Course: Mathematics for IT Professionals

Lecture 12
Integration

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Outline

The topics to be treated in this lecture are:

- Indefinite and definite Integrals
- Anti-derivative
- Integration formula
- Integration by parts
- Solved Examples

Lecture Learning Outcomes

At the end of the session, you will be able to

- understand the concept of Anti-derivatives
- know the difference between definite and indefinite integrals
- understand the concept of integration by parts
- use basic integration rules to find antiderivatives of functions

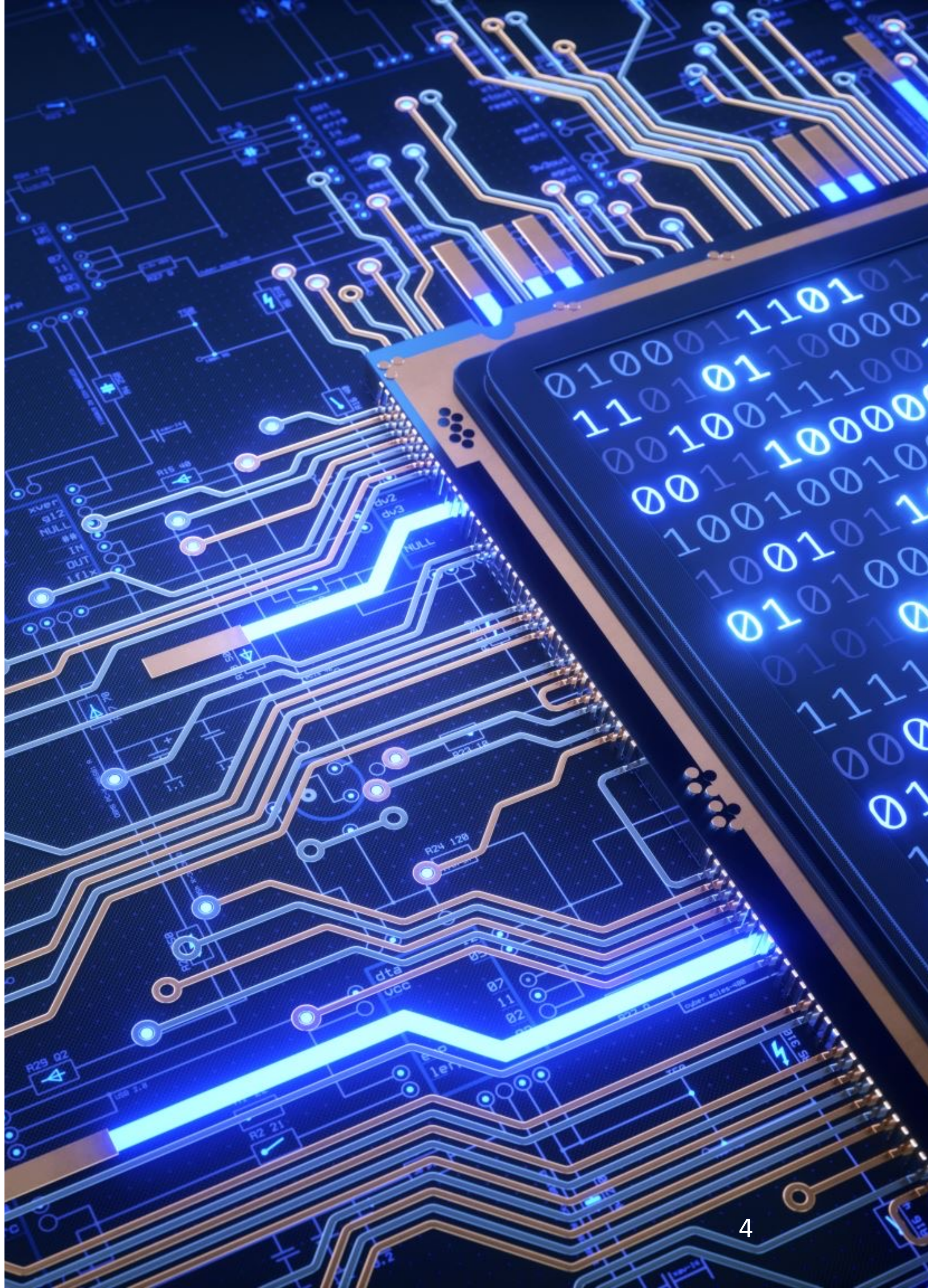
Introduction

- An **anti-derivative** is a function that reverses what the derivative does.
- To find a function F whose derivative is $f(x) = 3x^2$, you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2$$

> **Definition of Antiderivative**
>
> A function F is an **antiderivative** of f on an interval I when $F'(x) = f(x)$ for all x in I.

Brokate, M., Manchanda, P., & Siddiqi, A. H. (2019). *Calculus for scientists and engineers*. Springer Singapore.



Introduction

- We can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative.

  $G(x) = x^2 + C$ Family of all antiderivatives of $f(x) = 2x$

  where C is a constant. The constant C is called the **constant of integration.**

- The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

  $G'(x) = 2x$ Differential equation

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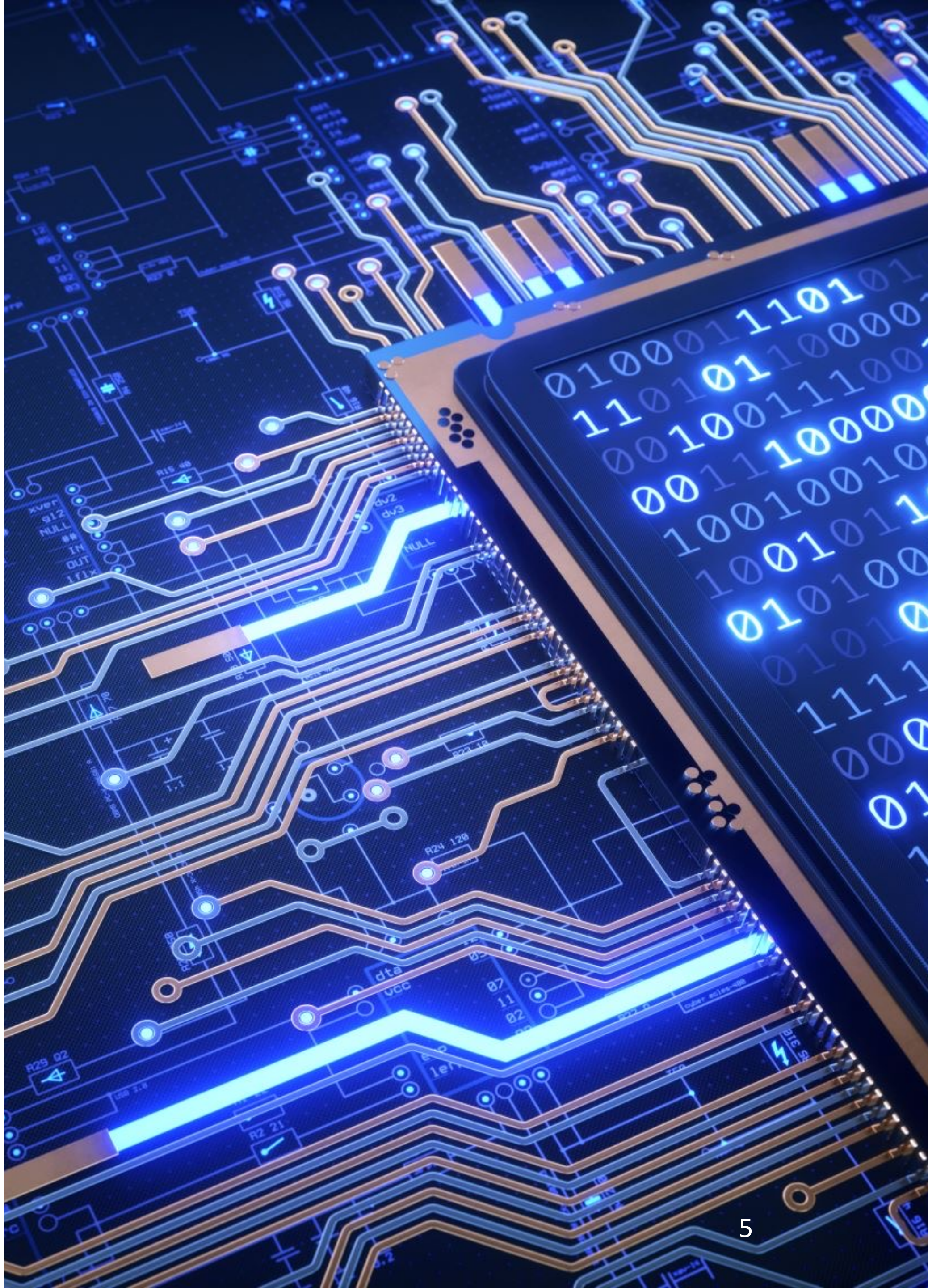


Worked Example I

Find the general solution of the differential equation $y' = 2$

Solution:

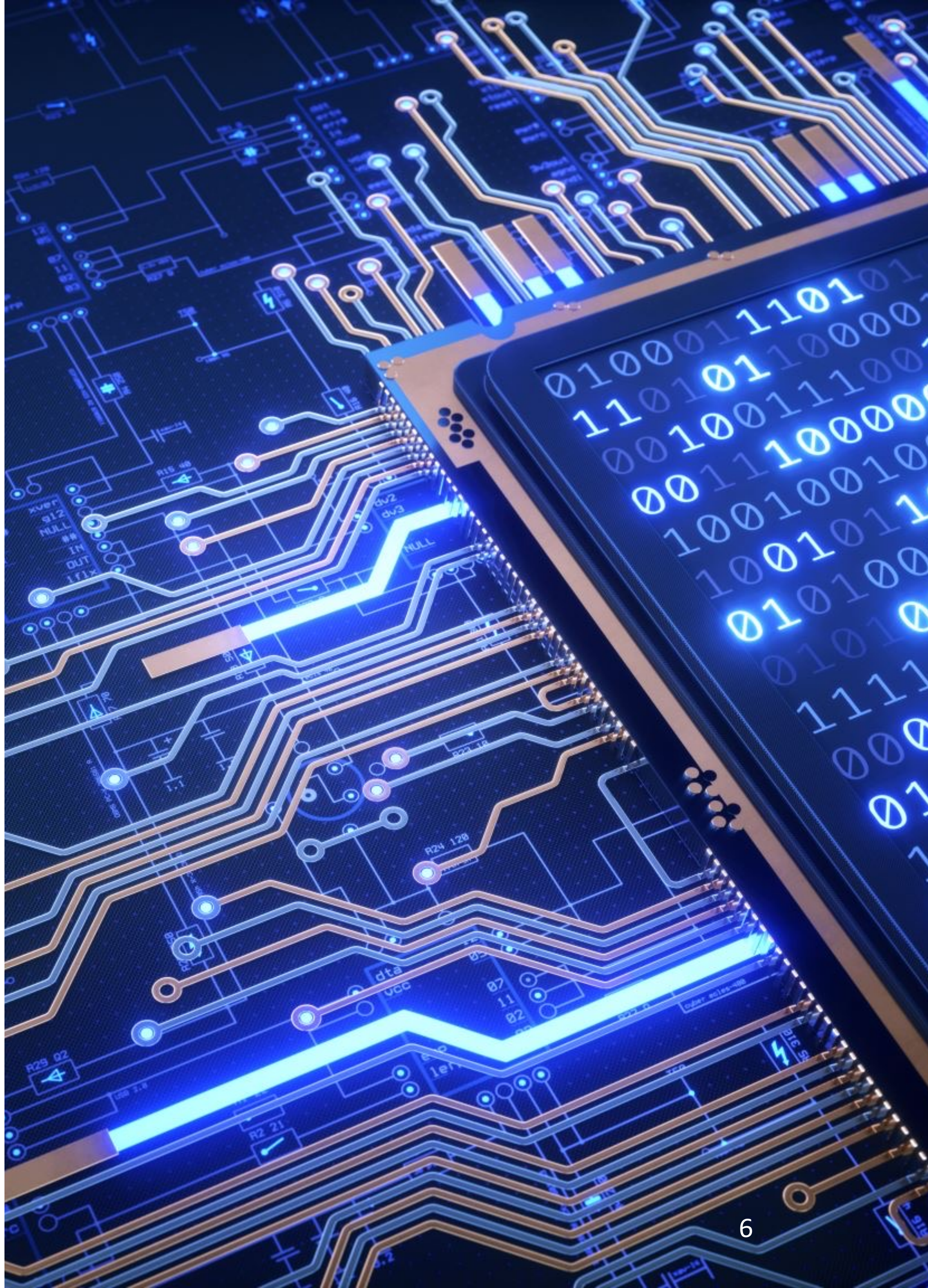
To begin, we need to find a function whose derivative is 2

One such function is

$y = 2x$ 2*x* is *an* antiderivative of 2

Now, we can conclude that the general solution of the differential equation is

$y = 2x + C$ General solution



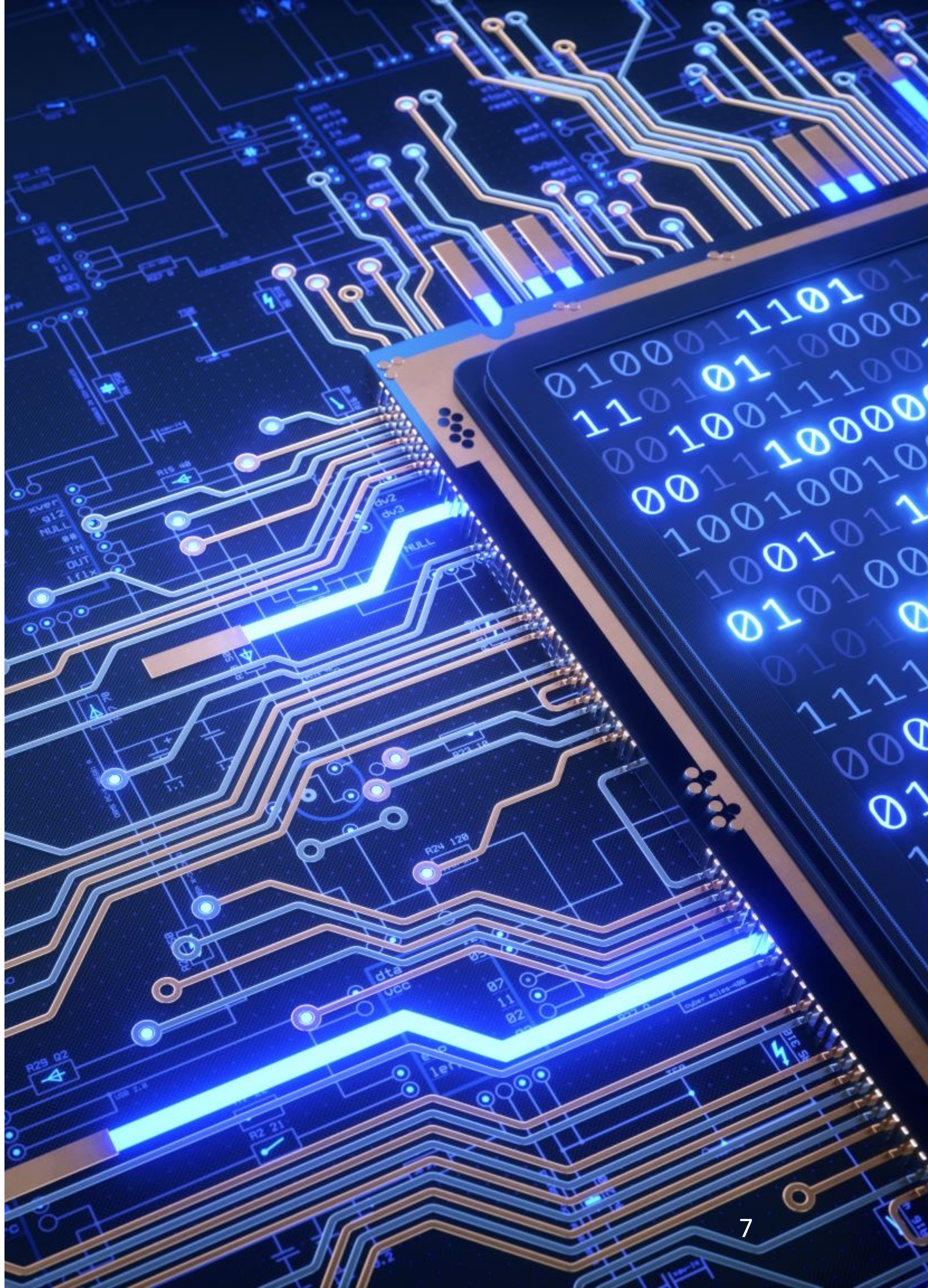
Notation of Anti-derivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\, dx$$



Notation of Anti-derivatives

Definition

The operation of finding the anti-derivative of a function f(x) is called **anti-differentiation** (or **Integration**) and is denoted by the integral sign

$$\int f(x)\,dx = F(x) + C$$

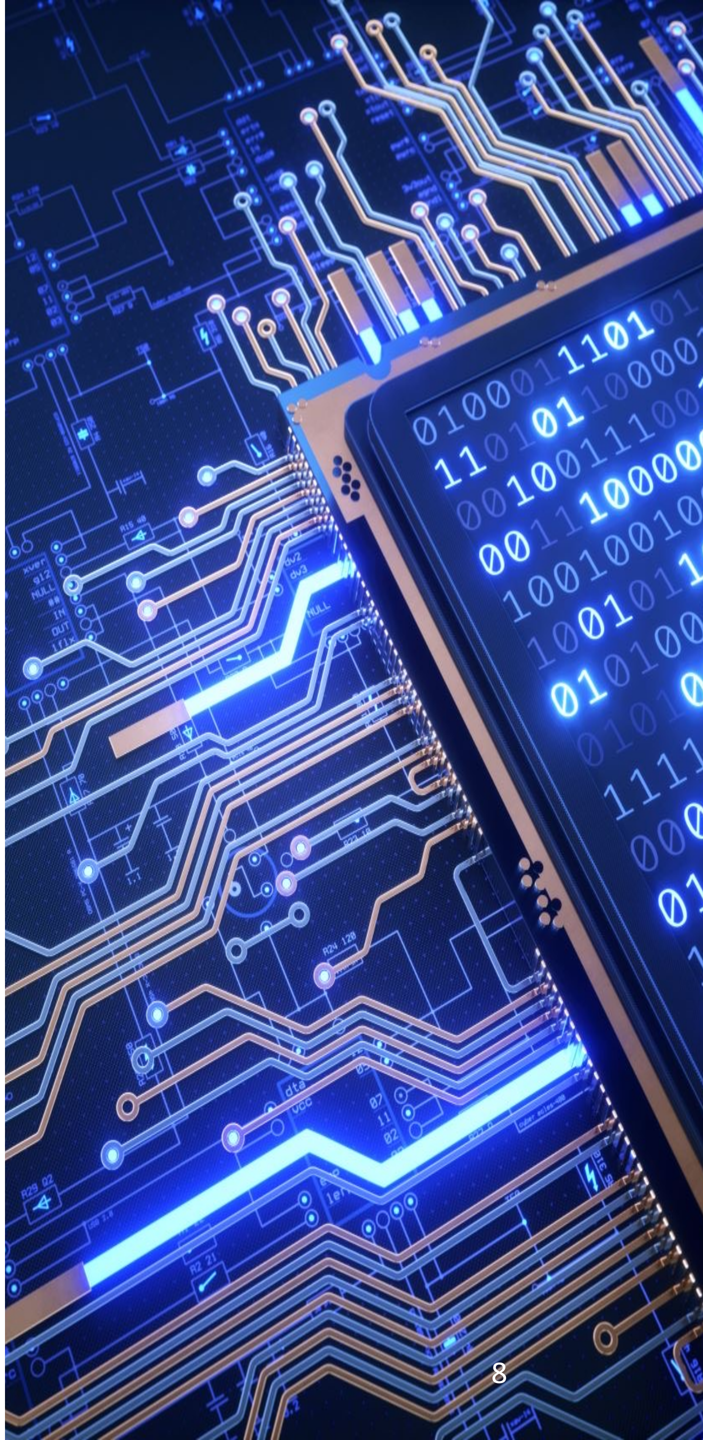
Integral sign

integrand

Variable of integration

Anti-derivative of f(x)

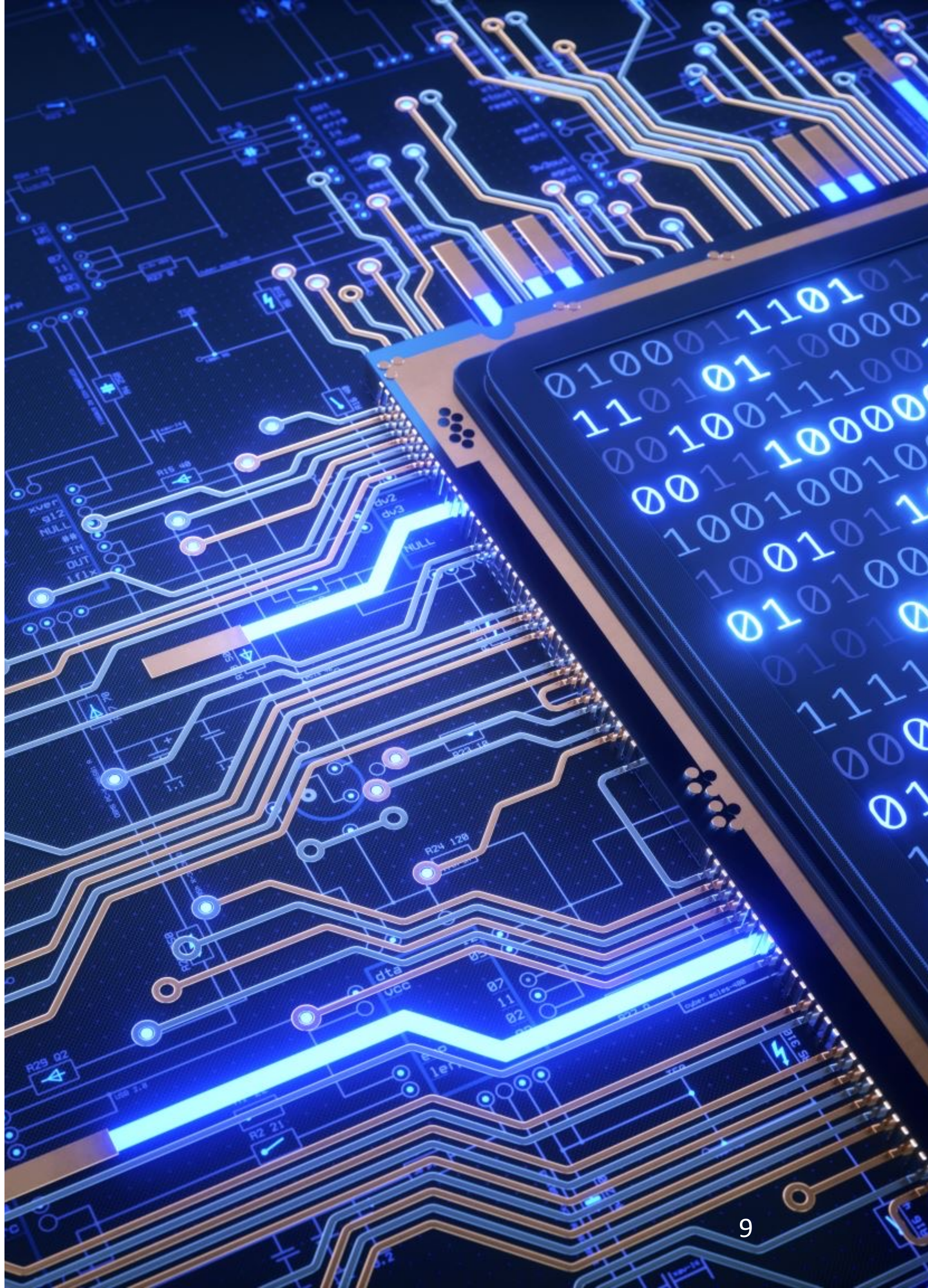
Constant of integration



Indefinite and Definite Integrals

Indefinite $$\int f(x)dx$$

Definite $$\int_{x_1}^{x_2} f(x)dx$$

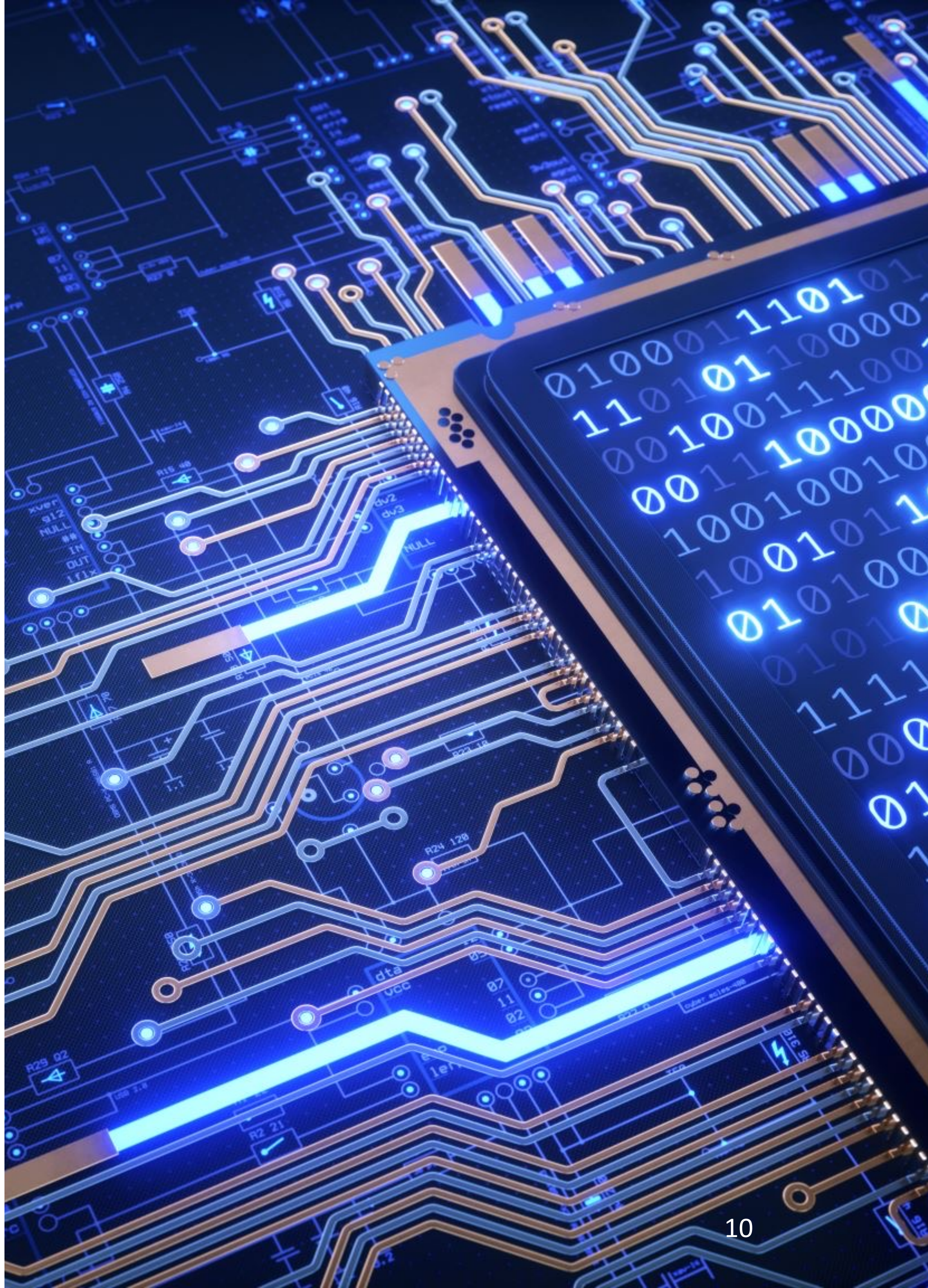


Indefinite and Definite Integrals

$$F(x) = \int f(x)dx$$

$$I = \int_a^b f(x)dx$$

$$I = F(x)\Big]_a^b = F(b) - F(a)$$



Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

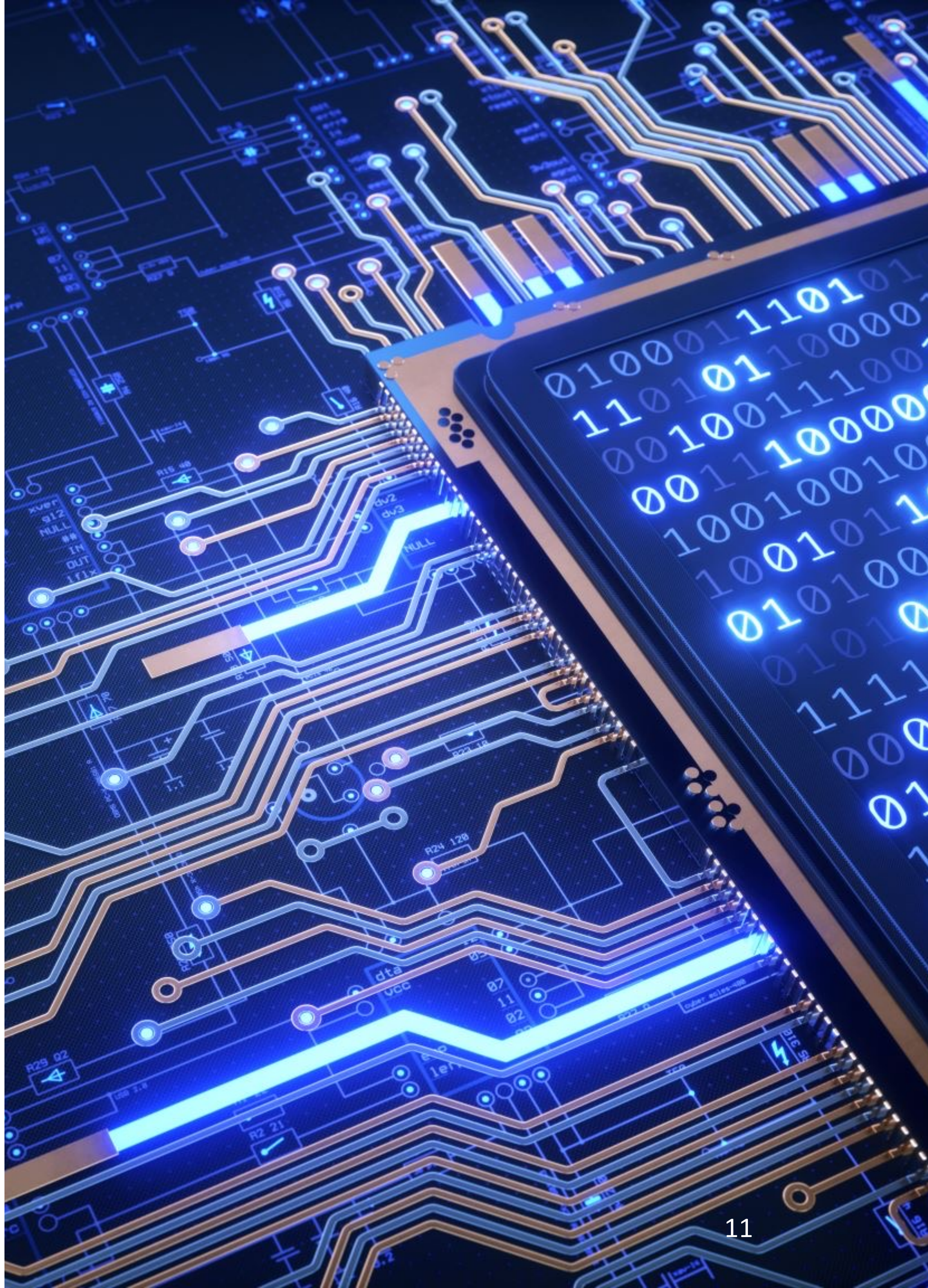
$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

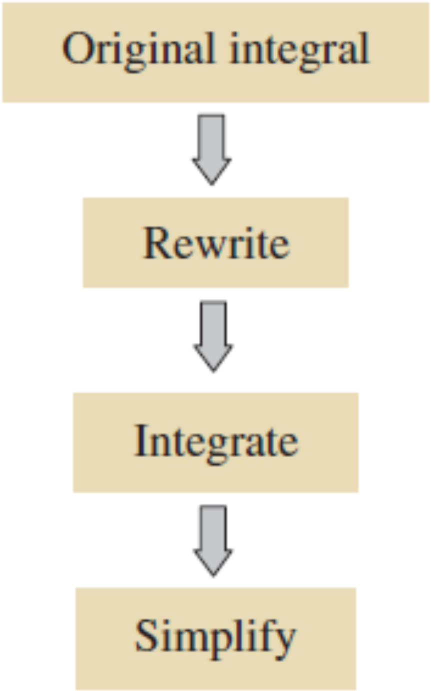
Moreover, if $\int f(x)dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.



Basic Integration Rules



Compute the antiderivative of $3x$

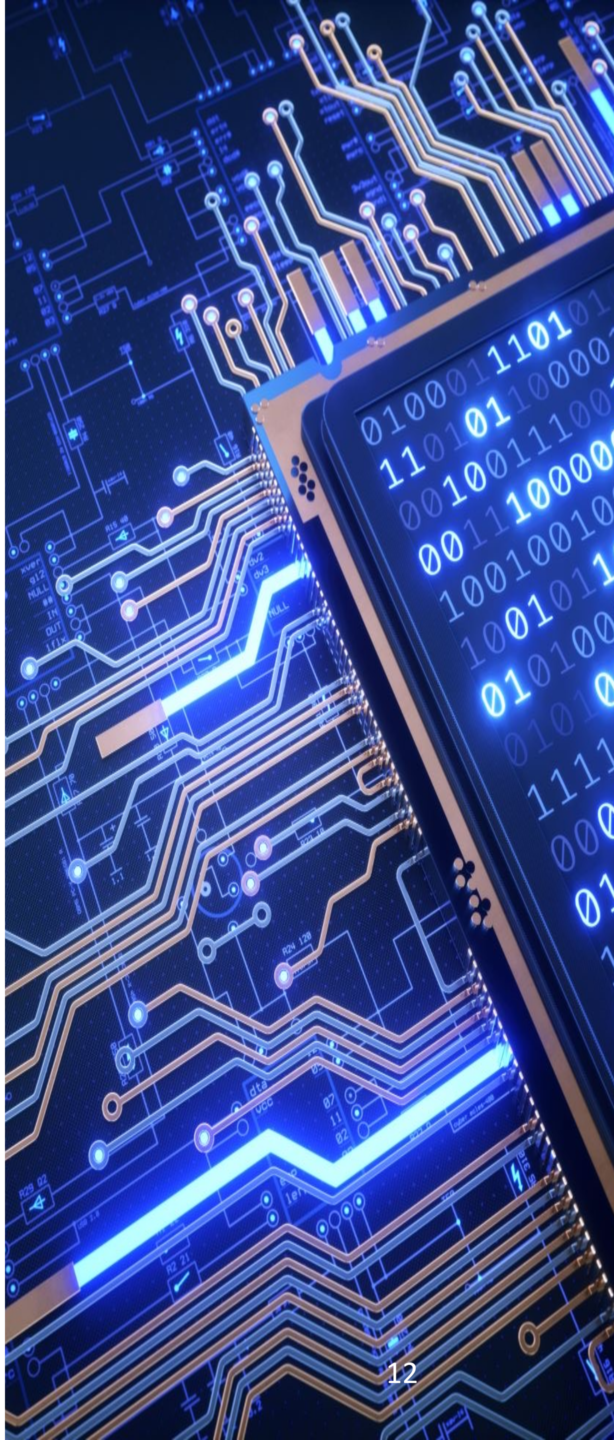
Solution:

$$\int 3x\,dx = 3\int x\,dx$$ Constant Multiple Rule

$$= 3\int x^1\,dx$$ Rewrite x as x^1.

$$= 3\left(\frac{x^2}{2}\right) + C$$ Power Rule ($n = 1$)

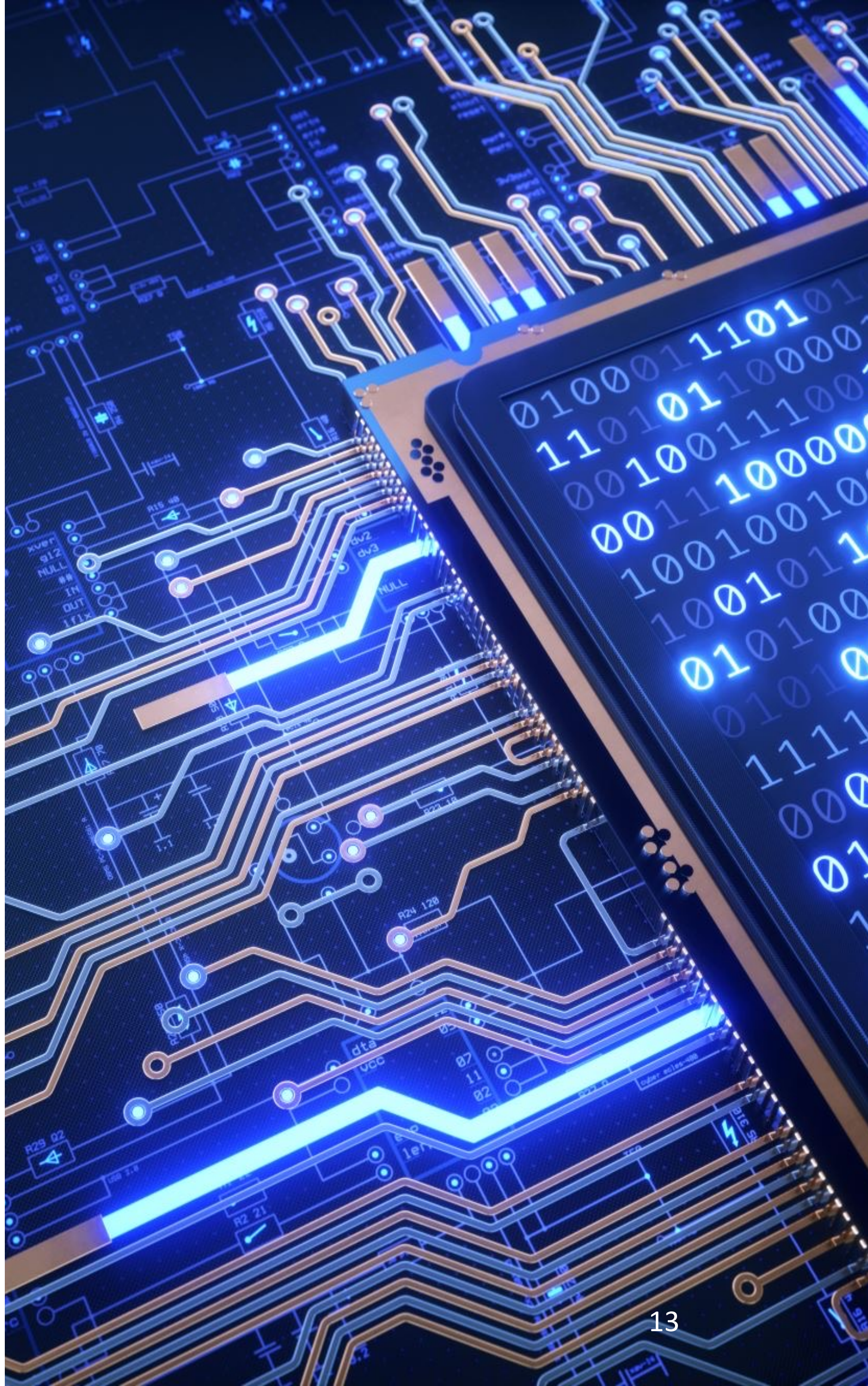
$$= \frac{3}{2}x^2 + C$$ Simplify.



Basic Integration Rules

Basic Integration Rules

Differentiation Formula	Integration Formula	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule



Basic Integration Rules

$$\frac{d}{dx}[\sin x] = \cos x \qquad \int \cos x \, dx = \sin x + C$$

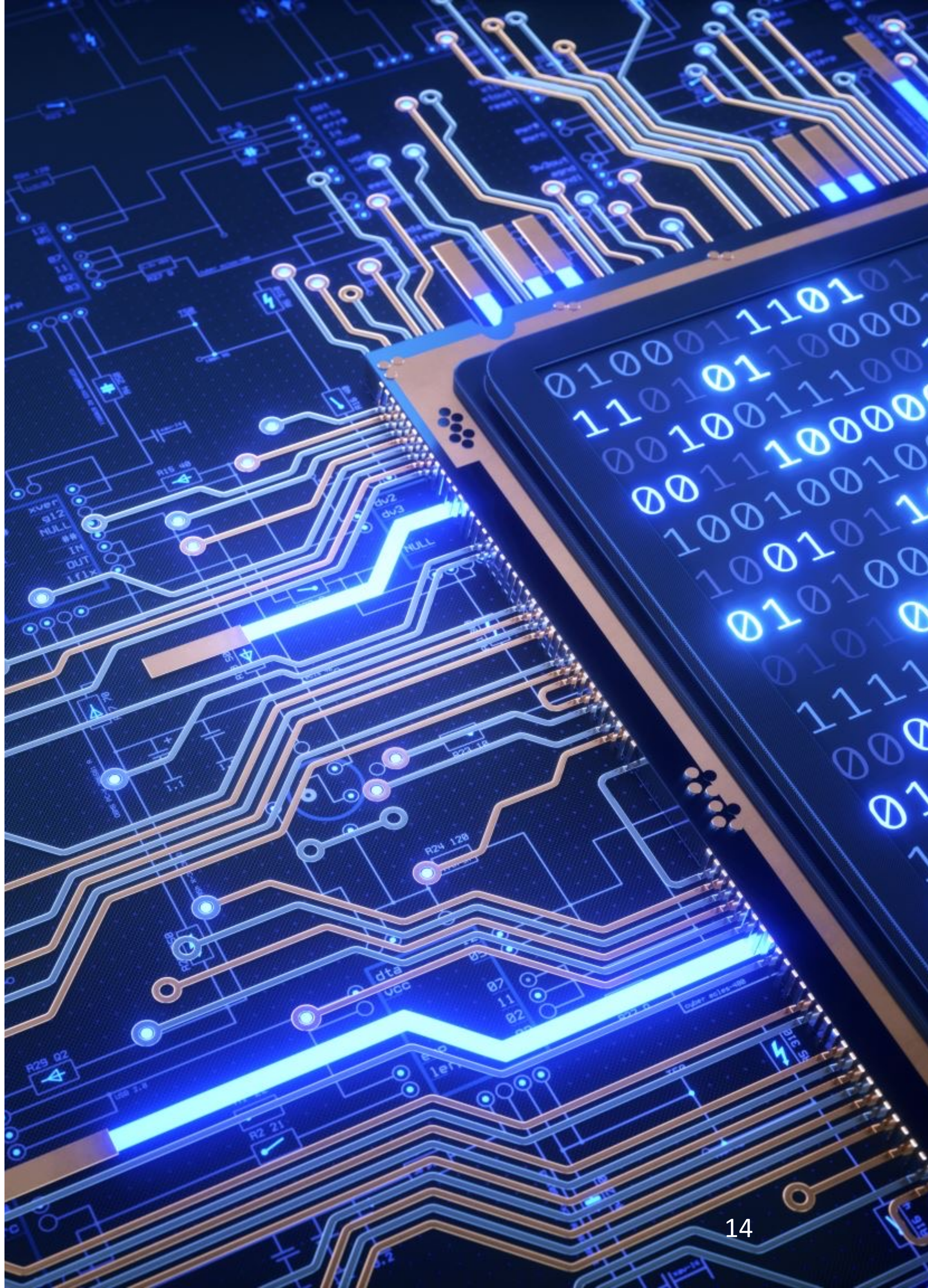
$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \sin x \, dx = -\cos x + C$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \int \sec^2 x \, dx = \tan x + C$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \int \sec x \tan x \, dx = \sec x + C$$

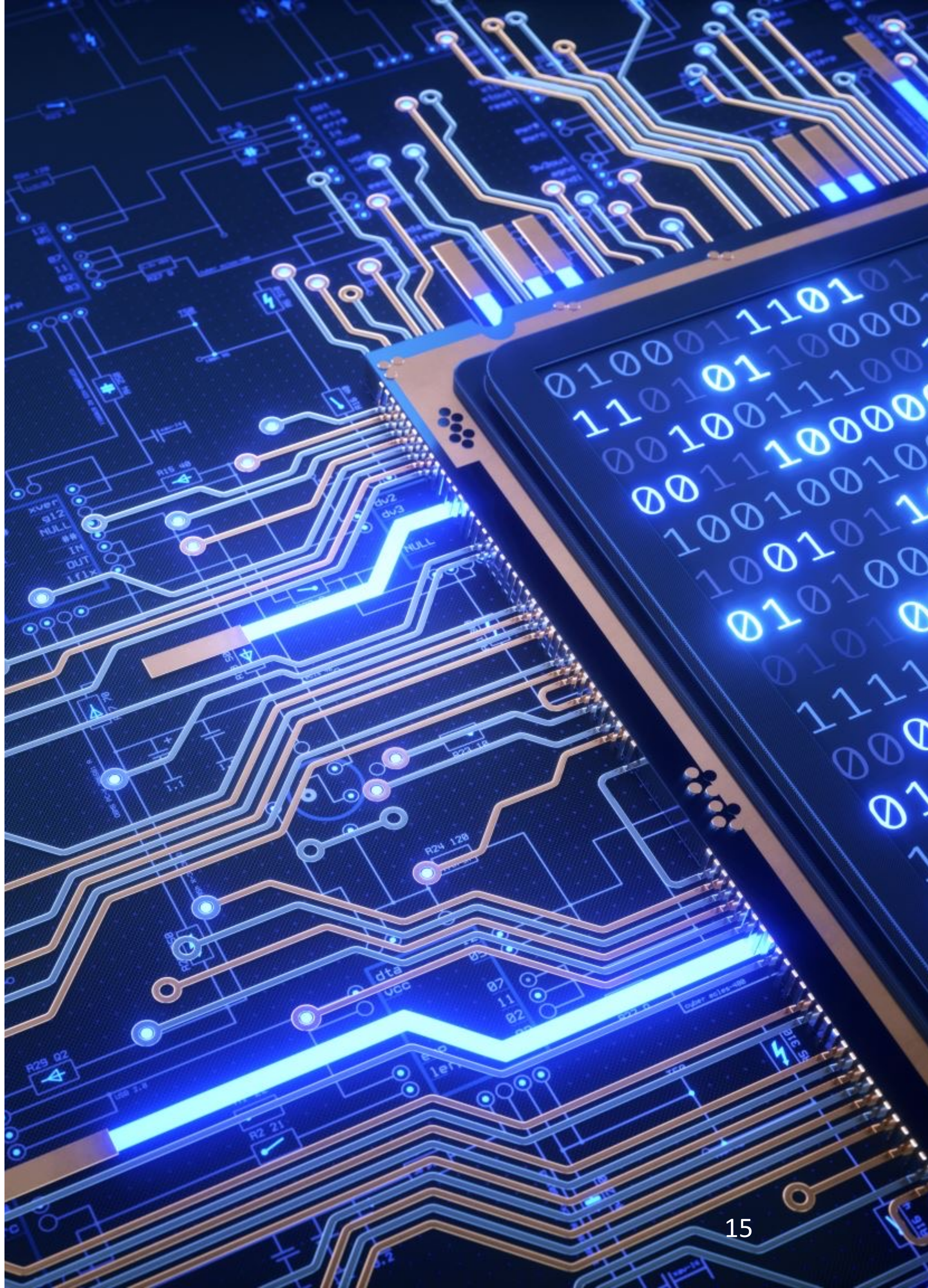
$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \int \csc x \cot x \, dx = -\csc x + C$$



Basic Integration Rules

e^{ax}	$\frac{e^{ax}}{a}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$\frac{1}{a}\sin ax$
$\sin^2 ax$	$\frac{1}{2}x - \frac{1}{4a}\sin 2ax$
$\cos^2 ax$	$\frac{1}{2}x + \frac{1}{4a}\sin 2ax$
$x\sin ax$	$\frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$
$x\cos ax$	$\frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$
$\sin ax\cos ax$	$\frac{1}{2a}\sin^2 ax$
$\sin ax\cos bx$ for $a^2 \neq b^2$	$-\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$
xe^{ax}	$\frac{e^{ax}}{a^2}(ax-1)$
$\ln x$	$x(\ln x - 1)$
$\frac{1}{ax^2+b}$	$\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right)$



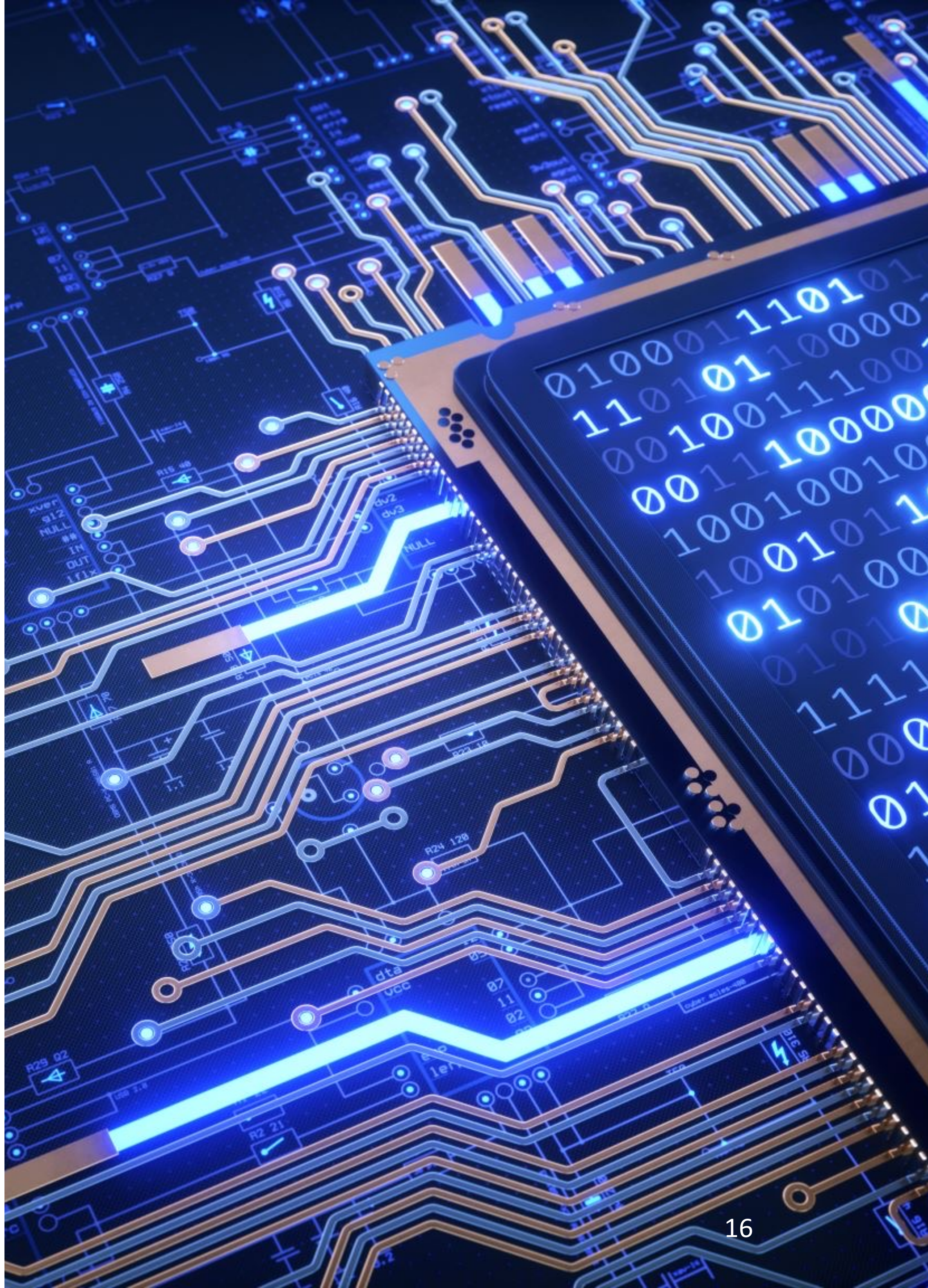
Example: Integrate the following

$$\int e^x \, dx$$

$$\int \frac{1}{x} \, dx$$

$$\int k \, dx$$

$$\int x^n \, dx$$



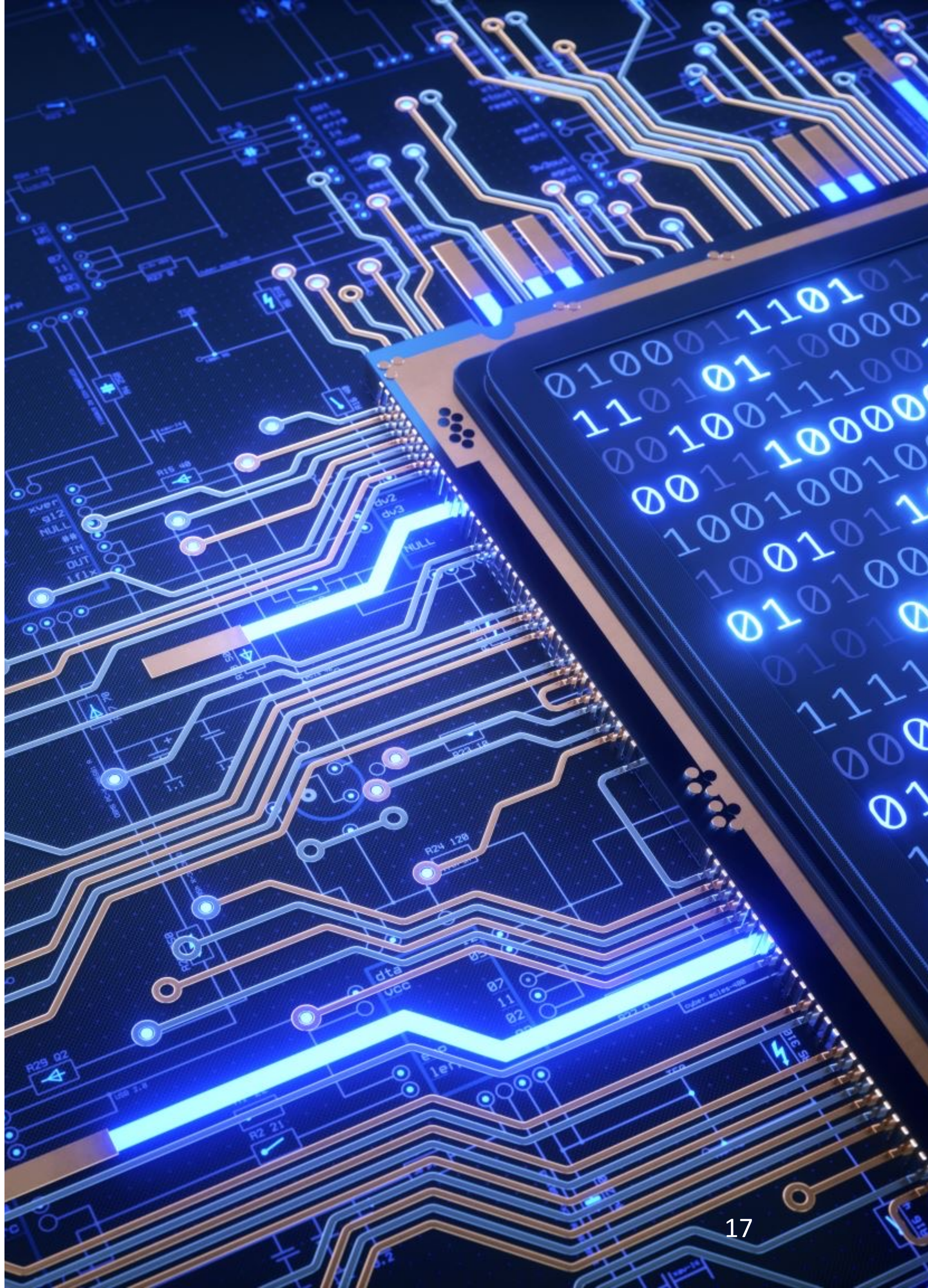
Solution

$$\int e^{x}\,dx = e^{x} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int k\,dx = kx + C$$

$$\int x^{n}\,dx = \frac{x^{n+1}}{n+1} + C$$



Integration by Parts

Start with the product rule:

$$\frac{d}{\cancel{dx}}(uv) = u\frac{dv}{\cancel{dx}} + v\frac{du}{\cancel{dx}}$$

$$d(uv) = u\ dv + v\ du$$

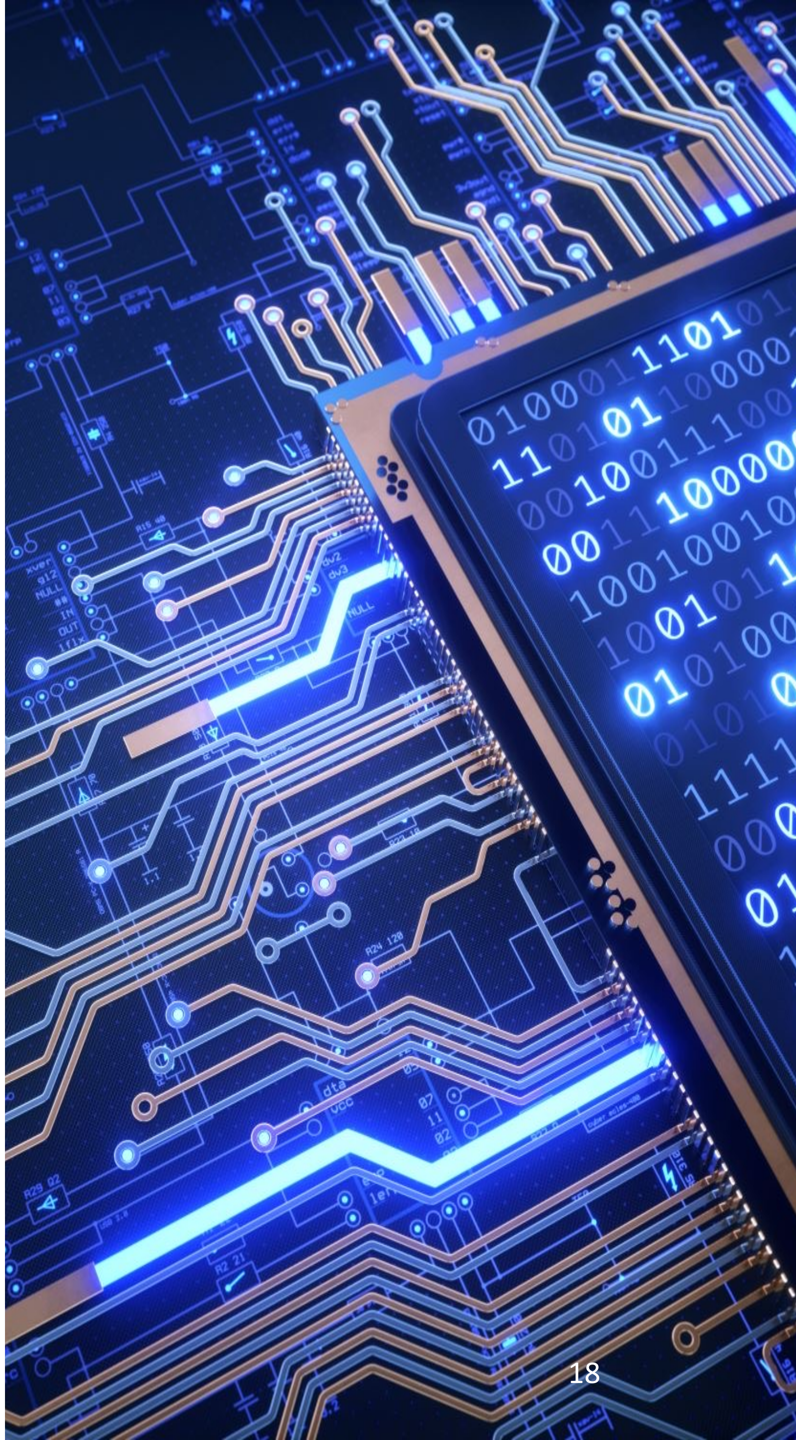
$$d(uv) - v\ du = u\ dv$$

$$u\ dv = d(uv) - v\ du$$

$$\int u\ dv = \int \left(d(uv) - v\ du\right)$$

$$\int u\ dv = \int \left(d(uv)\right) - \int v\ du$$

$$\boxed{\int u\ dv = uv - \int v\ du}$$



Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

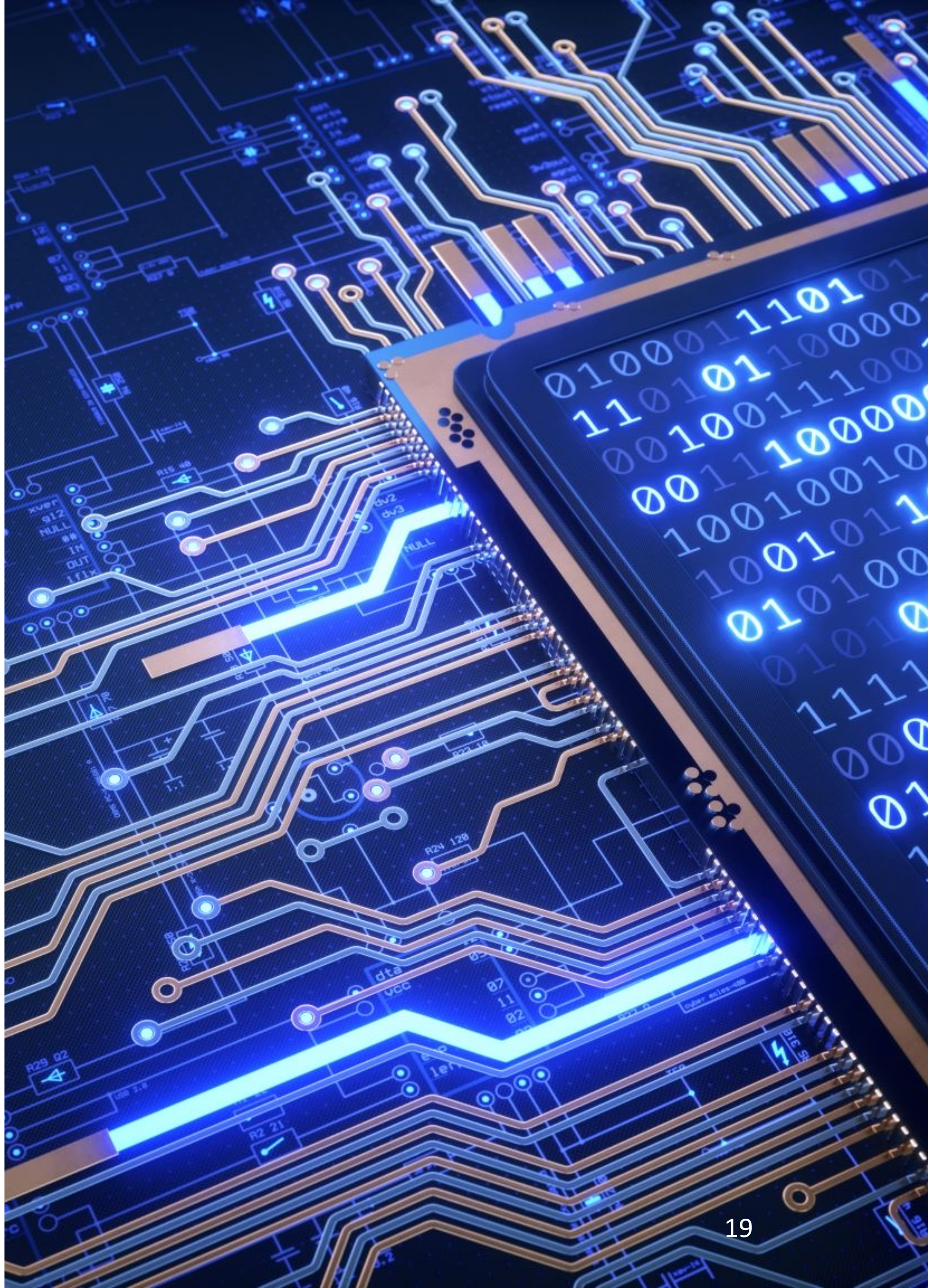
dv is easy to integrate

u differentiates to zero (usually)

The Integration by Parts formula is a "product rule" for integration.

Choose u in this order: LIPET

Logs, Inverse trig, Polynomial, Exponential, Trig



Integration by Parts

$$\int u\ dv = uv - \int v\ du$$

$$\int x \cdot \cos x\ dx$$

polynomial factor

LIPET

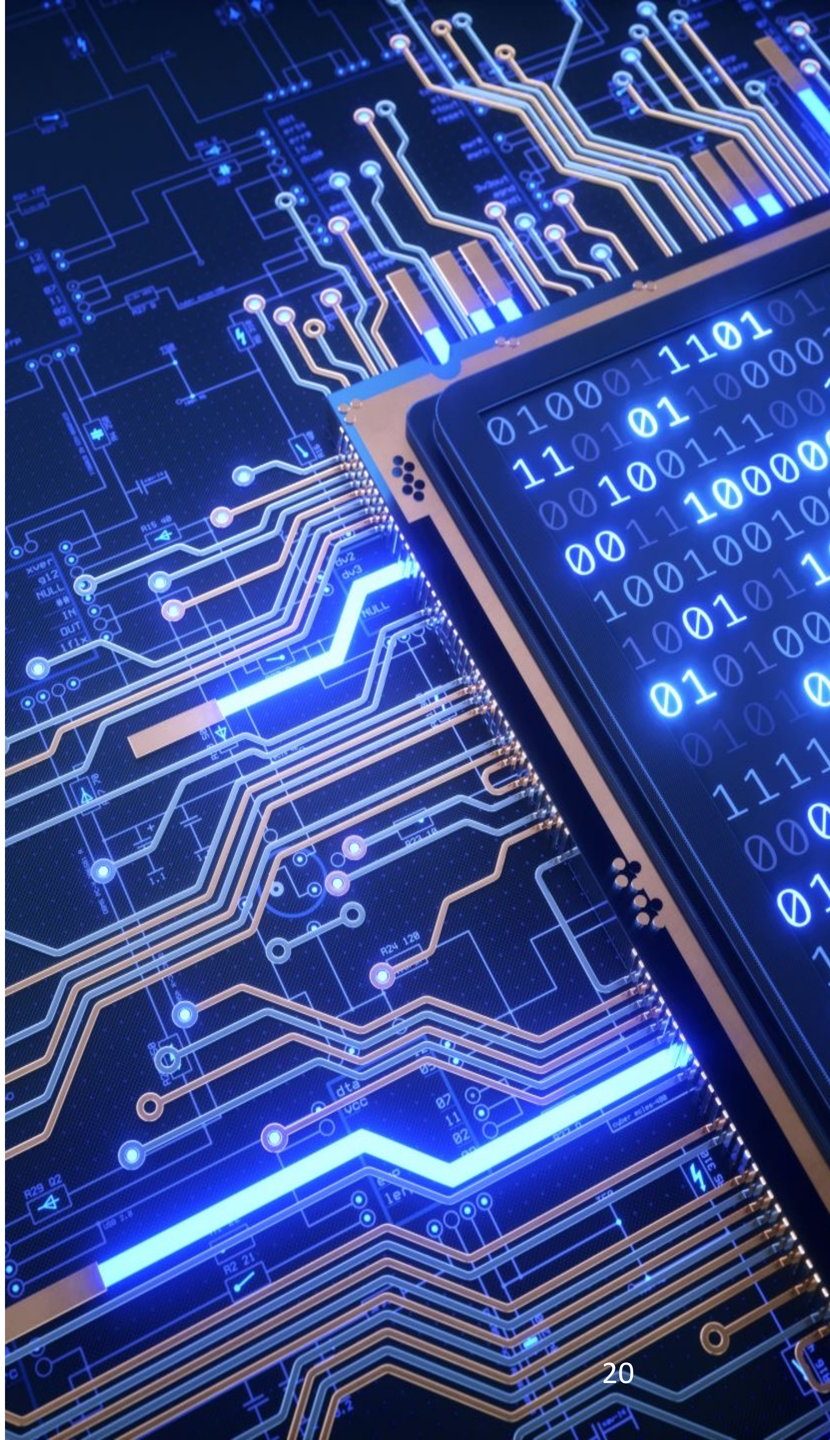
$u = x \qquad dv = \cos x\ dx$

$du = dx \qquad v = \sin x$

$$u\ v - \int v\ du$$

$$x \cdot \sin x - \int \sin x\ dx$$

$$x \cdot \sin x + \cos x + C$$



Integration by Parts

$$\int u\ dv = uv - \int v\ du$$

LIPET

$$\int \ln x\ dx$$

logarithmic factor

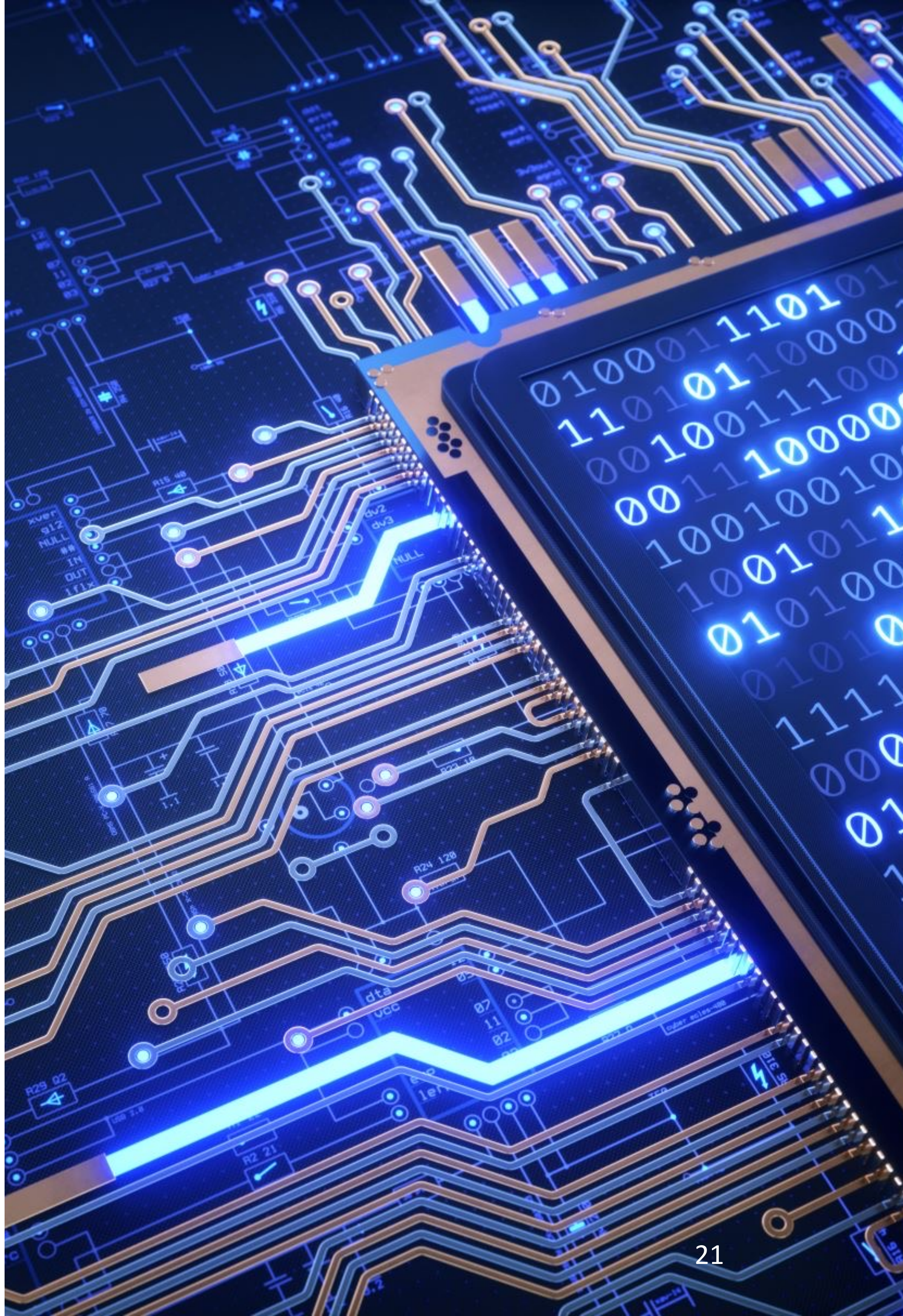
$$\Rightarrow \quad u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$u\ v - \int v\ \ du$$

$$\ln x \cdot x - \int x \cdot \frac{1}{x}\ dx$$

$$x \ln x - x + C$$



Integration by Parts

$$\int u\ dv = uv - \int v\ du$$

LIPET

$$\int x^2 e^x\ dx$$

$$u\ v - \int v\ \ du$$

$u = x^2$	$dv = e^x dx$
$du = 2x\ dx$	$v = e^x$

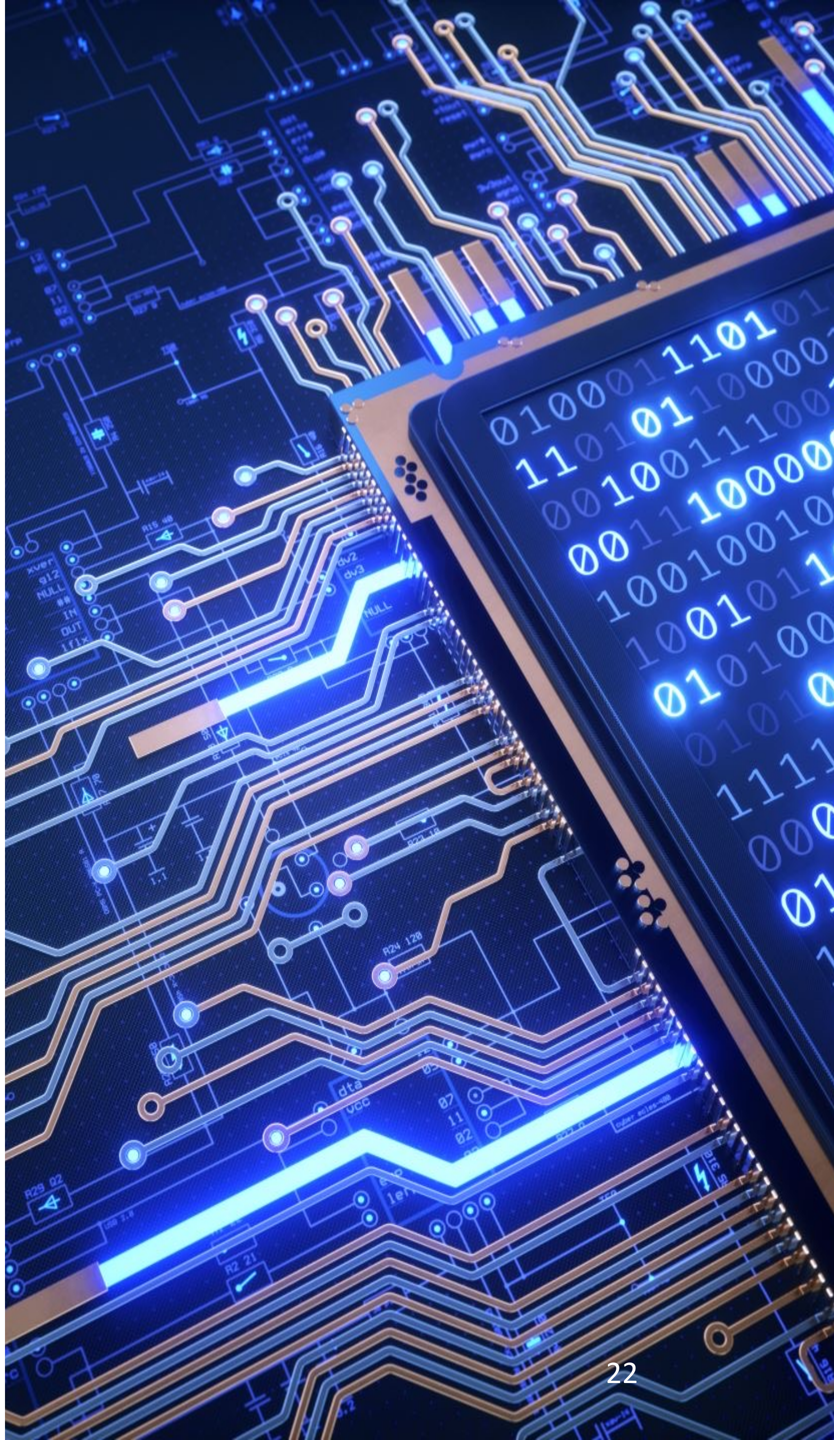
$$x^2 e^x - \int e^x \cdot 2x\ dx$$

$$x^2 e^x - 2\int x e^x\ dx$$

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

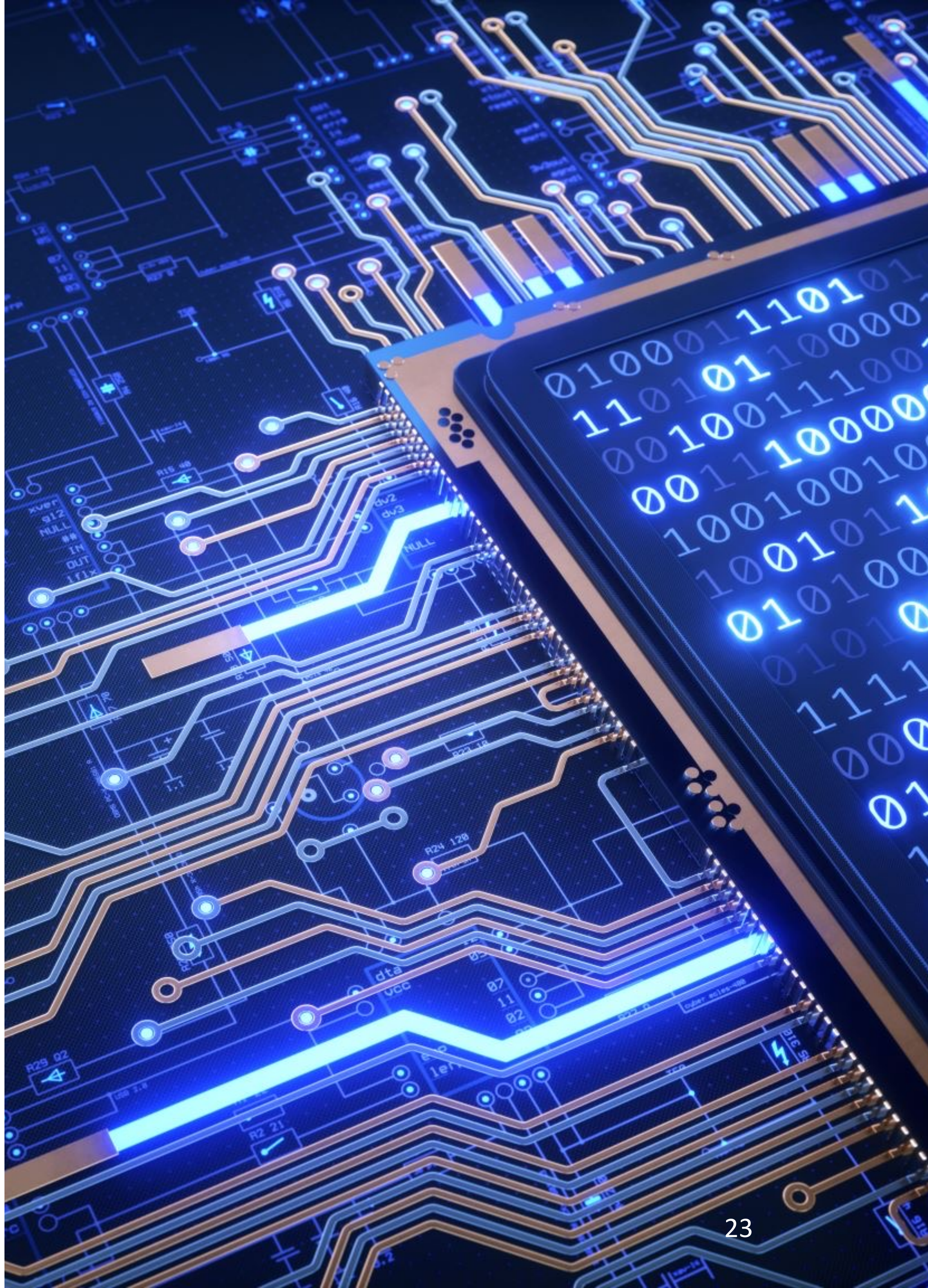
$$x^2 e^x - 2\left(x e^x - \int e^x dx \right)$$

$$x^2 e^x - 2x e^x + 2e^x + C$$



Exercise I

1. Integrate the following:
   i. $12e^{4x}$
   ii. $6x^2 + \frac{3}{x}$
   iii. $\int_0^{\pi} \sin x$
   iv. $12x\sin 2x$
   v. $\int_0^1 8xe^{-2x}$



Solution to Exercise I

i. $$z = \int 12e^{4x}dx = 12\frac{e^{4x}}{4} + C$$

$$= 3e^{4x} + C$$

ii. $$z = \int \left(6x^2 + \frac{3}{x}\right)dx$$

$$= \int 6x^2 dx + \int \frac{3}{x}dx$$

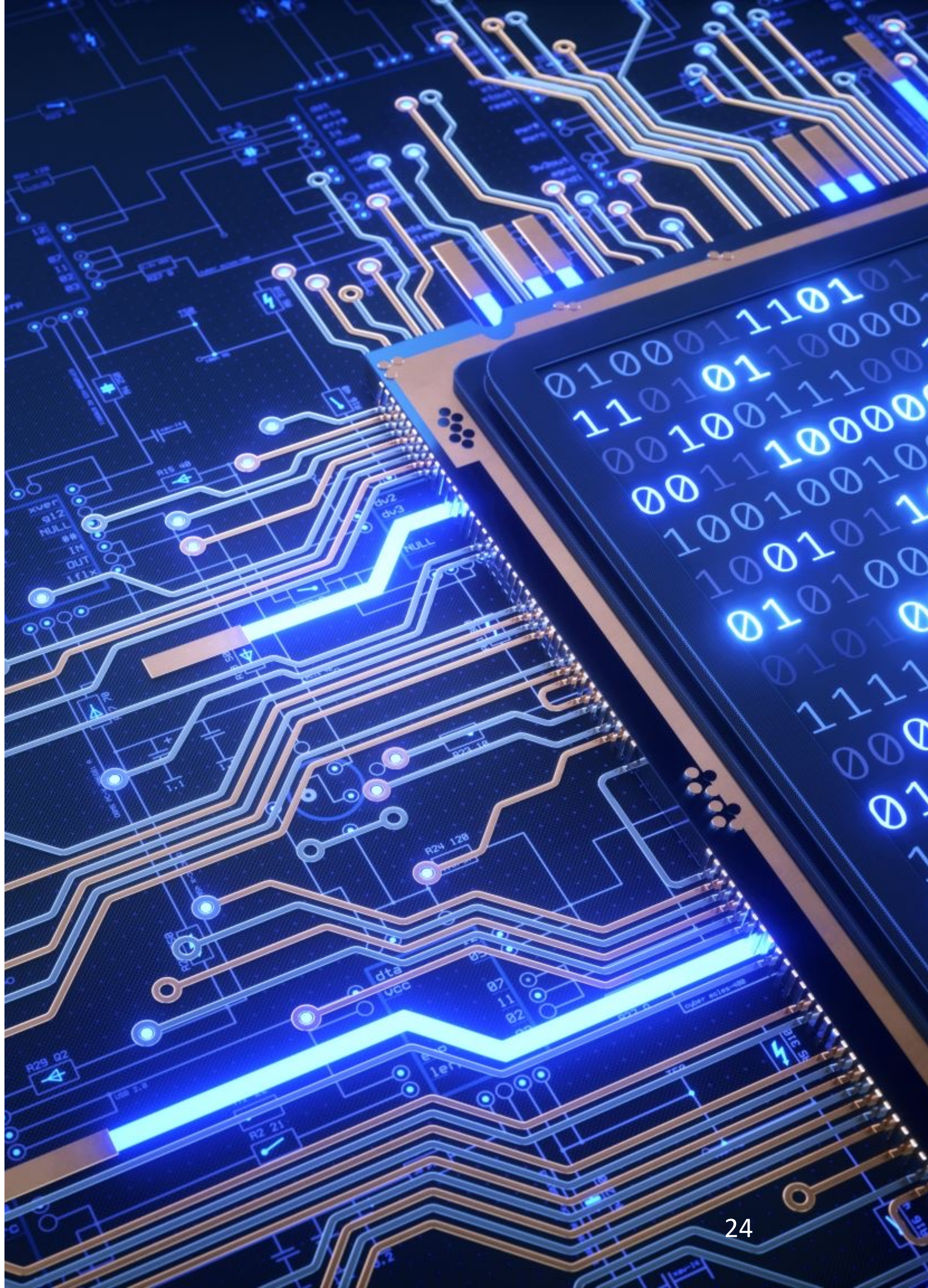
$$= \frac{6x^3}{3} + 3\ln x + C$$

$$= 2x^3 + 3\ln x + C$$

iii. $$I = \int_0^{\pi} \sin x dx = -\cos x\Big]_0^{\pi}$$

$$= -\cos\pi - (-\cos 0)$$

$$= -(-1) - (-1) = 2$$



Solution to Exercise I

iv. $$z = \int 12x \sin 2x dx$$

$$= 12\left(\frac{1}{(2)^2}\sin 2x - \frac{x}{2}\cos 2x\right) + C$$

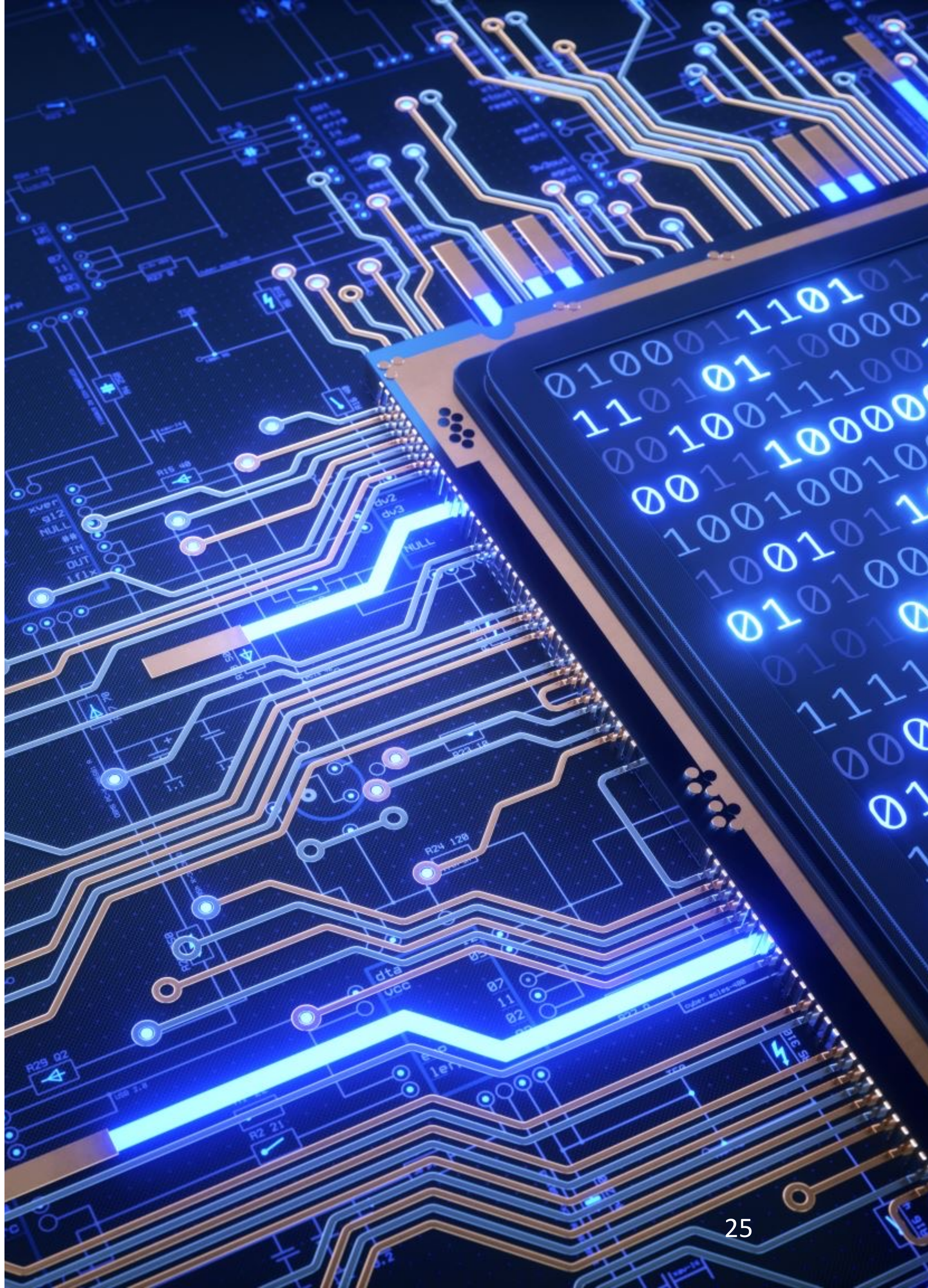
$$= 3\sin 2x - 6x\cos 2x + C$$

v. $$I = \int_0^1 8xe^{-2x} dx$$

$$= 8\frac{e^{-2x}}{(2)^2}[-2x-1]_0^1$$

$$= 2e^{-2}\left[-2(1)-1\right] - 2e^{-0}\left[0-1\right]$$

$$= -6e^{-2} + 2 = 1.188$$

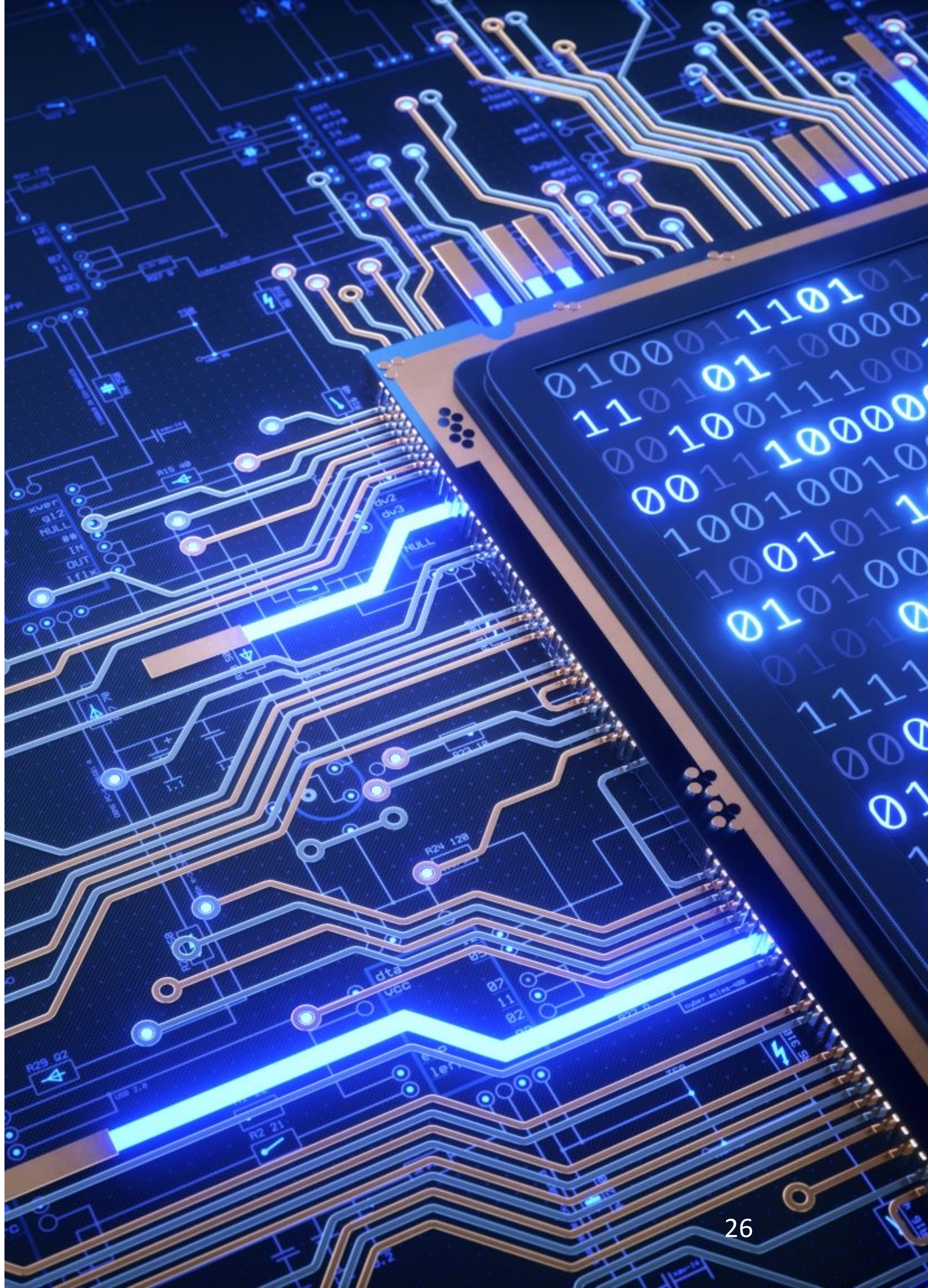


Exercise II

2. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

3. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

   Find the displacement of the particle during the time period $1 \le t \le 4$.



Solution to Exercise II

2. First, we need to write the integrand in a simpler form by carrying out the division:

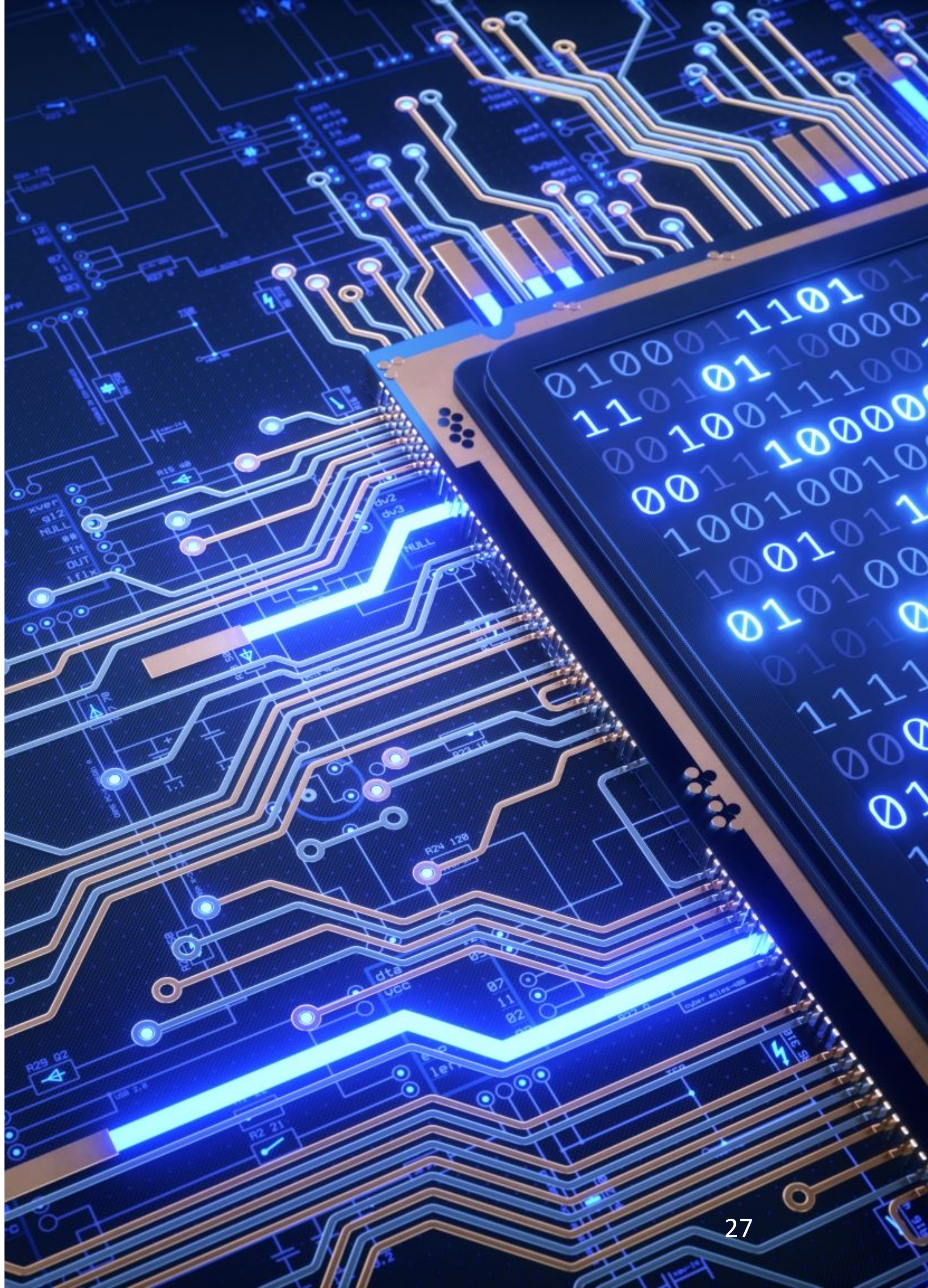
$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9$$

$$= 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$

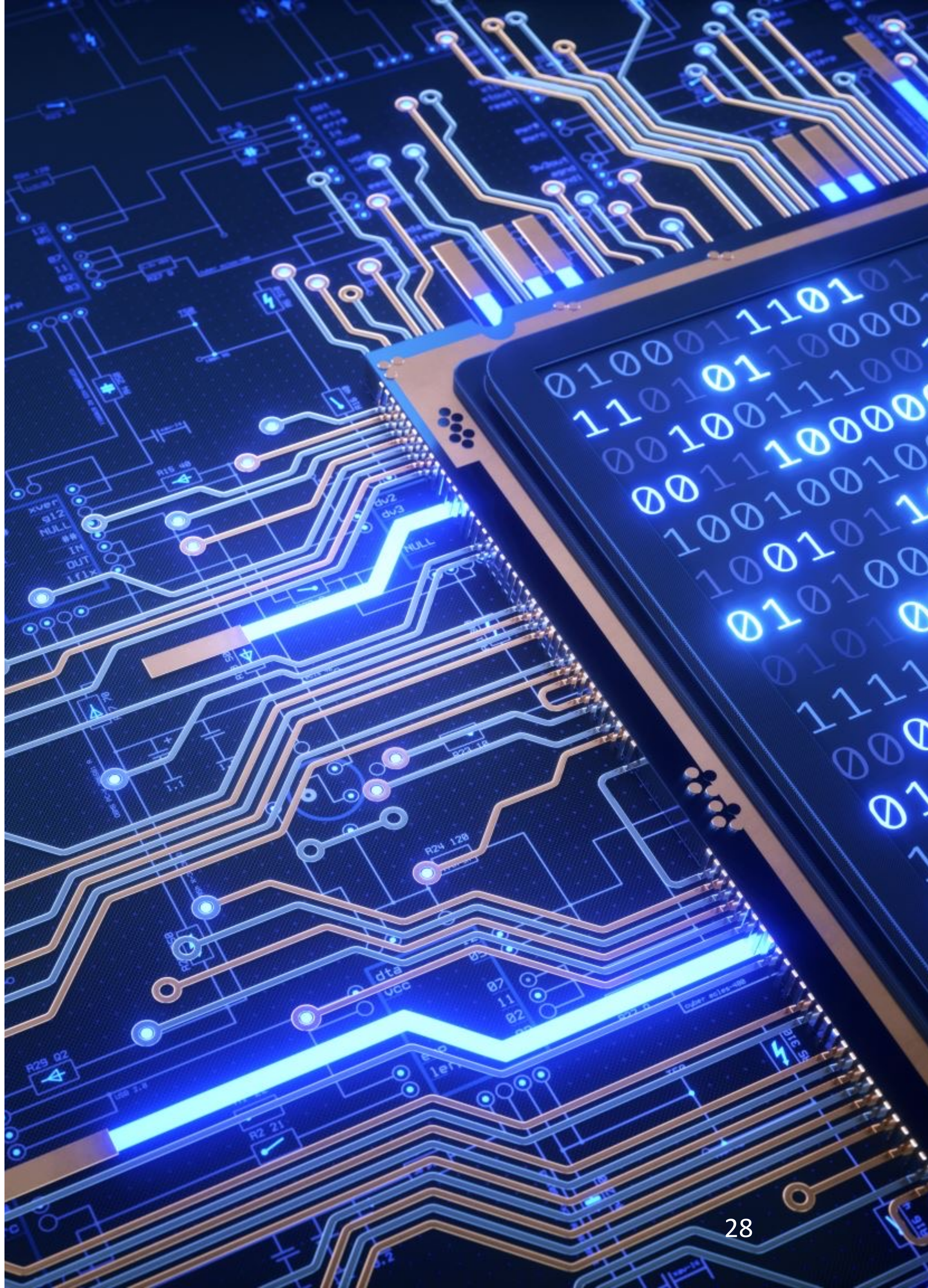


Solution to Exercise II

3. The displacement is

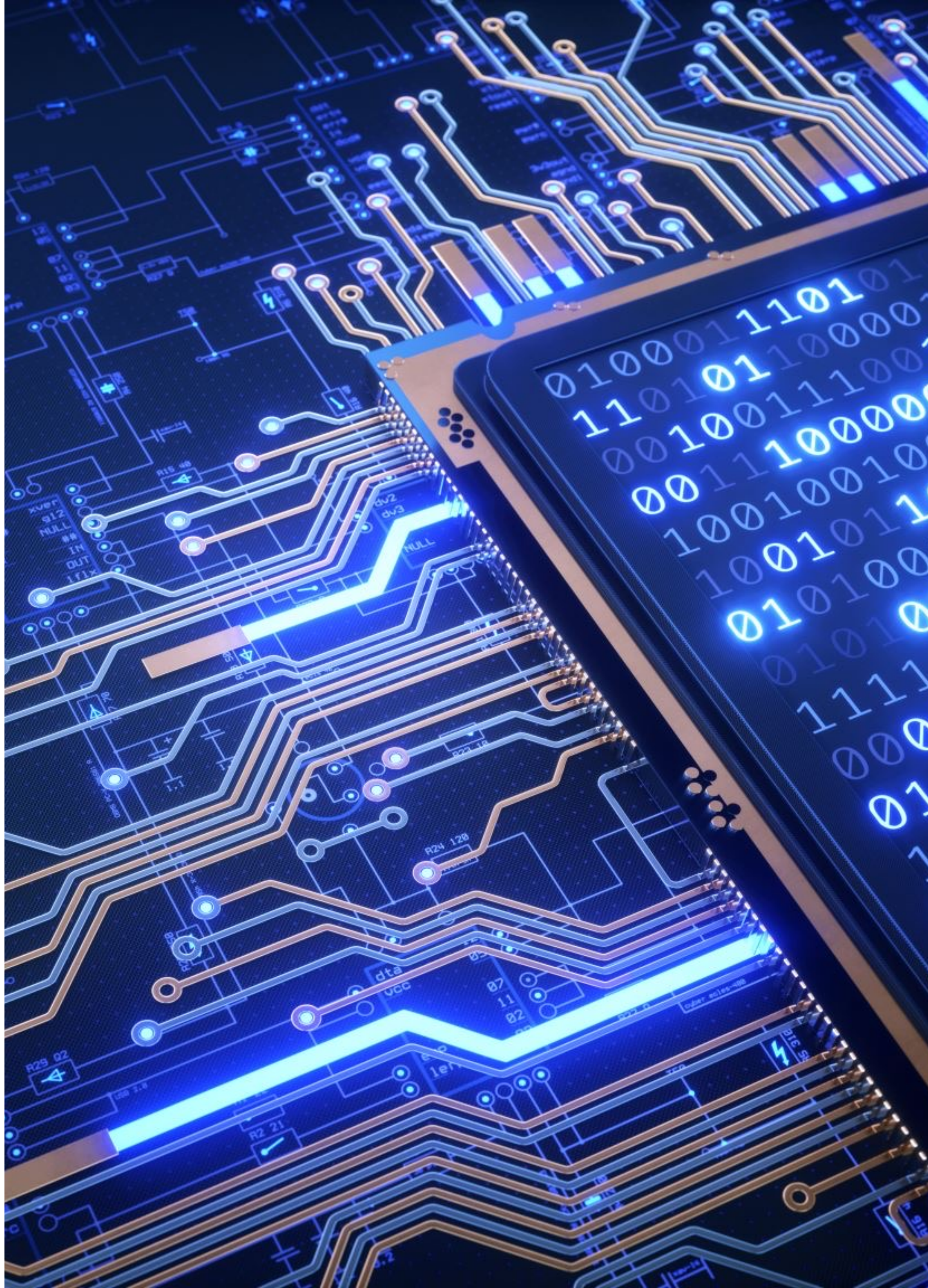
$$\begin{aligned} s(4) - s(1) &= \int_1^4 v(t)\,dt \\ &= \int_1^4 (t^2 - t - 6)\,dt \\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \\ &= -\frac{9}{2} \end{aligned}$$

This means that the particle moved 4.5m toward the left.



Applications of Integration

- Area Under Curves
- Volumes of Solids
- Lengths of Curves
- Center of Mass and Moments of Inertia
- Work and Energy
- Probability and Statistics
- Differential Equations
- Electricity and Magnetism
- Fluid Dynamics
- Economics and Finance

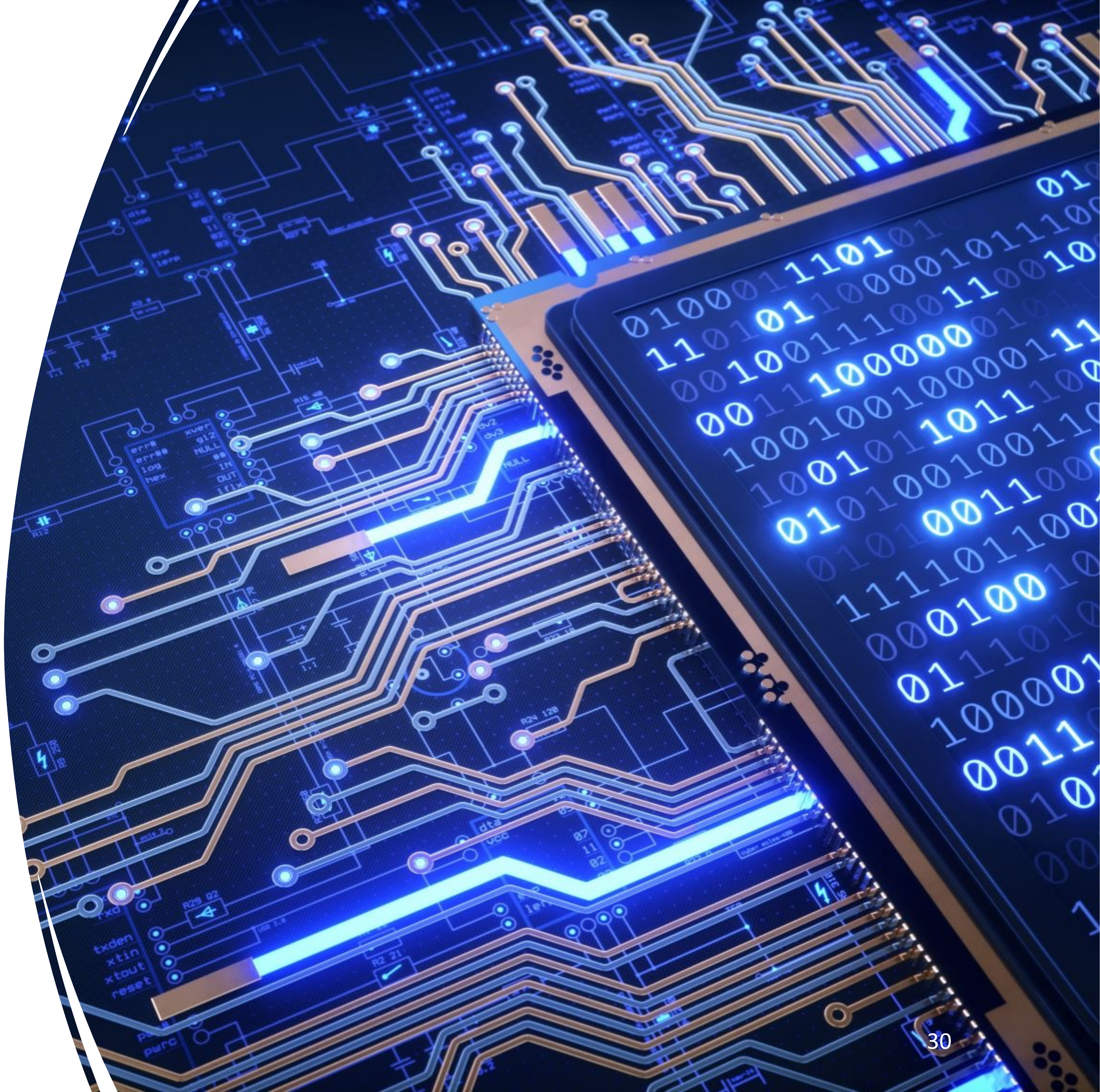


Summary

- **Antiderivative** of a function f is **a differentiable function F whose derivative is equal to the original function f**.
  - This can be stated symbolically as F' = f
- Definite vs Indefinite Integrals
- Integration by parts

$$\int u \, dv = uv - \int v \, du$$

- Applications of integration



Reference

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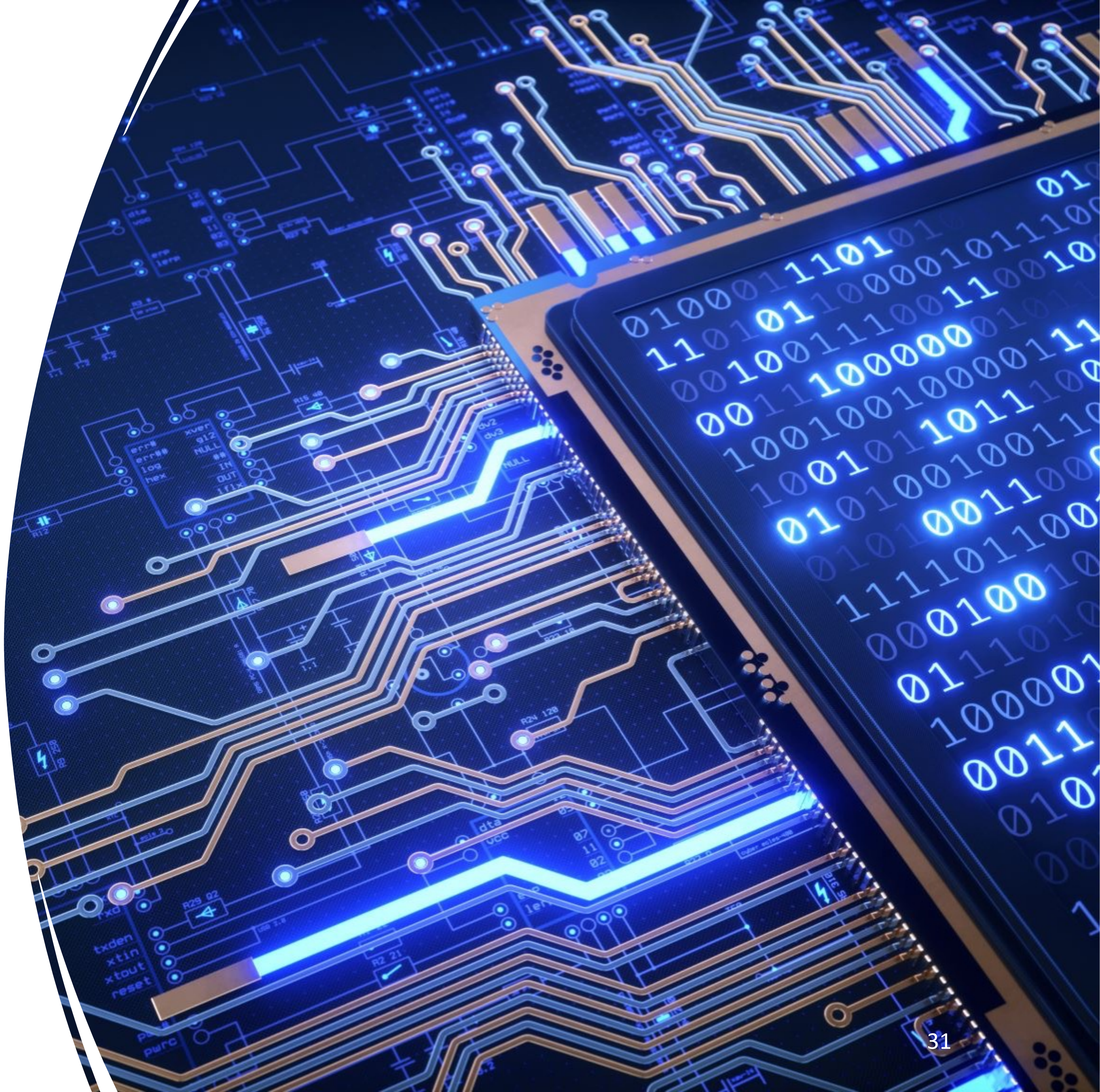


Thank you!