Business Mathematics

Lecture 1

Linear Equations

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Introduction to Lecture 1

This lecture introduces you to linear equations, solving linear equations in two and three unknowns, and their applications to solving economics and business-related problems. We shall demonstrate how linear equations are used in supply and demand analysis as just a snippet on how linear equations can be used to comprehend real-life business phenomena.

Further Readings

These notes have been derived from diverse resources. These resources are recommended for further reading to gain more insights on the application of linear equations to business or commercial arithmetic. These are (Jacques, 2006; P. Kahenya, 2017; P. N. Kahenya, 2021; Lay, 2003; Lay et al., 2016; Murray & Robert, 2009).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Define a linear equation

(ii) Solve linear systems involving two and three unknowns.

(iii) Apply linear equations in solving economics and business-related problems.

Definition of Terms

Definition 1: Linear equation

It is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$ where $x_1, x_2, \cdots, x_n$ are variables and $a_1, a_2, \cdots, a_n, k$ are known numbers, real or complex. For example;

a) $2x_1 + 3x_2 = 7$

b) $x_1 - 2x_2 + 4x_3 = 0$

c) $3x + 8y = 9$

d) $a + 2b + 4c - 5d = 0$

Linear equations can be visualized as straight lines in 2D and 3D. Linear equations can also be seen as planes in 3D. phenomena in economics and business are best comprehended when modeled or expressed as linear relationships.

For example, the supply and demand relationship may well be understood when expressed as a linear relationship. The gradient function of linear graphs is equally important and may express different meanings depending on what quantities are expressed in the linear equation..

Definition 2: System of linear equations.

A linear system is a collection of one or more linear equations involving the same variables. For example;

a) $$\begin{aligned} 3x + 6y &= 19 \\ x - 2y &= 7 \end{aligned}$$

b) $$\begin{aligned} 2p + 3q + 5r &= 8 \\ p + q - r &= 3 \\ 7p + 2q &= 11 \end{aligned}$$

Definition 3: Solutions of a linear system

A solution of a linear system is a list of numbers $s_1, s_2, \cdots, s_n$ that satisfy an equation in a linear system. That is given a linear system $a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$ then the set of all possible solutions $s_1, s_2, \cdots, s_n$ are substitutes of the variables $x_1, x_2, \cdots, x_n$.

A system of linear equations is either consistent or inconsistent. That is, it is consistent if it has a unique solution or infinitely many solutions, and it is inconsistent if it has no solutions.

Definition 4: Function

A function is a relation between a set of inputs, called the domain, and a set of possible outputs, called the codomain, with the property that each input is related to exactly one output. It can be denoted as;

$y = f(x)$ is y is a function of x i.e. $f: X \rightarrow Y, \forall x \in X, \forall y \in Y$

It is these linear functions that one can use to model real-world phenomena and analyze relationships between different quantities.

Solving Systems of Linear Equations

There exist different methods of solving systems of linear equations. Several web-based calculators, mobile apps, and scientific calculators can be used to solve systems of linear equations.

Substitution Method

Example 1: Solve the following linear system; $\begin{matrix} 2x + 3y = 19 \\ 3x - y = 12 \end{matrix}$

Solution: Substitution involving replacing one of the two unknowns and working with only one. You need to identify which is the easiest to replace. In our system if we consider equation;

$3x - y = 12$ then we can make y the subject to get $y = 3x - 12$.

We then replace y with $(3x - 12\ )$in equation $2x + 3y = 19$ to get;

$$2x + 3(3x - 12) = 19$$

Expanding the brackets to get; $2x + 9x - 36 = 19$

$$11x = 19 + 36$$

$$11x = 55 \therefore x = 5$$

Hence $y = 3x - 12 = 3(5) - 12 = 3$. The solution set is $\{x, y\} = \{5,3\}$.

Example 2: Solve the following linear system; $\begin{matrix} x + y - z = 5 \\ 3x - z = 10 \\ 2x - 3y + z = -1 \end{matrix}$

Solution: The system has 3 unknowns. Hence we must replace 2 unknowns systematically to remain with one unknown. We can first consider equation $3x - z = 10$ since it is easier to make z the subject of the linear equation to get $z = 3x - 10$ and then use $3x - 10$ to replace z in the first or third equation.

Suppose we replace z with $3x - 10$ in equation $x + y - z = 5$ to get;

$$x + y - (3x - 10) = 5$$

$$x + y - 3x + 10 = 5$$

$$y - 2x = -5 \cdots (*)$$

Next we replace z with $3x - 10$ in equation $2x - 3y + z = -1$ to get

$$2x - 3y + (3x - 10) = -1$$

$$2x - 3y + 3x - 10 = -1$$

$$5x - 3y = 9 \cdots (**)$$

Note that equations (*) and (**) have two unknowns x and y.

From equation (*) we have $y = 2x - 5$. We then replace y with $2x - 5$ in equation (**) to get

$$5x - 3(2x - 5) = 9$$

$$5x - 6x + 15 = 9$$

$$-x = -6 \therefore x = 6$$

Since $y = 2x - 5 \Rightarrow y = 2(6) - 5 = 12 = 5 = 7$

Also $z = 3x - 10 \Rightarrow z = 3(6) - 10 = 18 - 10 = 8$

The unique solution is $\{x, y, z\} = \{6,7,8\}$

Elimination Method

This method involves removing or eliminating one unknown from the system by either adding or subtracting a multiple of one equation to the or from a multiple of another, so that you eventually you are left with only one unknown.

Example 1: Solve the following system using elimination method; $\begin{matrix} 7x + 3y = 16 \\ 5x + 2y = 11 \end{matrix}$

Solution: One can start by eliminating either of the unknown. In our case we can start by eliminating the unknown x. We first multiply the first equation by 5 and the second equation by 7 and then subtract the resultant second equation from the resulting first equation i.e.

$$\begin{matrix} (7x + 3y = 16) \times 5 \\ (5x + 2y = 11) \times 7 \end{matrix} \Rightarrow \begin{array}{r} 35x + 15y = 80 \\ 35x + 14y = 77 \quad - \\ \hline \underline{y = 3} \end{array}$$

We can proceed to substitute y with 3 in any equation to get x i.e.

$$7x + 3(3) = 16 \Rightarrow 7x + 9 = 16$$

$$7x = 16 - 9 \Rightarrow 7x = 7 \therefore x = 1$$

Alternatively, we multiply the first equation by 2 and the second equation by 3 and then subtract the resultant second equation from the resulting first equation i.e.

$$\begin{matrix} (7x + 3y = 16) \times 2 \\ (5x + 2y = 11) \times 3 \end{matrix} \Rightarrow \begin{array}{r} 14x + 6y = 32 \\ 15x + 6y = 33 \quad - \\ \hline \underline{-x = -1} \end{array} \quad \therefore x = 1$$

Hence the solution set is $\{x, y\} = \{1,3\}$

Example 2: Solve the following linear system;

$$\begin{aligned} x + 3y - 2z &= -5 \\ 3x + y - 4z &= 0 \\ 5x + 2y + z &= 7 \end{aligned}$$

Solution: We multiply the first equation one with 3 and subtract equation two from the resultant equation one to eliminate x i.e.

$$(x + 3y - 2z = -5) \times 3$$
$$3x + y - 4z = 0$$

$$\begin{array}{r} 3x + 9y - 6z = -15 \\ 3x + y - 4z = 0 \\ \hline 8y - 2z = -15 \cdots (*) \end{array} \; -$$

We need another equation like (*). We multiply the first equation with 5 and subtract equation 3 to eliminate x again i.e.

$$(x + 3y - 2z = -5) \times 5$$
$$5x + 2y + z = 7$$

$$\begin{array}{r} 5x + 15y - 10z = -25 \\ 5x + 2y + z = 7 \\ \hline 13y - 11z = -32 \cdots (**) \end{array} \; -$$

We now have two equations (*) and (**) with unknowns y and z. To eliminate y we can multiply equation (*) by 13 and equation (**) by 8 and subtract the two.

$$(8y - 2z = -15) \times 13$$
$$(13y - 11z = -32) \times 8$$

To get; $\begin{array}{r} 104y - 26z = -195 \\ 104y - 88z = -256 \\ \hline 62z = 61 \end{array} \; - \quad \Rightarrow z = \frac{61}{62}$

We can then use substitution method to get the other unknowns. From (*) we have;

$$8y - 2\left(\frac{61}{62}\right) = -15 \quad \Rightarrow 8y - \frac{61}{31} = -15$$

$$8y = \frac{61}{31} - 15 \therefore y = -\frac{101}{62}$$

From equation $x + 3y - 2z = -5$ then we have;

$$x = 2z - 3y - 5 = 2\left(\frac{61}{62}\right) - 3\left(-\frac{101}{62}\right) - 5 = \frac{61}{31} + \frac{303}{62} - 5 = \frac{115}{62}$$

hence the solution set is $\{x, y, z\} = \left\{\frac{115}{62}, -\frac{101}{62}, \frac{61}{62}\right\}$

Example 3: Solve the following linear system;

$$\begin{aligned} 3x + 2y - z &= 10 \\ x + 3y - 5z &= 16 \\ 6x + 4y - 2z &= 20 \end{aligned}$$

Solution: Note that we can multiply the first equation with 2 and subtract the third equation to get;

$$\begin{array}{l} (3x + 2y - z = 10) \times 2 \\ \underline{6x + 4y - 2z = 20} \end{array}$$

To get;

$$\begin{array}{ll} 6x + 4y - 2z = 20 & \\ \underline{6x + 4y - 2z = 20} & - \\ 0 + 0 + 0 = 0 & \end{array}$$

We are getting $0 = 0$! Which is a valid statement. It is not possible to eliminate any unknown. In this situation we conclude the system has many solutions. One can use Gaussian elimination method to find the general solution of this statement. However, it implies that z is a free variable and can take any value.

Example 4: Use elimination method to solve the linear system;

$$\begin{aligned} 3x + 7y &= 16 \\ 6x + 14y &= 28 \end{aligned}$$

Solution: We can multiply the first equation with 2 and subtract the second equation to eliminate x. However, in the process we also eliminate y. That is,

$$\begin{array}{c} (3x + 7y = 16) \times 2 \\ 6x + 14y = 28 \end{array}$$

To get;

$$\begin{array}{cl} 6x + 14y = 32 & \\ \underline{6x + 14y = 28} & - \\ \underline{0 = 4} & \end{array}$$

$0 = 4$ is invalid. Hence the system is inconsistent, and it has no solutions.

Cramer's Rule

The Cramer's method utilizes the determinant of a matrix.

Now, given any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of matrix A denoted as

$$\det(A) = \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore by Cramer's rule, given a linear system; $\begin{matrix} a_1x + a_2y = c_1 \\ b_1x + b_2y = c_2 \end{matrix}$ where x and y are variables and $a_1, a_2, b_1, b_2, c_1, c_2$ are known Real or Complex numbers then in matrix form we have the system as;

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then;

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} c_1 & a_2 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{c_1b_2 - a_2c_2}{a_1b_2 - a_2b_1}$$

Where Δx is the determinant with respect to variable x, got by replacing the coefficients of x with the constants and finding the determinant of the resultant matric. While Δ is the determinant of the coefficient's matrix. Again, to get y we have;

$$y = \frac{\Delta y}{\Delta} = \frac{\begin{vmatrix} a_1 & c_1 \\ b_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1c_2 - b_1c_1}{a_1b_2 - a_2b_1}$$

Where Δy is the determinant with respect to variable y got by replacing the coefficients of y with the constants and finding the determinant of the resultant matric. While Δ is the determinant of the coefficient's matrix. Note that in both cases the denominator is the same.

Example 1: Solve the linear system; $\begin{matrix} 3x + 5y = 1 \\ 5x - 2y = 12 \end{matrix}$ using the Cramer's rule.

Solution: First write the system into matrix form to get; $\begin{pmatrix} 3 & 5 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$

Matrix $\begin{pmatrix} 3 & 5 \\ 5 & -2 \end{pmatrix}$ is the coefficients matrix since it consists of the coefficients of the variables x and y of the system. Of course, matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ is the variables matrix while matrix $\begin{pmatrix} 1 \\ 12 \end{pmatrix}$ is the constant matrix. Then;

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 1 & 5 \\ 12 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{-2 - 60}{-6 - 25} = \frac{-62}{-31} = 2$$

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{36 - 5}{-6 - 25} = \frac{31}{-31} = -1$$

hence the solution set is; $\{x, y\} = \{-1,2\}$

Example 2: Solve the linear system using the Cramer's rule; $\begin{matrix} 5x - 2y = 13 \\ 10x - 4y = 23 \end{matrix}$

Solution: In matrix form we have; $\begin{pmatrix} 5 & -2 \\ 10 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 23 \end{pmatrix}$

Hence;

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 13 & -2 \\ 23 & -4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 10 & -4 \end{vmatrix}} = \frac{-52 + 46}{-20 + 20} = \frac{-6}{0} \text{ invalid}$$

Note that the determinant of the coefficient's matrix is zero i.e. the coefficients matrix $\begin{pmatrix} 5 & -2 \\ 10 & -4 \end{pmatrix}$ is a singular matrix. Such matrices have no inverse. Therefore, our system is inconsistent. It has no solution.

Example 3: Solve the linear system using the Cramer's rule; $\begin{matrix} x - 3y = 13 \\ 2x - 6y = 26 \end{matrix}$

Solution: In matrix form we have; $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 26 \end{pmatrix}$

Hence;

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 13 & -3 \\ 26 & -6 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{-78 + 78}{-6 + 6} = \frac{0}{0} \text{ indeterminate}$$

When you get such an indeterminate form $\frac{0}{0}$, then the system is consistent but has infinitely many solutions. A closer observation will see that the second equation is a multiple of the first equation.

Remark: If the coefficients matrix is a singular matrix i.e. its determinant is zero, then the system may have infinitely many solutions or no solutions. One need to investigate further using different methods to determine which is the case.

Example 4: Use Cramer's rule to solve the following linear system with three unknowns;

$$\begin{gathered} 3x-2y+z=-5 \\ x+y-3z=10 \\ x-4y+7z=-25 \end{gathered}$$

Solution: Rewriting the system in matrix form we get;

$$\begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -5 \\ 10 \\ -25 \end{pmatrix}$$

Then the determinant of the coefficients matrix

$$\Delta=\begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -3 \\ 1 & -4 & 7 \end{vmatrix}=3\begin{vmatrix} 1 & -3 \\ -4 & 7 \end{vmatrix}-(-2)\begin{vmatrix} 1 & -3 \\ 1 & 7 \end{vmatrix}+1\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix}=3(-5)+2(10)+1(-5)=0$$

We need to find the determinant with respect to $x, \Delta x$ i.e.

$$\Delta x=\begin{vmatrix} -5 & -2 & 1 \\ 10 & 1 & -3 \\ -25 & -4 & 7 \end{vmatrix}=-5\begin{vmatrix} 1 & -3 \\ -4 & 7 \end{vmatrix}-(-2)\begin{vmatrix} 10 & -3 \\ -25 & 7 \end{vmatrix}+1\begin{vmatrix} 10 & 1 \\ -25 & -4 \end{vmatrix}$$

$$=-5(-5)+2(-5)+1(-15)=0$$

This implies that $x=\frac{\Delta x}{\Delta}=\frac{0}{0}$. This is an indeterminant form. We can conclude our system has infinitely many solutions. We can assume that $z=\alpha$ where $\alpha \in \mathbb{R}$ i.e. any real number. Then we can multiply equation $x+y-3z=10$ by 3 and subtract it from equation $3x-2y+z=-5$ to eliminate x and hence be able to get y (note that $z=\alpha$). That is;

$$\begin{aligned} &3x-2y+z=-5 \\ &(x+y-3z=10)\times 3 \\ &x-4y+7z=-25 \end{aligned}$$

To get;
$$\begin{array}{r} 3x-2y+z=-5 \\ 3x+3y-9z=30 \quad - \\ \hline -5y+10z=-35 \cdots (*) \end{array}$$

From equation (*) we have; $5y=35+10z=35+10\alpha$

Divide every term by 5 to get; $y=7+2\alpha$

Finally to get x we replace y and z in equation $x+y-3z=10$ (you can also use the other two equations) to get; $x=10-y+3z=10-(7+2\alpha)+3\alpha=10-7-2\alpha+3\alpha=3+\alpha$

The general solution of the system is; $\{x,y,z\}=\{\alpha+3,2\alpha+7,\alpha\}, \alpha \in \mathbb{R}$

Graphical method

Linear systems with unique solutions will have their graph functions intersecting at a point. Systems with infinitely many solutions will end up being one single line i.e. coincident line. While inconsistent systems will have parallel lines. Plotting linear equations with more than 2 unknowns may not be practical unless one uses a computer algebraic system such as GeoGebra, Maple, among others.

Example 1: Use graphical method to solve the linear system; $\begin{matrix} 4x + y = 11 \\ x + 2y = 8 \end{matrix}$

Solution: To plot the graph manually, we require a table of integral values of x (domain) and the corresponding values of y (codomain). That is, we can take from – 2 to 4 and find the corresponding values of y.

Equation; $4x + y = 11 \Rightarrow y = 11 - 4x$							
x	-2	-1	0	1	2	3	4
y	19	15	11	7	3	-1	-5
Equation $x + 2y = 8 \Rightarrow y = \frac{8-x}{2}$							
x	-2	-1	0	1	2	3	4
y	5	4.5	4	3.5	3	2.5	2

Next we plot the points on the xy-plane to get the graph below. The two functions intersect at point p(2, 3). Hence our solution set is $\{x, y\} = \{2,3\}$

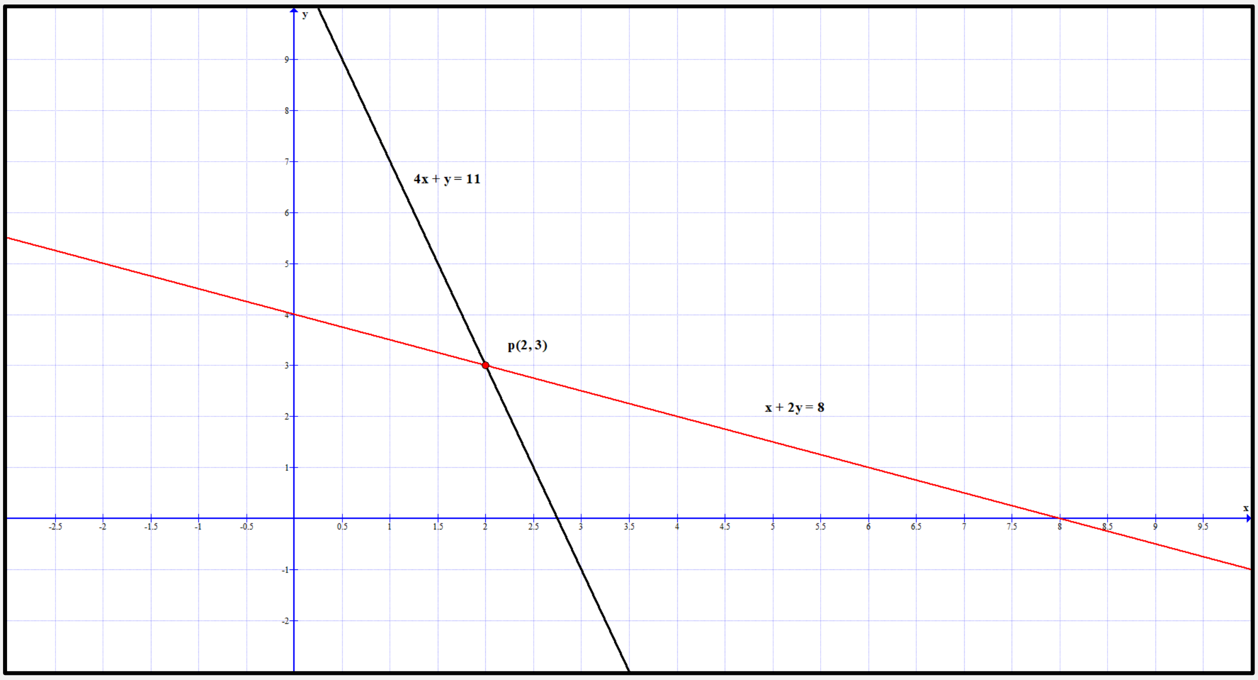


Figure 1: Intersection lines

Example 2: Use graphical method to solve the following system; $\begin{array}{c} 4x + y = 5 \\ 8x + 2y = 16 \end{array}$

Solution: We plot the functions over the domain $-2 \le x \le 4$

Equation; $4x + y = 5 \Rightarrow y = 5 - 4x$							
x	-2	-1	0	1	2	3	4
y	13	9	5	1	-3	-7	-11
Equation $8x + 2y = 16 \Rightarrow y = \frac{16-8x}{2} = 8 - 4x \therefore y = 8 - 4x$							
x	-2	-1	0	1	2	3	4
y	16	12	8	4	0	-4	-8

Next we plot the two functions, and we get parallel lines. This means that the system has no solution.

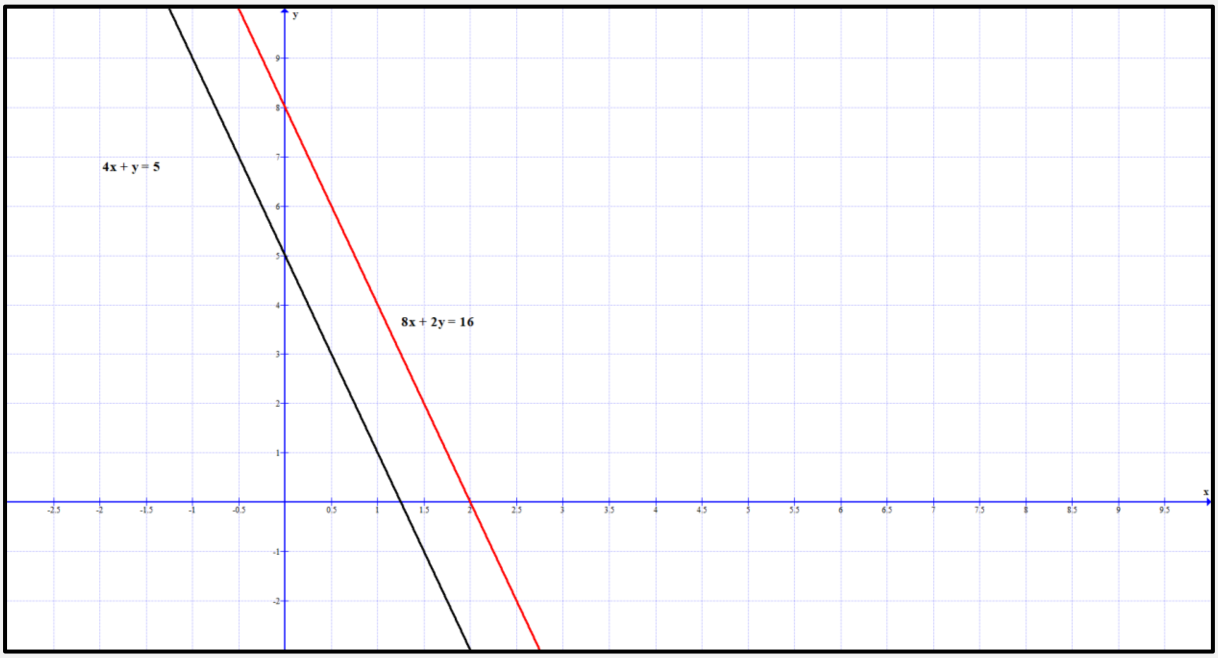


Figure 2: Parallel lines

Demand and Supply Analysis

In this section we demonstrate with simple examples how linear equations can be used to model and solve real-world problems in economics and business.

Example 1: A function is a relationship between the independent and dependent variables. A function denoted $y = f(x)$ meaning that y is a function of x, implies that y depends on x.

For example, the quantity demanded Q of a good depends on the market price P. Hence we can say that Q is a function of P i.e. Q depends on P and denote it as;

$$Q = f(P)$$

In fact, this is the Demand function.

Note the inverse function, that is, $P = g(Q)$. We can proceed to plot this on a plane with P on the vertical axis and Q on the horizontal axis. It can be hypothesized that;

$\mathrm{P} = \mathrm{a}(\mathrm{Q}) + \mathrm{b}$ for some parameters a and b.

In normal circumstances, the relation between market price of a commodity and the demand is more complicated than the representation in the linear function above. However, using a linear function makes it convenient in analysis of a problem i.e. modelling.

'Modelling is the process of identifying key features of the real-world and making appropriate simplifications and assumptions' (Jacques, 2006). Models help in understanding and predicating phenomena. Hence the graph of the demand function $\mathrm{P} = -\mathrm{a}Q + \mathrm{b}$ shows that the demand of a commodity will fall as the market price increases. In other words, variable P is a decreasing function of Q. Note that the gradient of the curve is negative.

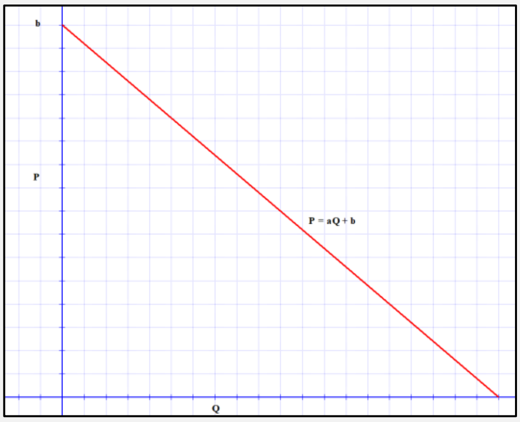


Figure 3: Linear graph - Demand Function

Example 2: Given the demand function $P = -3Q + 10$, determine the value of P when Q is 2 and Q when P is 4.

Solution:

i) $P = -3(2) + 10 = 10 - 6 = 4$

ii) $4 = -3Q + 10 \Rightarrow 3Q = 6 \therefore Q = 2$

Example 3: The supply function graph represents the relationship between the quantity Q of commodity produced in the market and the price P of the commodity. Unlike the demand function curve, in the supply function curve, as the prices rises, the supply rises too. The graph has a positive gradient. P is an increasing function of Q.

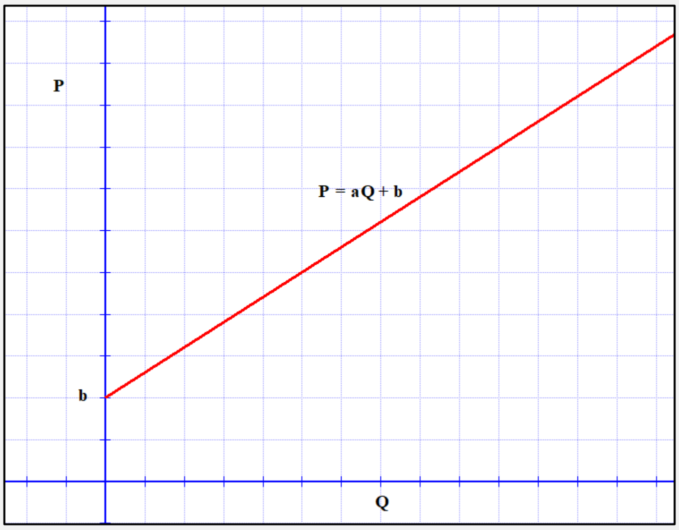


Figure 4: Linear Graph - Supply Function

Example 4: Plotting both the Demand and Supply functions on the same axes displays two intersecting lines. The point of intersection of the two functions is called the point of equilibrium. This is where the Quantity Supplied equals the Quantity Demanded. At this point we have the Price P_0 and Quantity Q_0 i.e. the equilibrium price and quantity.

Example 5: The demand and supply functions of a commodity are given by;

$$P = -3Q_D + 40$$
$$P = \frac{1}{3}Q_S + 20$$

Where P, Q_D, Q_S denote the price, quantity demanded, and quantity supplied respectively.

a) Determine the equilibrium price and quantity.

b) Find the effect on the equilibrium if there is a fixed tax of 4 shillings on each item.

Solution:

a) In equilibrium, $Q_D = Q_S$, hence we can let $Q_D = Q_S = Q$

$\Rightarrow P = -3Q + 40 \cdots \text{(i)}$ and $P = \frac{1}{3}Q + 20 \cdots \text{(ii)}$

Hence, $-3Q + 40 = \frac{1}{3}Q + 20$

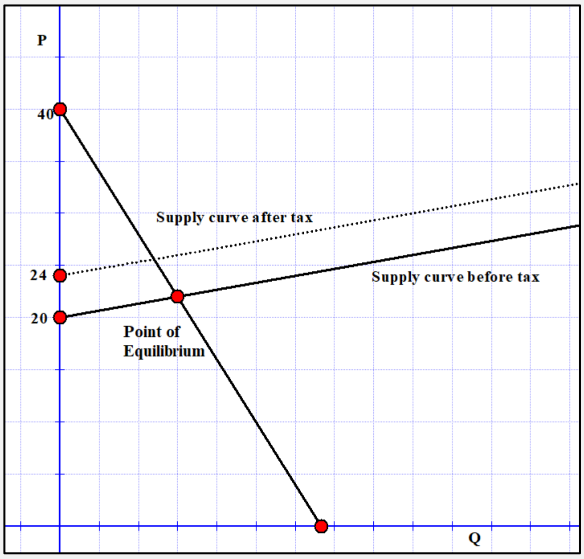
$$20 = 3Q + \frac{1}{3}Q = \frac{10}{3}Q \therefore Q = 6 \Rightarrow P = \frac{1}{3}(6) + 20 = 22 \therefore \{P, Q\} = \{22,6\}$$

b) We subtract the 4 shillings from the sale of the good supplied i.e. $P - 4$. Hence we have

$P - 4 = \frac{1}{3}Q_S + 20 \Rightarrow P = \frac{1}{3}Q_S + 24$

Again, since at equilibrium $Q_S = Q_D$ we have; $\frac{1}{3}Q_S + 24 = -3Q_D + 40$

$\frac{10}{3}Q = 16 \therefore Q = 4.8 \Rightarrow P = \frac{1}{3}(4.8) + 24 = 25.6$



Note that the price after tax is 25.6 shillings. This is an additional 3.6 shillings which the consumer pays, and hence the remaining 0.4 shillings must be paid by the producer of the commodity.

Example 6: Consider the linear system below of the demand and supply functions for two interdependent items, and then find the equilibrium price and quantity.

$$Q_{D1} = 20 - 5P_1 + P_2 \cdots \text{(i)}$$
$$Q_{D2} = 10 + 2P_1 - 2P_2 \cdots \text{(ii)}$$
$$Q_{S1} = -4 + 2P_1 \cdots \text{(iii)}$$
$$Q_{S2} = -2 + 3P_2 \cdots \text{(iv)}$$

With Q_{Di}, Q_{Si}, and P_i is the quantity demand, quantity supplied, and price of items i respectively.

Solution: At the point of equilibrium, the quantity demanded equals the quantity supplied for each of the items. Hence we have; $Q_{D1} = Q_{S1}; Q_{D2} = Q_{S2}$

We can let $Q_{D1} = Q_{S1} = Q_1$ and $Q_{D2} = Q_{S2} = Q_2$.

Hence our equations (i) and (iii) for Item 1, becomes;

$Q_1 = 20 - 5P_1 + P_2$ and $Q_1 = -4 + 2P_1$

$\Rightarrow 20 - 5P_1 + P_2 = -4 + 2P_1$ (subtract $2P_1$ from both sides to get)

$20 - 7P_1 + P_2 = -4$ (Subtract 20 from both sides to get)

$$-7P_1 + P_2 = -24 \cdots (*)$$

Again, equations (ii) and (iv) for Item 2 becomes;

$Q_2 = 10 + 2P_1 - 2P_2$ and $Q_2 = -2 + 3P_2$

$\Rightarrow 10 + 2P_1 - 2P_2 = -2 + 3P_2$ (subtract $3P_2$ from both sides to get)

$10 + 2P_1 - 5P_2 = -2$ (subtract 10 from both sides to get)

$$2P_1 - 5P_2 = -12 \cdots (**)$$

We proceed to solve for P_1 and P_2 in equations (*) and (**). We can use substitution method (or any other of your choice), by making P_2 the subject in equation (*) i.e. $P_2 = 7P_1 - 24$. Then we replace P_2 in equation (**) with $7P_1 - 24$ to get;

$2P_1 - 5(7P_1 - 24) = -12$ (Expand the brackets to get)

$2P_1 - 35P_1 + 120 = -12$ (simplify to get)

$-33P_1 + 120 = -12$ (subtract 120 from both sides to get)

$$-33P_1 = -132 \therefore P_1 = 4$$

$$\therefore P_2 = 7P_1 - 24 = 28 - 24 = 4$$

We can next plugin these prices into the original equations to get;

$$Q_1 = 20 - 5P_1 + P_2 = 20 - 2\,(4) + 4 = 8$$

$$Q_2 = 10 + 2P_1 - 2P_2 = 10 + 2(4) - 2(4) = 10$$

Exercise

1) Find the determinant of the following matrices;

a) $\begin{pmatrix} 3 & 1 \\ 2 & -7 \end{pmatrix}$ c) $\begin{pmatrix} 0 & -1 \\ 0 & -7 \end{pmatrix}$ e) $\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 & 1 \\ 4 & 5 & 9 \\ -6 & 1 & 3 \end{pmatrix}$ d) $\begin{pmatrix} 21 & 10 & 11 \\ -4 & 15 & 9 \\ 6 & 1 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 2 & 3 & 1 & 5 \\ 0 & -1 & 2 & 6 \\ 0 & 1 & 4 & 5 \\ 3 & 7 & 0 & 3 \end{pmatrix}$

2) Solve the following linear systems using substitution method.

a) $\begin{matrix} 2x + 3y = 8 \\ 4x - y = 2 \end{matrix}$ b) $\begin{matrix} x + y + z = 17 \\ x - 2z = -15 \\ 2x + y + 3z = 38 \end{matrix}$ c) $\begin{matrix} x + 3y = -3 \\ x + y = 1 \end{matrix}$

3) Solve the following linear systems using purely elimination method

a) $\begin{matrix} 4x + 5y = 13 \\ 8x + 10y = 27 \end{matrix}$ b) $\begin{matrix} 2x + 3y + 4z = 10 \\ 4x + 6y + 8z = 20 \\ x - 2y - z = 13 \end{matrix}$ c) $\begin{matrix} 6p - 2q = 2 \\ p + 3q = 17 \end{matrix}$

4) Solve the following linear systems using Cramer's rule

a) $\begin{matrix} 3x + 2y = 9 \\ x + 7y = 22 \end{matrix}$ b) $\begin{matrix} x + 3y + 2z = 12 \\ 2x + 6y + 4y = 28 \\ x + 3y + 2z = 16 \end{matrix}$ c) $\begin{matrix} 5x + 3y = 25 \\ 10x + 6y = 32 \end{matrix}$

5) Draw the graphs of the following systems to find the solution set.

a) $\begin{matrix} p = 2q + 4 \\ p = -3q + 8 \end{matrix}$ b) $\begin{matrix} 3x = 2y + 4 \\ 5y = 10x - 3 \end{matrix}$ c) $\begin{matrix} y = 3x + 2 \\ y = 9 - 5x \end{matrix}$

6) Solve questions 10 to 13 in Practice Problems in (Jacques, 2006, p. 64).

7) Use a computer algebra system of your choice to confirm your solutions in questions 1 to 5 above.

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