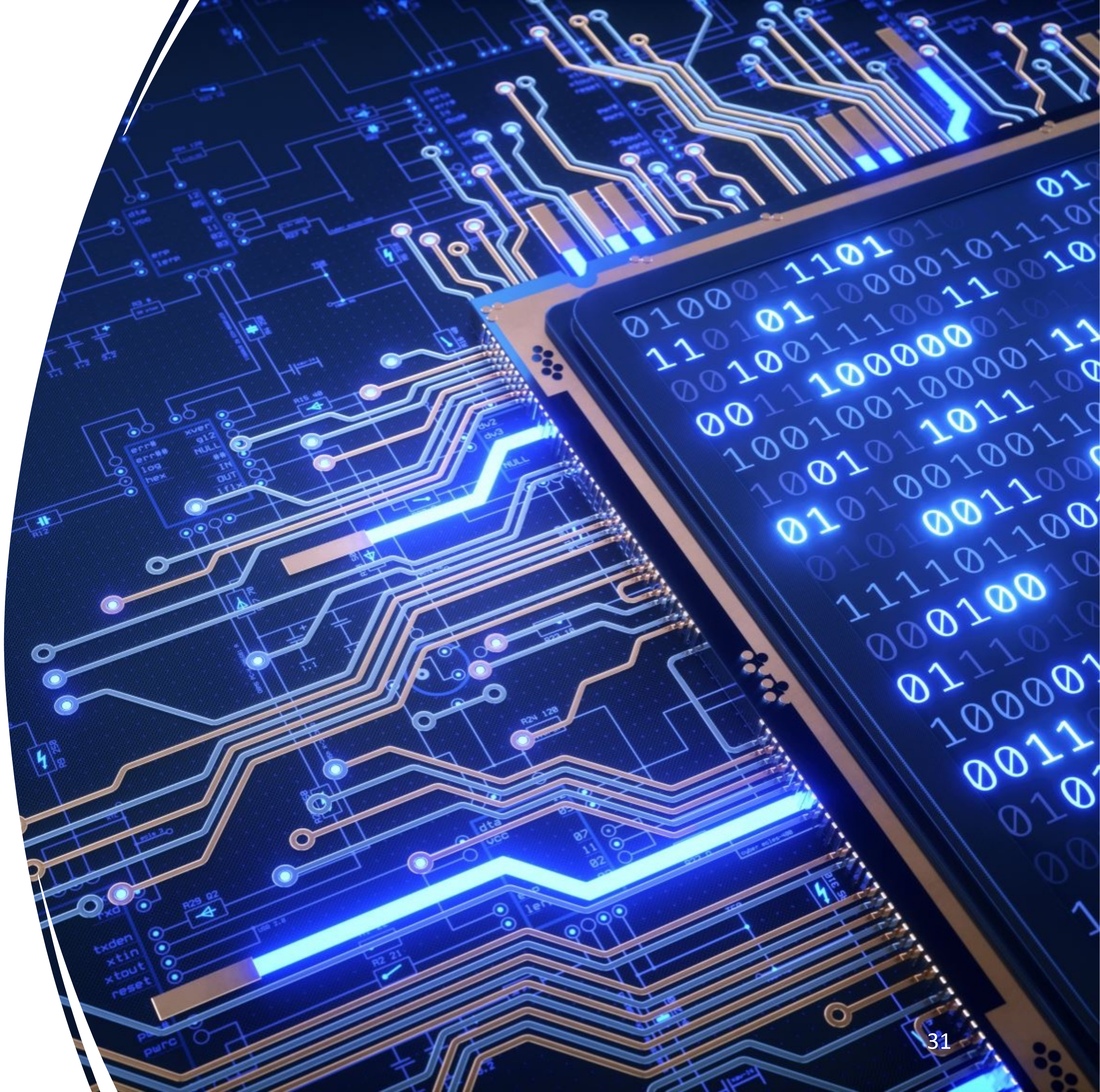
Applications of Differentiation

- Rate of Change
- Optimization
- Curve Sketching
- Related Rates Problems
- Physics
- Engineering
- Economics
- Biology
- Finance
- Statistics



Summary

- Calculus's differentiation concept deals with determining the rate at which one variable changes in relation to another.
  - Limit
  - Seven Differentiation Rules
    - Differentiating a constant
    - Power rule
    - Constant multiple rule
    - Sum and difference rule
    - Product rule
    - Quotient rule
    - Chain rule
  - Application of differentiation



Reference

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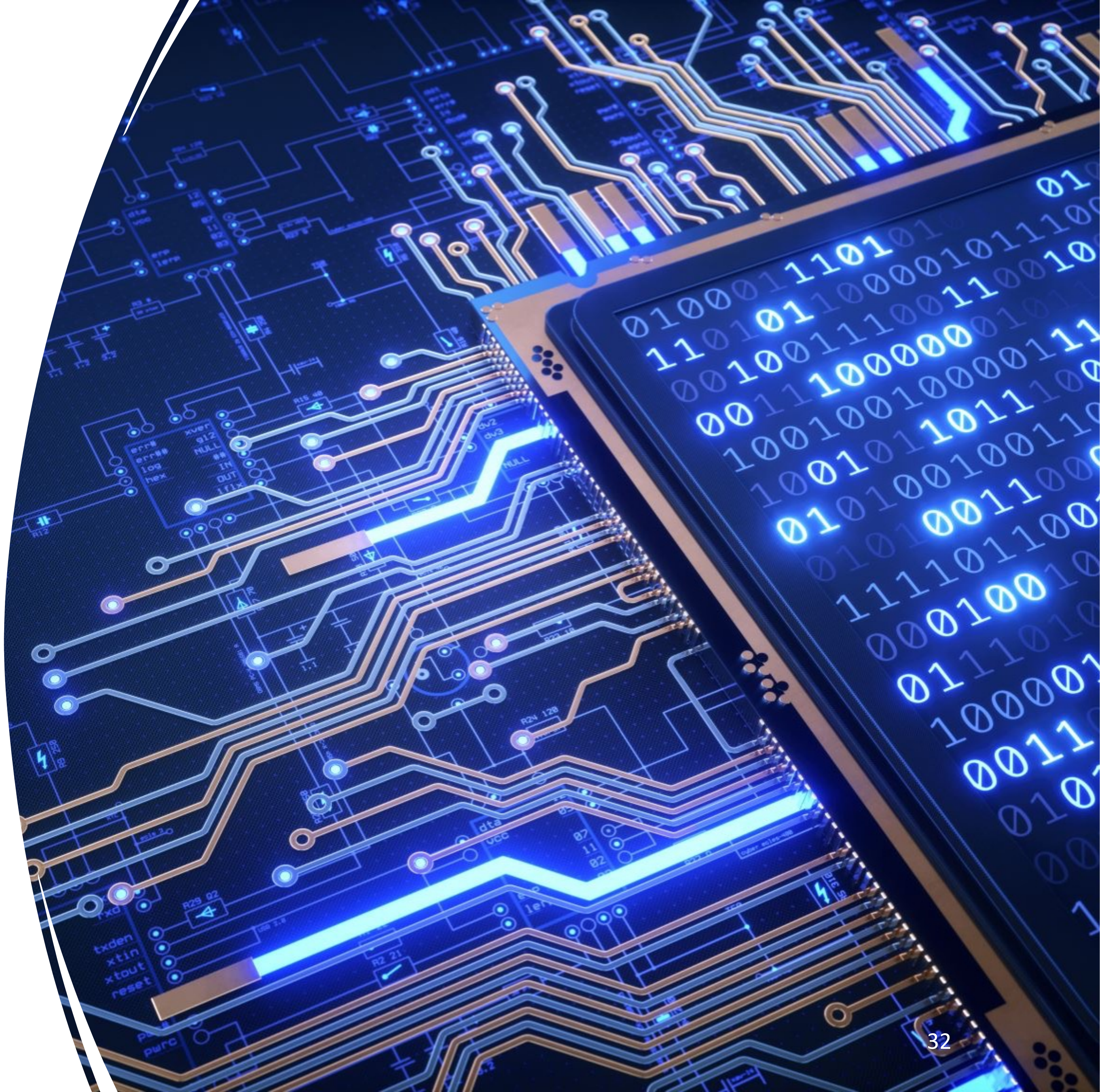

See you next time!