Business Mathematics

Lecture 1

Linear Equations

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Introduction to Business Mathematics

- Business mathematics focus on equipping you with basic knowledge and skills in economics and business mathematics.
- The course offer crucial knowledge and skills in application of mathematical concepts in economics and business.
- It offers important background prerequisite for advance courses in finance, economics, accounting, operational research, and business mathematics.
- It covers topics on application of basic calculus in business mathematics; application of matrix algebra; input-output analysis; probability and decision making; linear equations and programming; and mathematics of finance.

Course Intended Outcomes

At the end of this course that you will be able to:

a) Apply the concept of linear and quadratic equations, and basic calculus in solving business problems.

b) Use Leontief model to solve input-output problems.

c) Formulate and solve linear programming problems.

d) Apply probability in decision making.

e) Describe the dynamics of economic systems.

Introduction to Lecture 1

- This lecture introduces you to linear equations, solving linear equations in two and three unknowns, and their applications to solving economics and business-related problems.

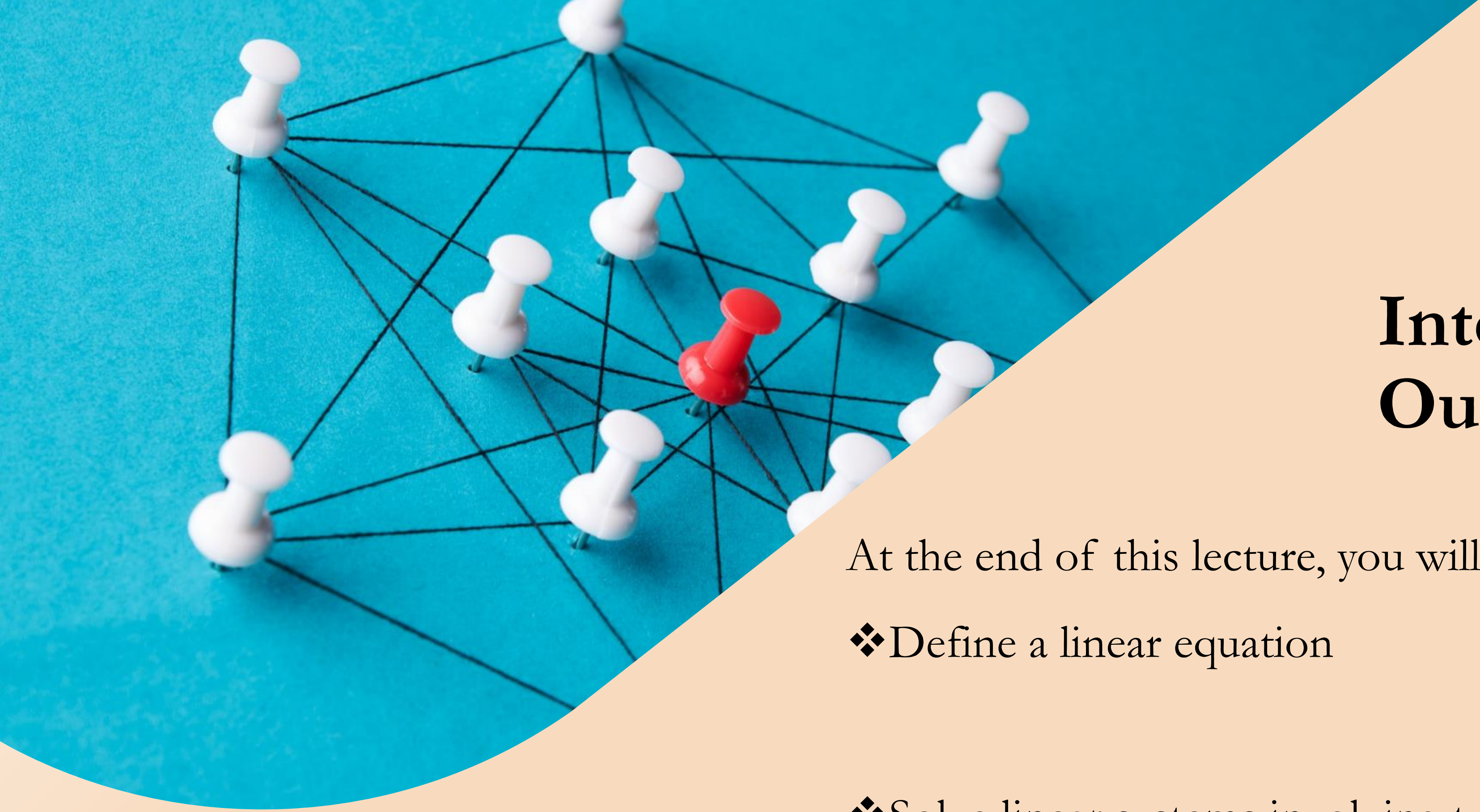
- We shall demonstrate how linear equations are used in supply and demand analysis as just a snippet on how linear equations can be used to comprehend real-life business phenomena.

Further Readings

- These notes have been derived from diverse resources.

- The resources are recommended for further reading to gain more insights on the application of linear equations to business or commercial arithmetic.

- These resources are (Jacques, 2006; P. Kahenya, 2017; P. N. Kahenya, 2021; Lay, 2003; Lay et al., 2016; Murray & Robert, 2009).



Intended Learning Outcomes

At the end of this lecture, you will be able to;

- ❖Define a linear equation
- ❖Solve linear systems involving two and three unknowns
- ❖Apply linear equations in solving economics and business-related problems

Definition of Terms

Definition 1: Linear equation

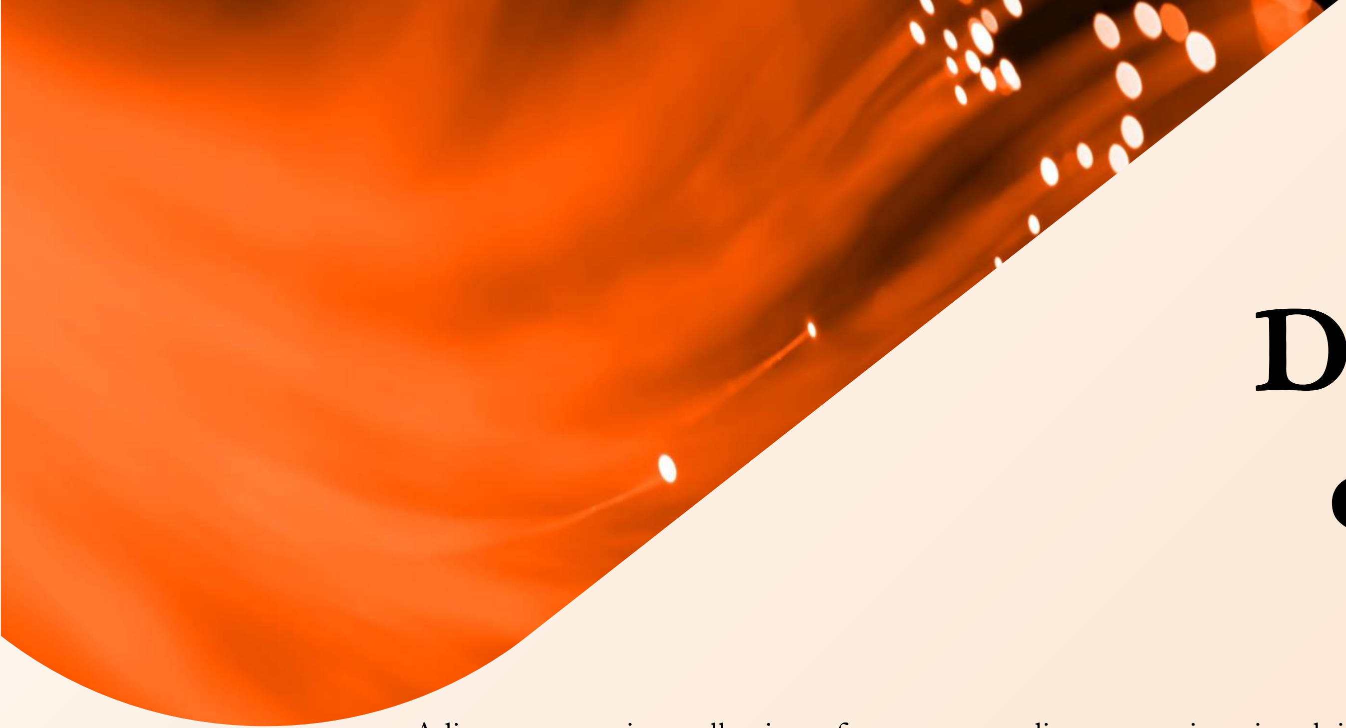
- It is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$ where $x_1, x_2, \cdots, x_n$ are variables and $a_1, a_2, \cdots, a_n, k$ are known numbers, real or complex.
- For example;

  a) $2x_1 + 3x_2 = 7$

  b) $x_1 - 2x_2 + 4x_3 = 0$

  c) $3x + 8y = 9$

  d) $a + 2b + 4c - 5d = 0$



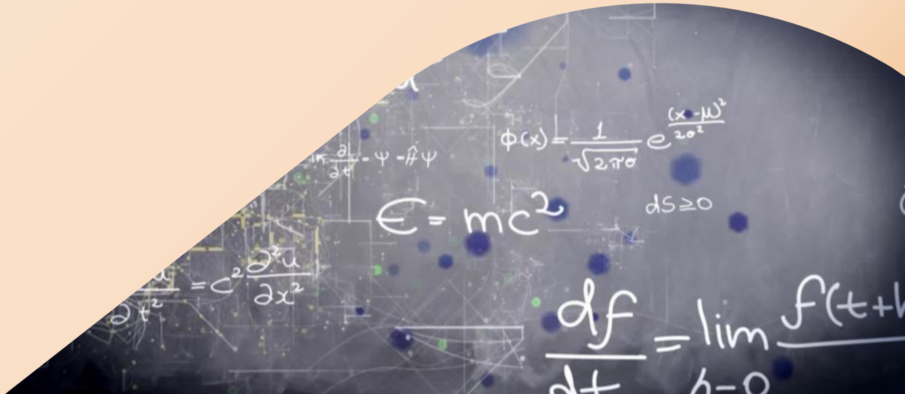
Definition 2: System of linear equations

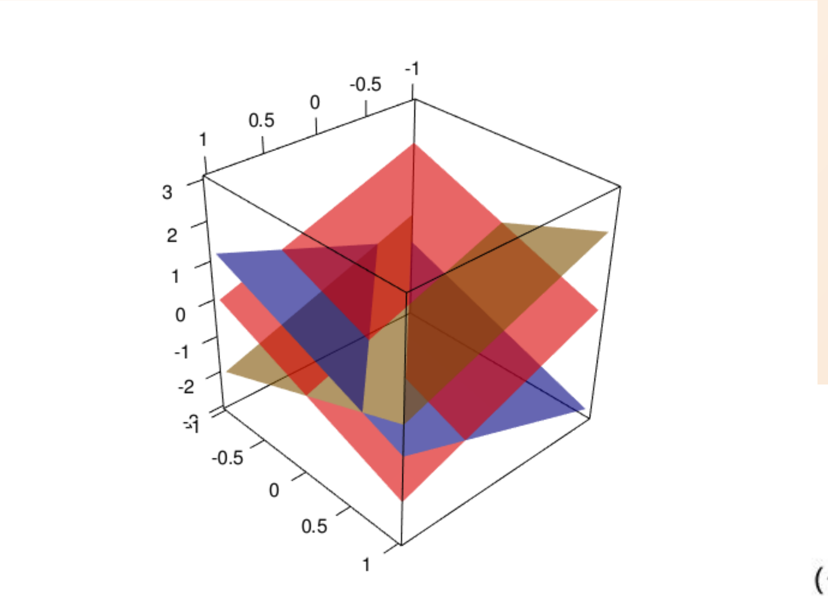
A linear system is a collection of one or more linear equations involving the same variables

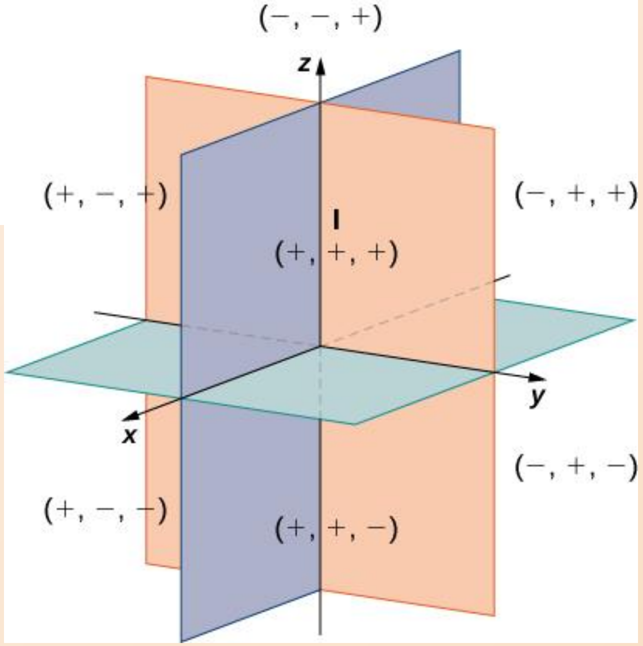
For example,

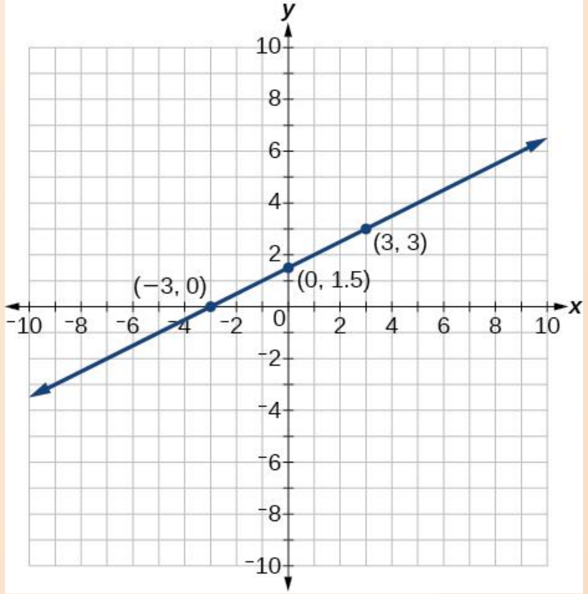
$$3x + 6y = 19$$
$$x - 2y = 7$$

$$2p + 3q + 5r = 8$$
$$p + q - r = 3$$
$$7p + 2q = 11$$




-1
-0.5
0
0.5
1
3
2
1
0
-1
-2
-0.5
0
0.5
1


(−, −, +)
z
(+, −, +)
(−, +, +)
I
(+, +, +)
y
x
(−, +, −)
(+, −, −)
(+, +, −)


y
x
(3, 3)
(−3, 0)
(0, 1.5)

Definition 3: Solutions of a linear system

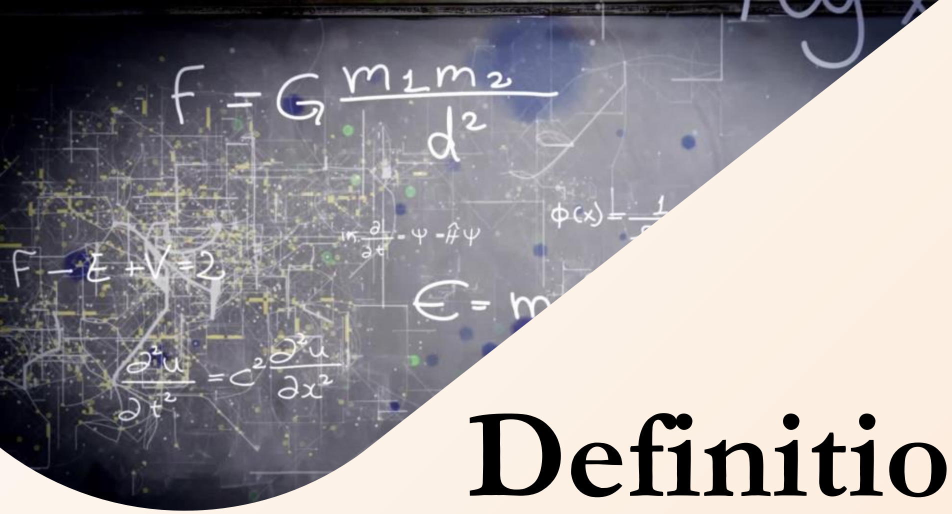
- A solution of a linear system is a list of numbers $s_1, s_2, \cdots, s_n$ that satisfy an equation in a linear system.

- That is given a linear system $a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$ then the set of all possible solutions $s_1, s_2, \cdots, s_n$ are substitutes of the variables $x_1, x_2, \cdots, x_n$

- A system of linear equations is either consistent or inconsistent

- That is, it is consistent if it has a unique solution or infinitely many solutions,

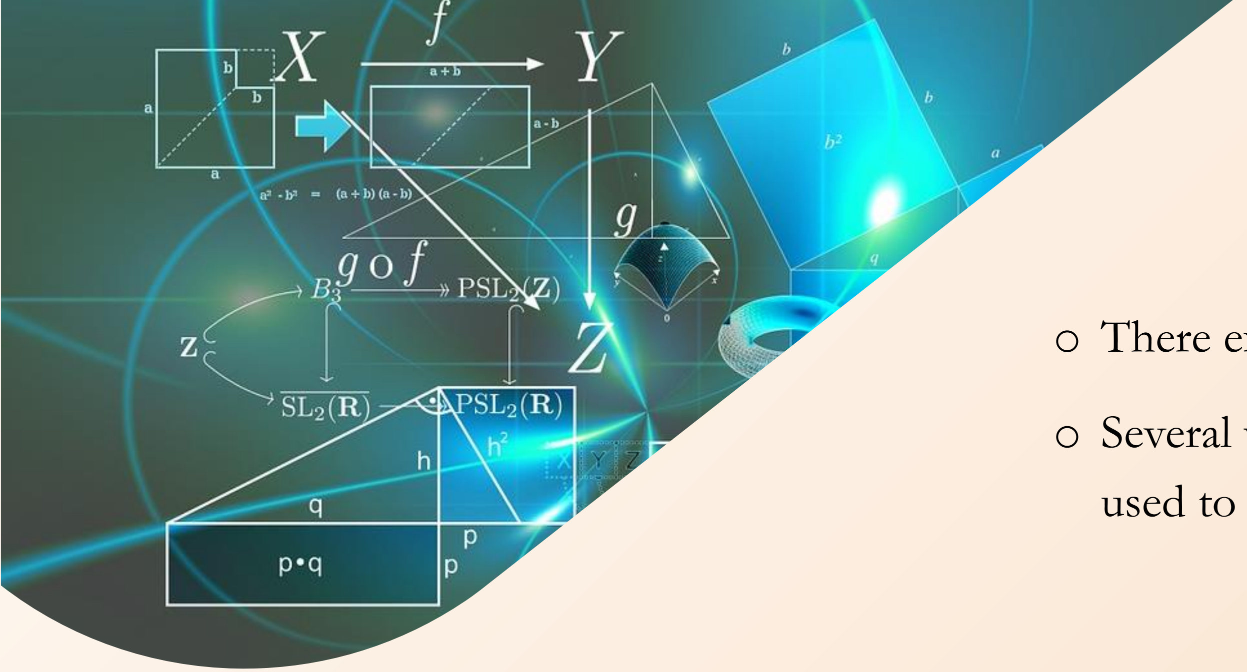
and it is inconsistent if it has no solutions



Definition 4: Function

- A function is a relation between a set of inputs, called the domain, and a set of possible outputs, called the codomain, with the property that each input is related to exactly one output
- It can be denoted as; $y = f(x)$ is y is a function of x i.e. $f: X \rightarrow Y, \forall x \in X, \forall y \in Y$
- It is these linear functions that one can use to model real-world phenomena and analyze relationships between different quantities

Solving Systems of Linear Equations



- There exist different methods of solving systems of linear equations
- Several web-based calculators, mobile apps, and scientific calculators can be used to solve systems of linear equations

Substitution Method

Example 1: Solve the following linear system

$$2x + 3y = 19$$
$$3x - y = 12$$

Solution: Substitution involving replacing one of the two unknowns and working with only one

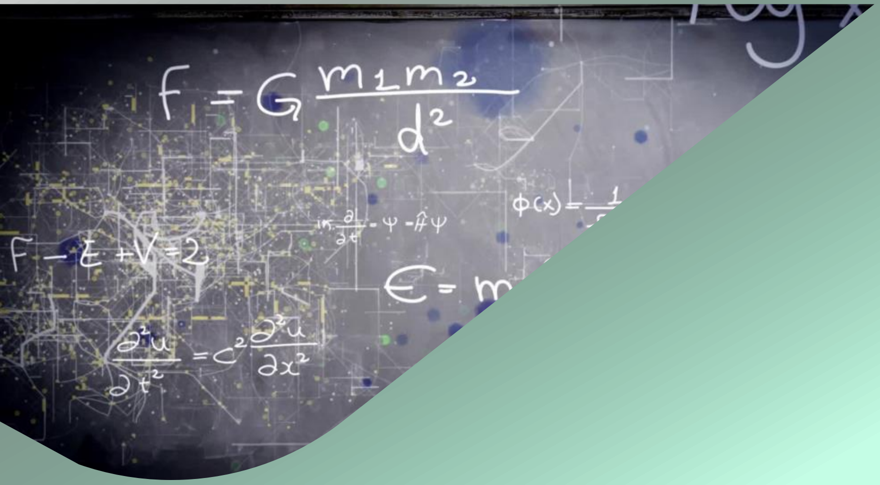
$$2x + 3y = 19$$
$$3x - y = 12$$

Substitution Method

Example 2: Solve the following linear system

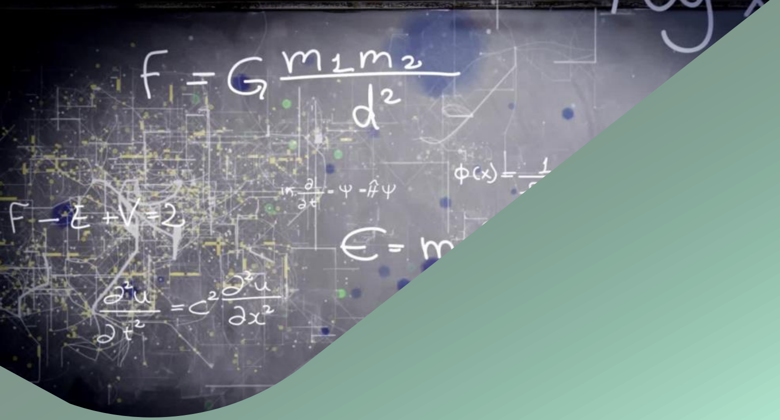
$$x + y - z = 5$$
$$3x - z = 10$$
$$2x - 3y + z = -1$$

$$x + y - z = 5$$
$$3x - z = 10$$
$$2x - 3y + z = -1$$



Elimination Method

This method involves removing or eliminating one unknown from the system by either adding or subtracting a multiple of one equation to the or from a multiple of another, so that you eventually you are left with only one unknown.



Elimination Method

Example 1: Solve the following system using elimination method;

$$7x + 3y = 16$$
$$5x + 2y = 11$$

Solution: One can start by eliminating either of the unknown. In our case we can start by eliminating the unknown x. We first multiply the first equation by 5 and the second equation by 7 and then subtract the resultant second equation from the resulting first equation.

Elimination Method

Example 1 contd: Solve the following system using elimination method;

$7x + 3y = 16$
$5x + 2y = 11$

$$(7x + 3y = 16) \times 5$$
$$(5x + 2y = 11) \times 7$$

Example 2: Solve the following system using elimination method;

$$x + 3y - 2z = -5$$
$$3x + y - 4z = 0$$
$$5x + 2y + z = 7$$

Elimination Method

Example 2 contd.: Solve the following system using elimination method;

$$x + 3y - 2z = -5$$
$$3x + y - 4z = 0$$
$$5x + 2y + z = 7$$

Elimination Method

Example 4: Use elimination method to solve the linear system;

$$\begin{aligned} 3x + 7y &= 16 \\ 6x + 14y &= 28 \end{aligned}$$

Solution:

Cramer's Rule

The Cramer's method utilizes the determinant of a matrix.

Now, given any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of matrix A denoted as

$$\det(A) = \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

Cramer's Rule

Cramer's rule, given a linear system; $\begin{array}{l} a_1x + a_2y = c_1 \\ b_1x + b_2y = c_2 \end{array}$ where x and y are variables and $a_1, a_2, b_1, b_2, c_1, c_2$ are known Real or Complex numbers then in matrix form we have the system as;

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then;

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} c_1 & a_2 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{c_1b_2 - a_2c_2}{a_1b_2 - a_2b_1}$$

$$y = \frac{\Delta y}{\Delta} = \frac{\begin{vmatrix} a_1 & c_1 \\ b_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1c_2 - b_1c_1}{a_1b_2 - a_2b_1}$$

Cramer’s Rule

Example 1: Solve the linear system; $\begin{array}{c} 3x + 5y = 1 \\ 5x - 2y = 12 \end{array}$

using the Cramer’s rule.

Solution:

Cramer's Rule

Example 2 : Solve the linear system using the Cramer's rule; $\begin{matrix} 5x - 2y = 13 \\ 10x - 4y = 23 \end{matrix}$

Solution:

Cramer's Rule

Example 3 : Solve the linear system using the Cramer's rule;

$$\begin{aligned} x - 3y &= 13 \\ 2x - 6y &= 26 \end{aligned}$$

Solution:

Cramer's Rule

Example 4: Use Cramer's rule to solve the following linear system with three unknowns;

$$\begin{aligned} 3x - 2y + z &= -5 \\ x + y - 3z &= 10 \\ x - 4y + 7z &= -25 \end{aligned}$$

- **Solution:** Then the determinant of the coefficients matrix

$$\Delta = \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -3 \\ 1 & -4 & 7 \end{vmatrix} = 3\begin{vmatrix} 1 & -3 \\ -4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} = 3(-5) + 2(10) + 1(-5) = 0$$

$$\Delta x = \begin{vmatrix} -5 & -2 & 1 \\ 10 & 1 & -3 \\ -25 & -4 & 7 \end{vmatrix} = -5\begin{vmatrix} 1 & -3 \\ -4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 10 & -3 \\ -25 & 7 \end{vmatrix} + 1\begin{vmatrix} 10 & 1 \\ -25 & -4 \end{vmatrix}$$

$$= -5(-5) + 2(-5) + 1(-15) = 0$$

$$x = \frac{\Delta x}{\Delta} = \frac{0}{0}$$

Cramer's Rule

Example 4 contd: Use Cramer's rule to solve the following linear system with three unknowns;

$$3x - 2y + z = -5$$
$$x + y - 3z = 10$$
$$x - 4y + 7z = -25$$

We can conclude our system has infinitely many solutions.

We can assume that $z = \alpha$ where $\alpha \in \mathbb{R}$ i.e. any real number.

Then we can multiply equation $x + y - 3z = 10$ by 3 and subtract it from equation $3x - 2y + z = -5$ to eliminate x and hence be able to get y (note that $z = \alpha$).

Cramer's Rule

Example 4 contd: Use Cramer's rule to solve the following linear system with three unknowns;

$$3x - 2y + z = -5$$
$$x + y - 3z = 10$$
$$x - 4y + 7z = -25$$

Graphical method

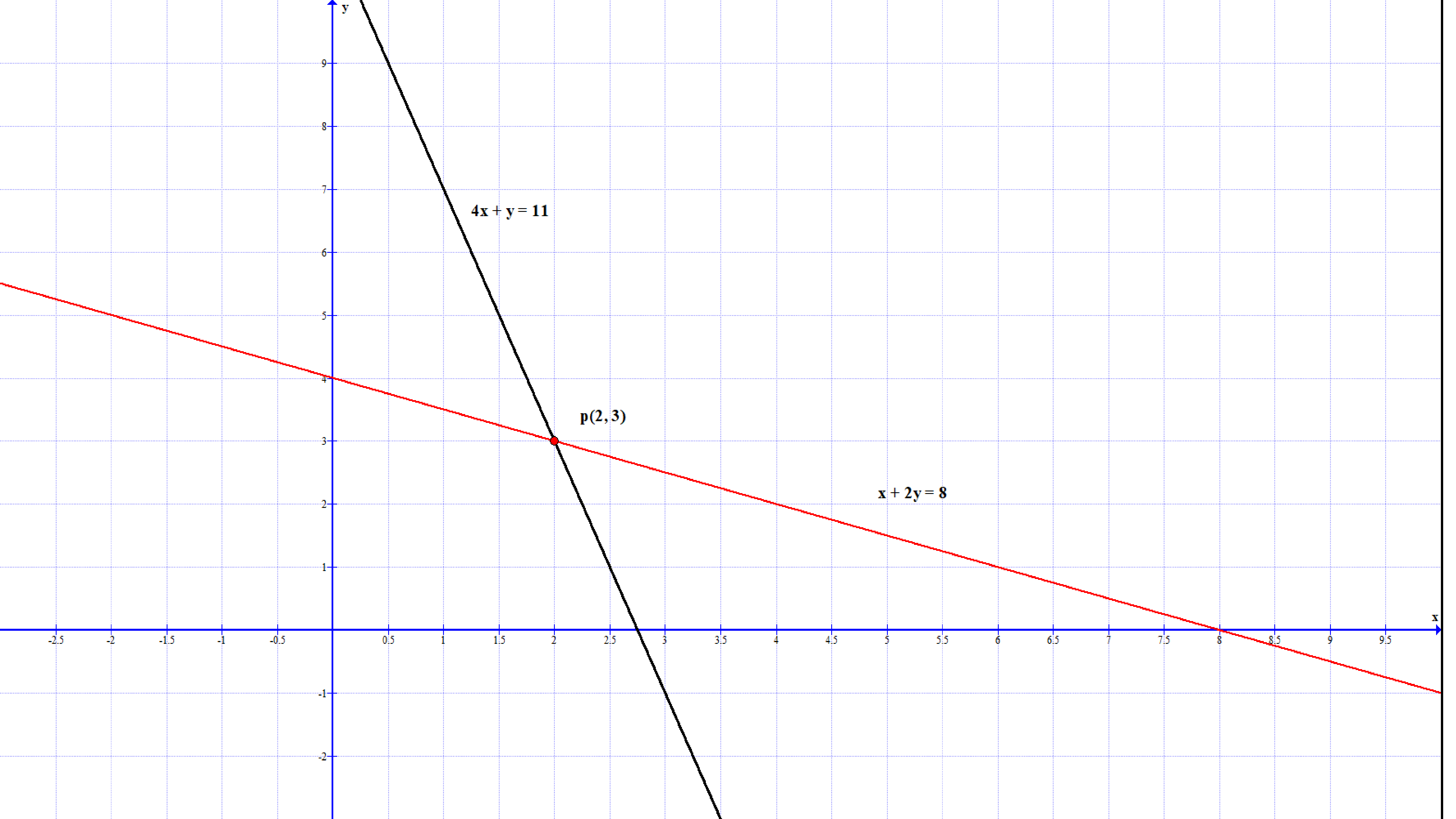
Example 1: Use graphical method to solve the linear system;
$$\begin{gathered} 4x + y = 11 \\ x + 2y = 8 \end{gathered}$$

Solution: To plot the graph manually, we require a table of integral values of x (domain) and the corresponding values of y (codomain).

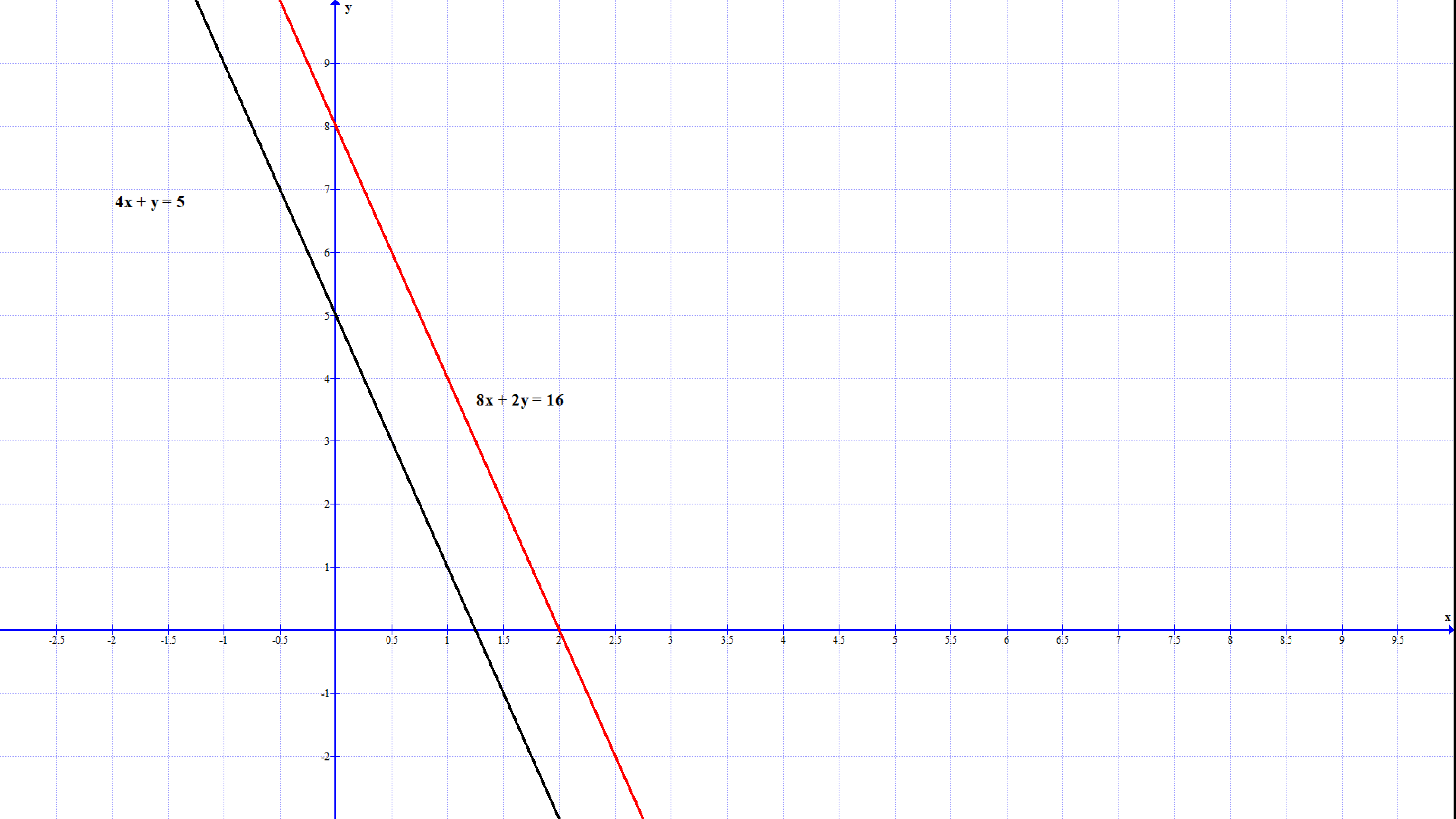
That is, we can take from – 2 to 4 and find the corresponding values of y.

Graphical method

Equation; $4x + y = 11 \Rightarrow y = 11 - 4x$							
x	-2	-1	0	1	2	3	4
y	19	15	11	7	3	-1	-5
Equation $x + 2y = 8 \Rightarrow y = \frac{8-x}{2}$							
x	-2	-1	0	1	2	3	4
y	5	4.5	4	3.5	3	2.5	2



y
x
4x + y = 11
p(2, 3)
x + 2y = 8



y
x
4x + y = 5
8x + 2y = 16

Demand and Supply Analysis

- A function is a relationship between the independent and dependent variables
- For example, the quantity demanded Q of a good depends on the market price P. Hence, we can say that Q is a function of P i.e. Q depends on P and denote it as;

$$Q = f(P)$$

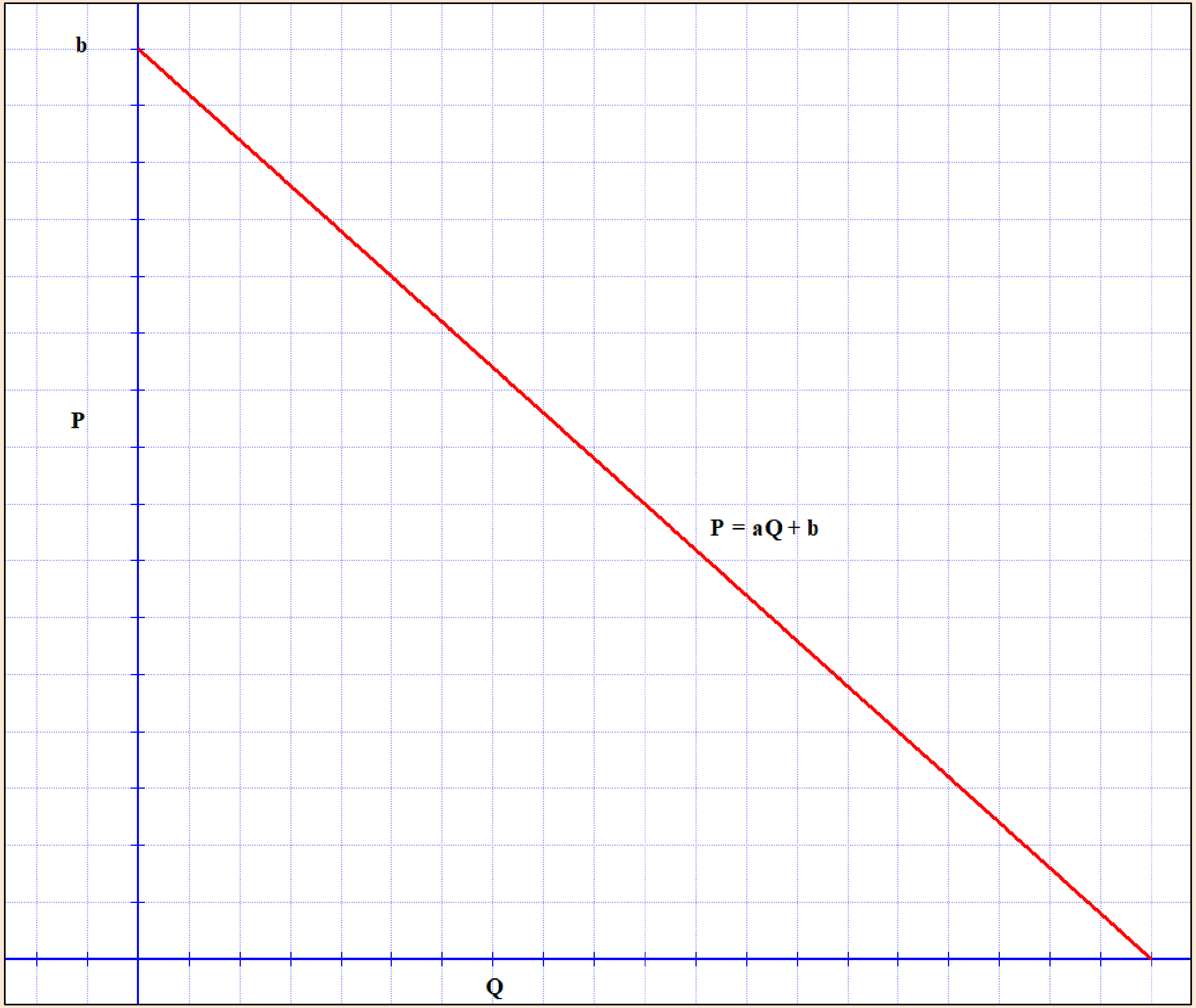
- Note the inverse function, that is, $P = g(Q)$.
- It can be hypothesized that; $\mathrm{P} = \mathrm{a}(\mathrm{Q}) + \mathrm{b}$ for some parameters a and b.
- In normal circumstances, the relation between market price of a commodity and the demand is more complicated than the representation in the linear function above.
- However, using a linear function makes it convenient in analysis of a problem i.e. modelling.

Example : Given the demand function $P = -3Q + 10$, determine the value of P when Q is 2 and Q when P is 4.

Solution:

i) $P = -3(2) + 10 = 10 - 6 = 4$

ii) $4 = -3Q + 10 \Rightarrow 3Q = 6 \therefore Q = 2$

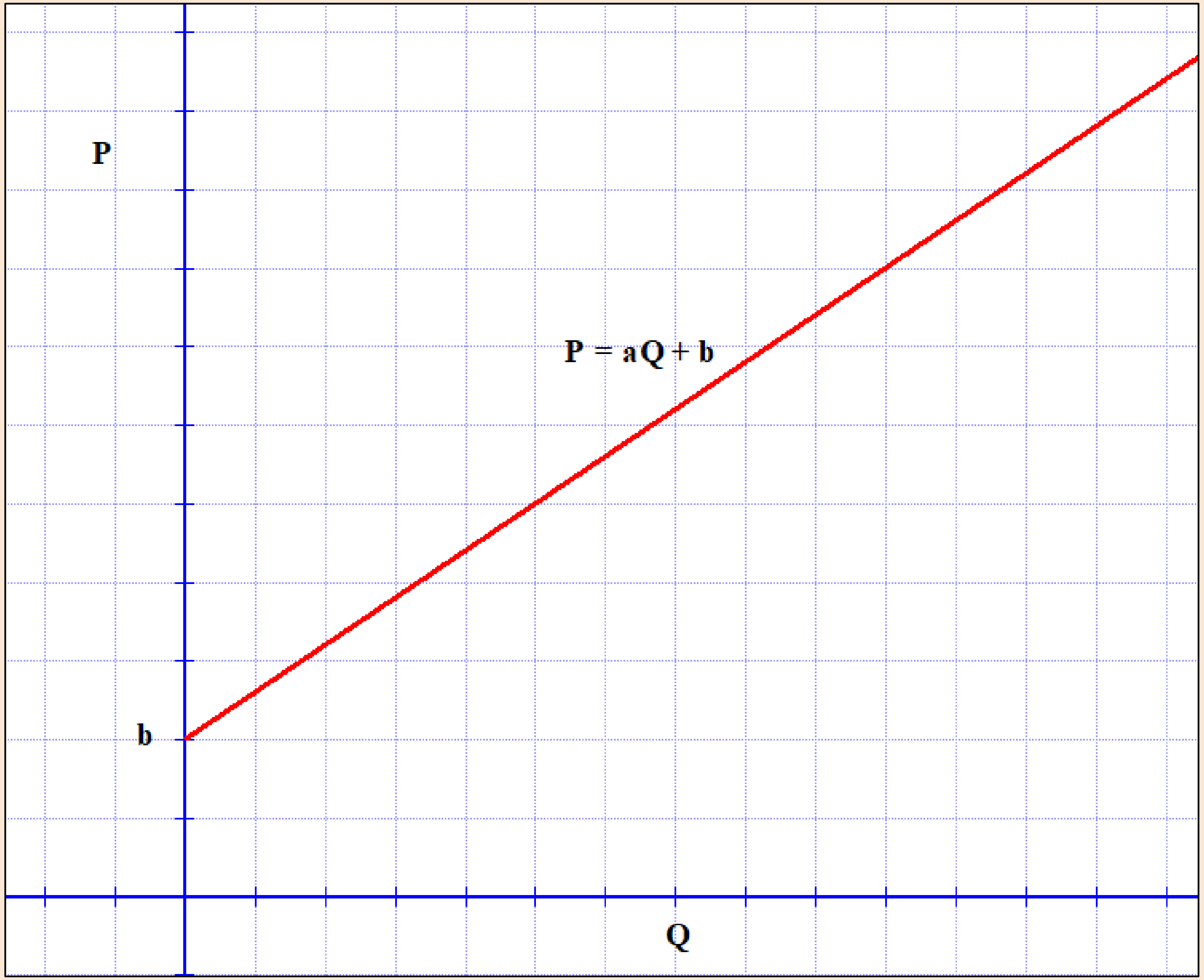


Example : Given the supply function $P = 2Q + 5$, determine the value of P when Q is 2 and Q when P is 15.

Solution:

i) $P = 2(2) + 5 = 9$

ii) $15 = 2Q + 5 \Rightarrow 2Q = 10 \therefore Q = 5$



Plotting both the Demand and Supply functions on the same axes displays two intersecting lines.

The point of intersection of the two functions is called the point of equilibrium.

This is where the Quantity Supplied equals the Quantity Demanded.

At this point we have the Price P_0 and Quantity Q_0 i.e. the equilibrium price and quantity

Example

The demand and supply functions of a commodity are given by;

$$P = -3Q_D + 40$$
$$P = \frac{1}{3}Q_S + 20$$

Where P, Q_D, Q_S denote the price, quantity demanded, and quantity supplied, respectively.

a) Determine the equilibrium price and quantity.

b) Find the effect on the equilibrium if there is a fixed tax of 4 shillings on each item.

Solution:

$$P = -3Q_D + 40$$

$$P = \frac{1}{3}Q_S + 20$$

In equilibrium, $Q_D = Q_S$, hence we can let $Q_D = Q_S = Q$

$$\Rightarrow P = -3Q + 40 \cdots \text{(i) and } P = \frac{1}{3}Q + 20 \cdots \text{(ii)}$$

Hence, $-3Q + 40 = \frac{1}{3}Q + 20$

$$20 = 3Q + \frac{1}{3}Q = \frac{10}{3}Q \therefore Q = 6$$

$$\Rightarrow P = \frac{1}{3}(6) + 20 = 22$$

$$\therefore \{P, Q\} = \{22,6\}$$

We subtract the 4 shillings from the sale of the good supplied i.e. $P - 4$.

Hence, we have $P - 4 = \frac{1}{3}Q_S + 20 \Rightarrow P = \frac{1}{3}Q_S + 24$

Again, since at equilibrium $Q_S = Q_D$ we have; $\frac{1}{3}Q_S + 24 = -3Q_D + 40$

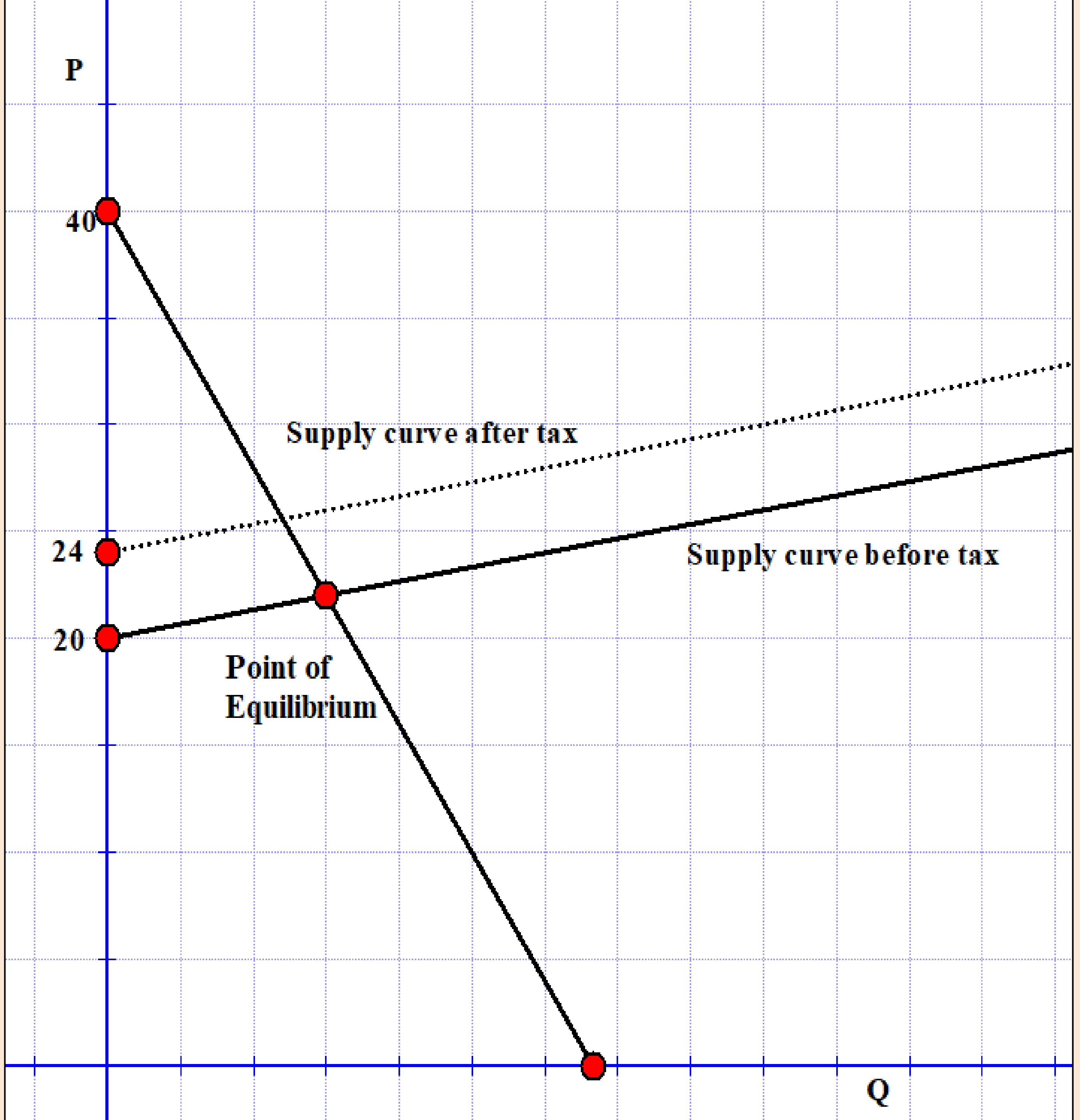
$\frac{10}{3}Q = 16 \therefore Q = 4.8$

$$\Rightarrow P = \frac{1}{3}(4.8) + 24 = 25.6$$

Note that the price after tax is 25.6 shillings.

This is an additional 3.6 shillings which the consumer pays.

Hence the remaining 0.4 shillings must be paid by the producer of the commodity



Example: Consider the linear system below of the demand and supply functions for two interdependent items, and then find the equilibrium price and quantity.

$$Q_{D1} = 20 - 5P_1 + P_2 \cdots \text{(i)}$$
$$Q_{D2} = 10 + 2P_1 - 2P_2 \cdots \text{(ii)}$$
$$Q_{S1} = -4 + 2P_1 \cdots \text{(iii)}$$
$$Q_{S2} = -2 + 3P_2 \cdots \text{(iv)}$$

With Q_{Di}, Q_{Si}, and P_i is the quantity demand, quantity supplied, and price of items i respectively.

Example contd: At the point of equilibrium, the quantity demanded equals the quantity supplied for each of the items.

Hence, we have; $Q_{D1} = Q_{S1}; Q_{D2} = Q_{S2}$

We can let $Q_{D1} = Q_{S1} = Q_1$ and $Q_{D2} = Q_{S2} = Q_2$.

Hence our equations (i) and (iii) for Item 1, becomes;

$Q_1 = 20 - 5P_1 + P_2$ and $Q_1 = -4 + 2P_1$

$\Rightarrow 20 - 5P_1 + P_2 = -4 + 2P_1$ (subtract $2P_1$ from both sides to get)

$20 - 7P_1 + P_2 = -4$ (Subtract 20 from both sides to get)

$$-7P_1 + P_2 = -24 \cdots (*)$$

Again, equations (ii) and (iv) for Item 2 becomes;

$Q_2 = 10 + 2P_1 - 2P_2$ and $Q_2 = -2 + 3P_2$

$\Rightarrow 10 + 2P_1 - 2P_2 = -2 + 3P_2$ (subtract $3P_2$ from both sides to get)

$10 + 2P_1 - 5P_2 = -2$ (subtract 10 from both sides to get)

$$2P_1 - 5P_2 = -12 \cdots (**)$$

We proceed to solve for P_1 and P_2 in equations (*) and (**). We can use substitution method (or any other of your choice), by making P_2 the subject in equation (*) i.e. $P_2 = 7P_1 - 24$. Then we replace P_2 in equation (**) with $7P_1 - 24$ to get;

$2P_1 - 5(7P_1 - 24) = -12$ (Expand the brackets to get)

$2P_1 - 35P_1 + 120 = -12$ (simplify to get)

$-33P_1 + 120 = -12$ (subtract 120 from both sides to get)

$$-33P_1 = -132 \therefore P_1 = 4$$

$$\therefore P_2 = 7P_1 - 24 = 28 - 24 = 4$$

We can next plugin these prices into the original equations to get;

$$Q_1 = 20 - 5P_1 + P_2 = 20 - 2\,(4) + 4 = 8$$

$$Q_2 = 10 + 2P_1 - 2P_2 = 10 + 2(4) - 2(4) = 10$$

References

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