Business Mathematics

Lecture 2

Non-linear Equations

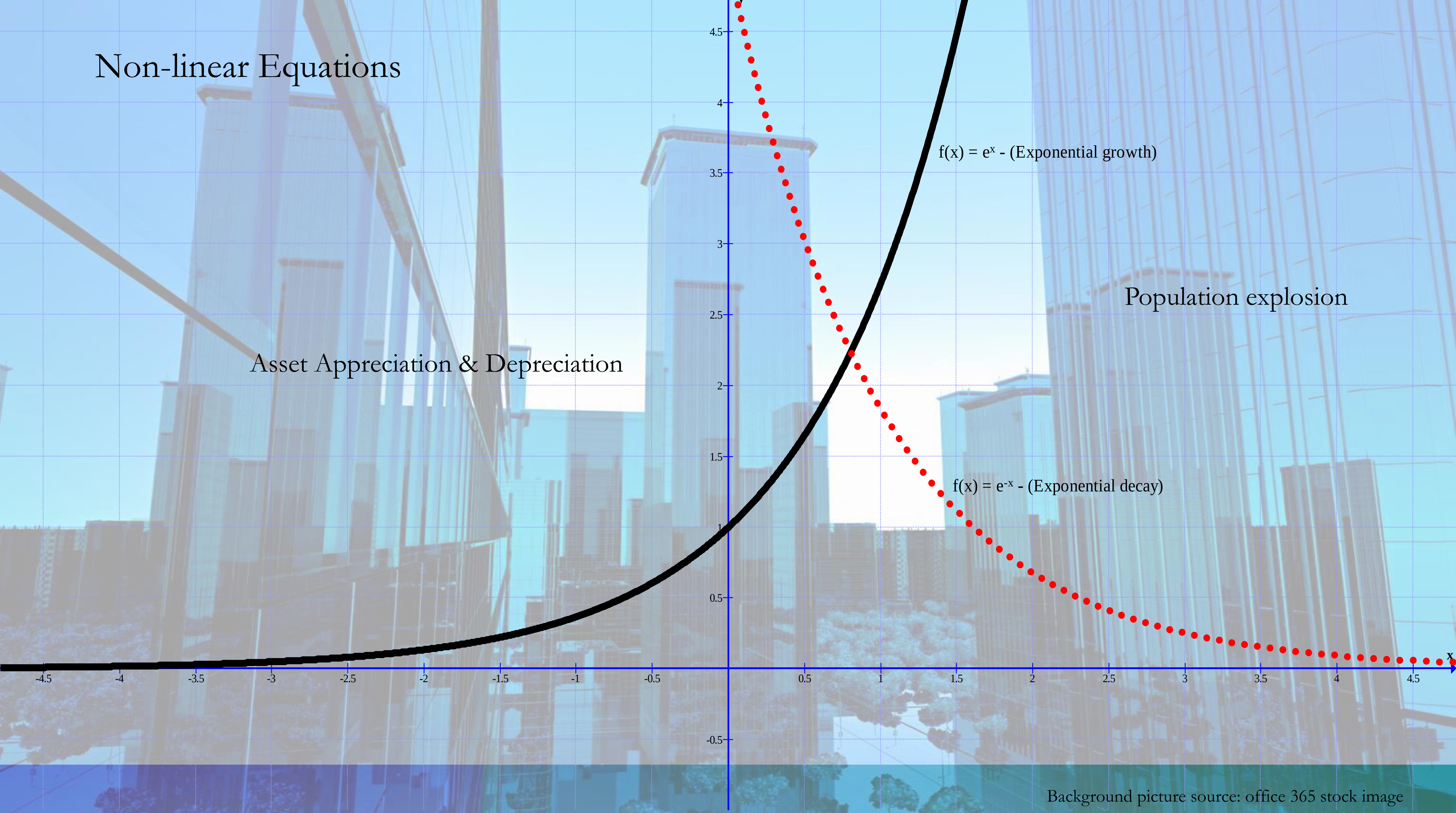
Kahenya, N.P

INTRODUCTION TO LECTURE 2

- This lecture introduces you to non-linear equations, and particularly the quadratic equations, solving quadratic equations, and their applications to solving economics and business-related problems.

- We shall demonstrate how quadratic, hyperbolic, logarithmic, and exponential functions are used in understanding the cost, revenue, and profit analysis, optimization problems, and break-even analysis

Non-linear Equations



Asset Appreciation & Depreciation

Population explosion

Background picture source: office 365 stock image

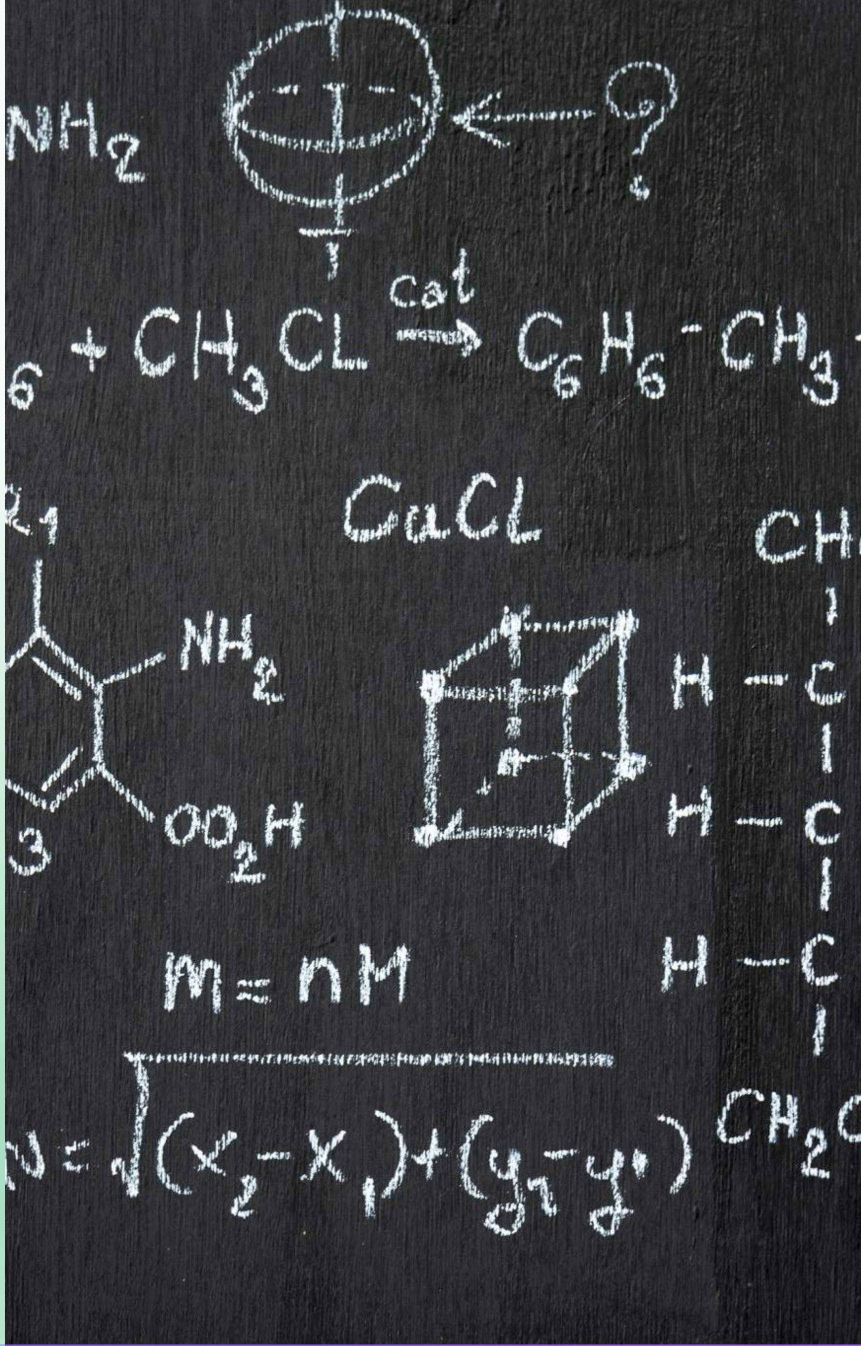
Further Readings

- These notes have been derived from diverse resources
- These are (Jacques, 2006; P. Kahenya, 2017; P. N. Kahenya, 2021; Murray & Robert, 2009)

Intended Learning Outcomes

At the end of this lecture, you will be able to

- Define a quadratic equation
- Solve quadratic equations
- Apply non-linear functions to solve economics and business-related problems



Definition 1: Polynomial Function

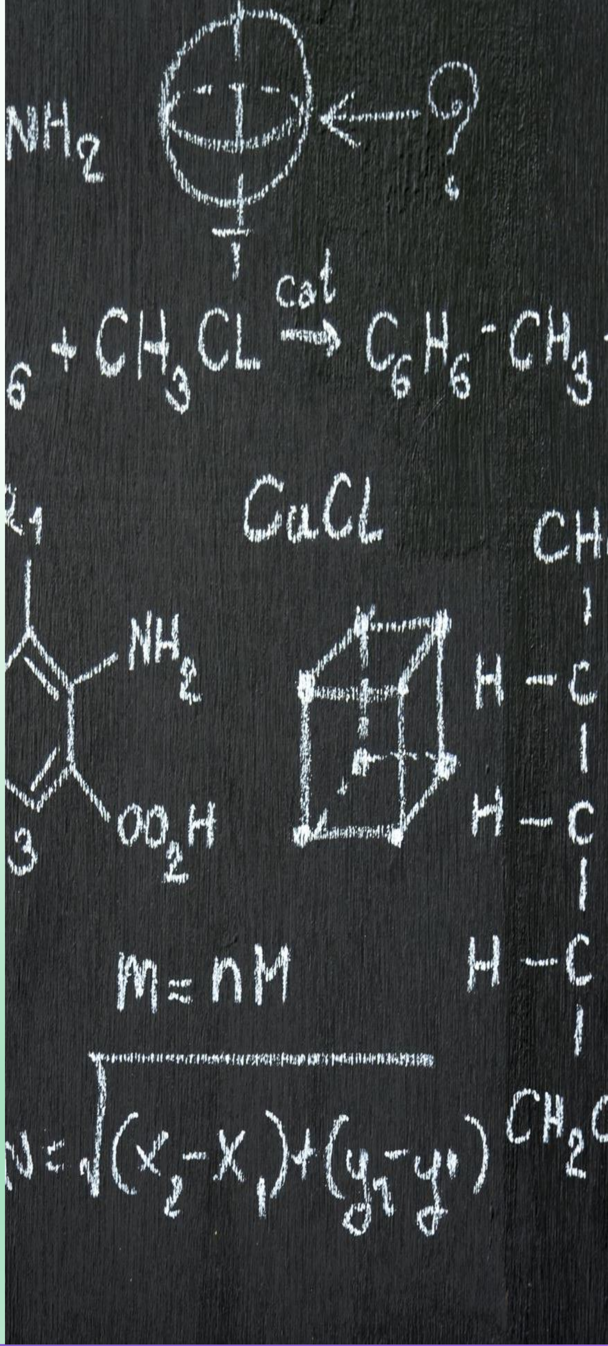
- A polynomial function is an expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad x^0 = 1$$

where x is variable, $a_n, a_{n-1}, \cdots a_1, a_0$, are constants or coefficients, n is a non-negative integer representing the degree of the polynomial, and $x^n, x^{n-1}, \cdots, x$ are terms of the polynomial function with corresponding exponents.

$x^3 + x^2 + x = k$

- In our last week lesson, we dealt with linear equations and functions

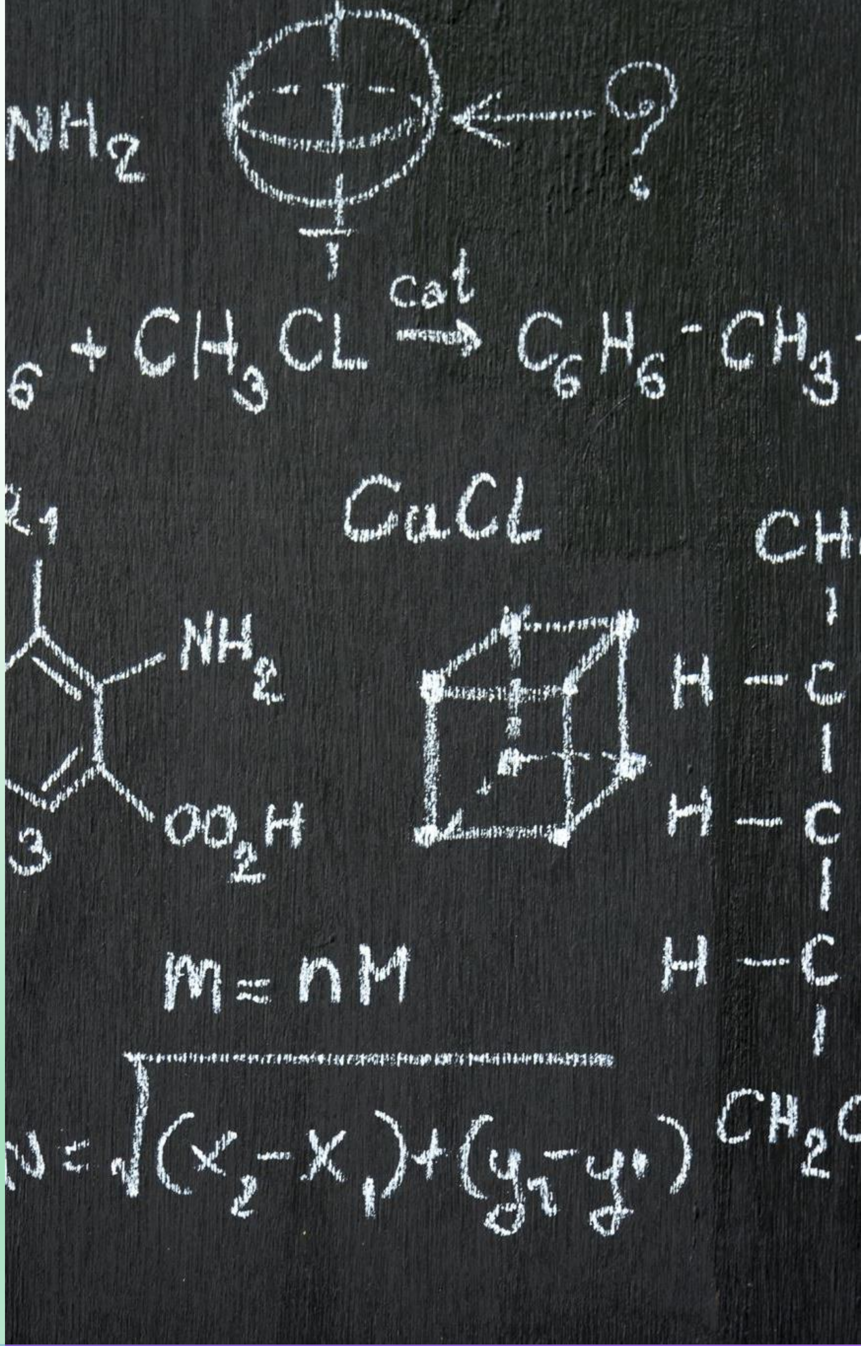


Definition 2: Quadratic Equation

It is a polynomial of degree 2 of the form $a_0 + a_1x + a_2x^2 = 0$ where x is a variable and a_0, a_1, a_2 are known numbers, real or complex with $a_2 \neq 0$. For example;

a) $4x^2 + 3x - 9 = 0$

b) $x^2 - 2x + 4 = 0$



Definition 3: Quadratic Function

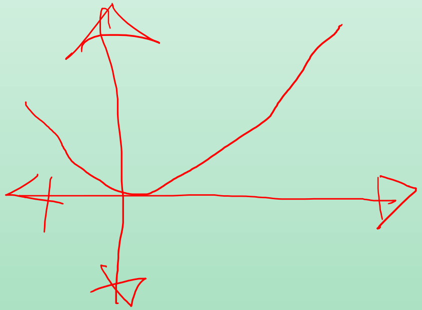
A quadratic function in x is a second-degree polynomial of the form

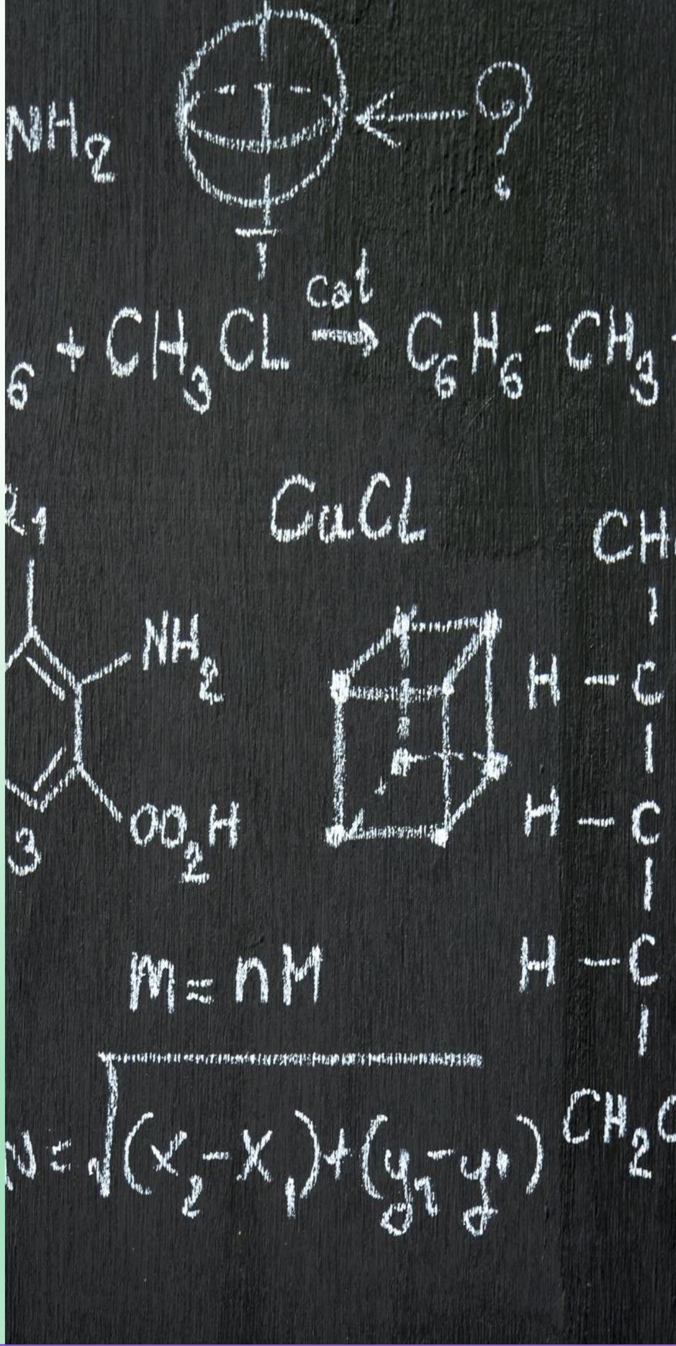
$f(x) = a_2x^2 + a_1x + a_0$ where x is a variable and a_0, a_1, a_2 are known numbers, real or complex with $a_2 \neq 0$.

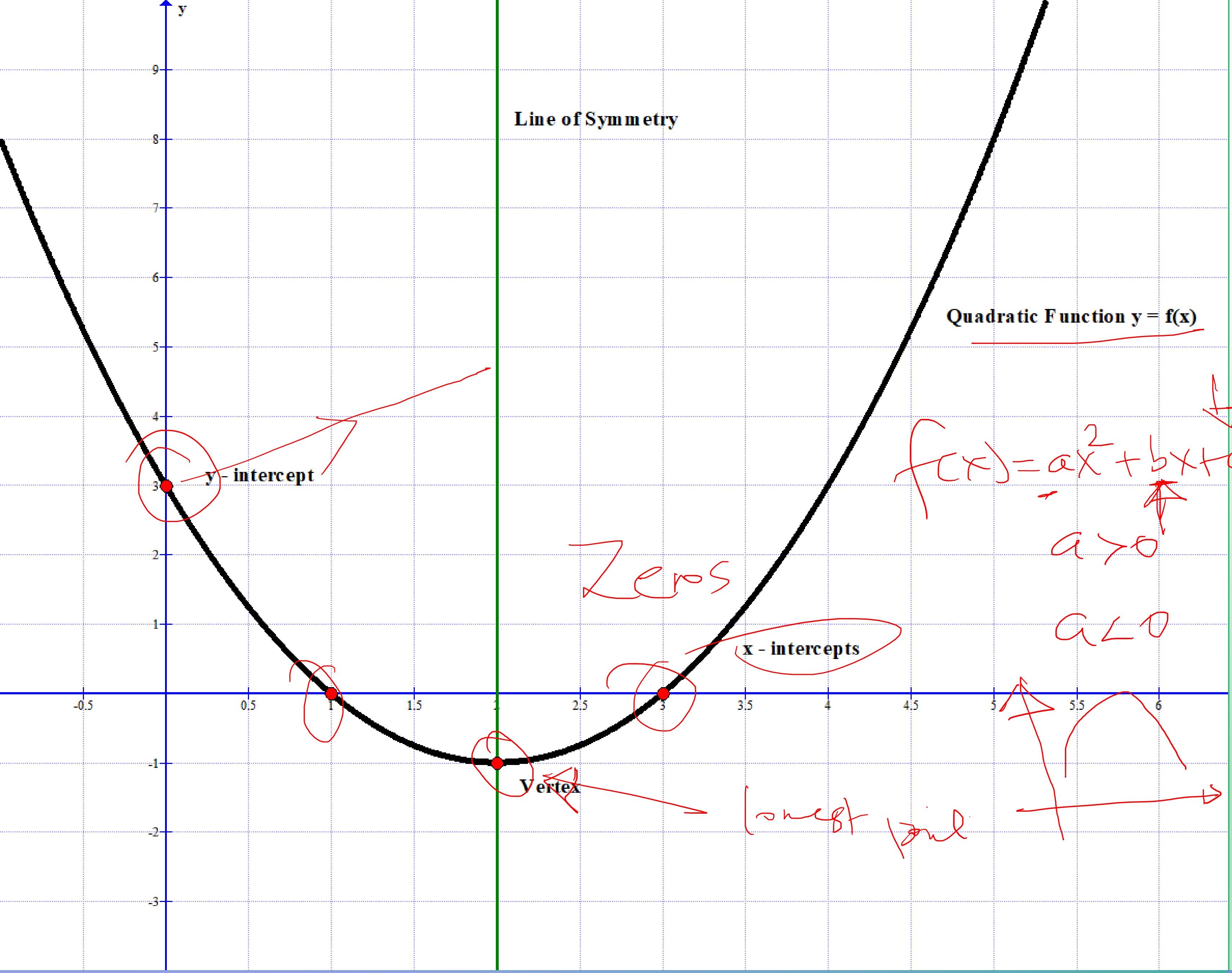
For example;

$$f(x) = 3x^2 + 2x + 4$$

$$g(x) = -\frac{1}{2}x^2 + 5x$$



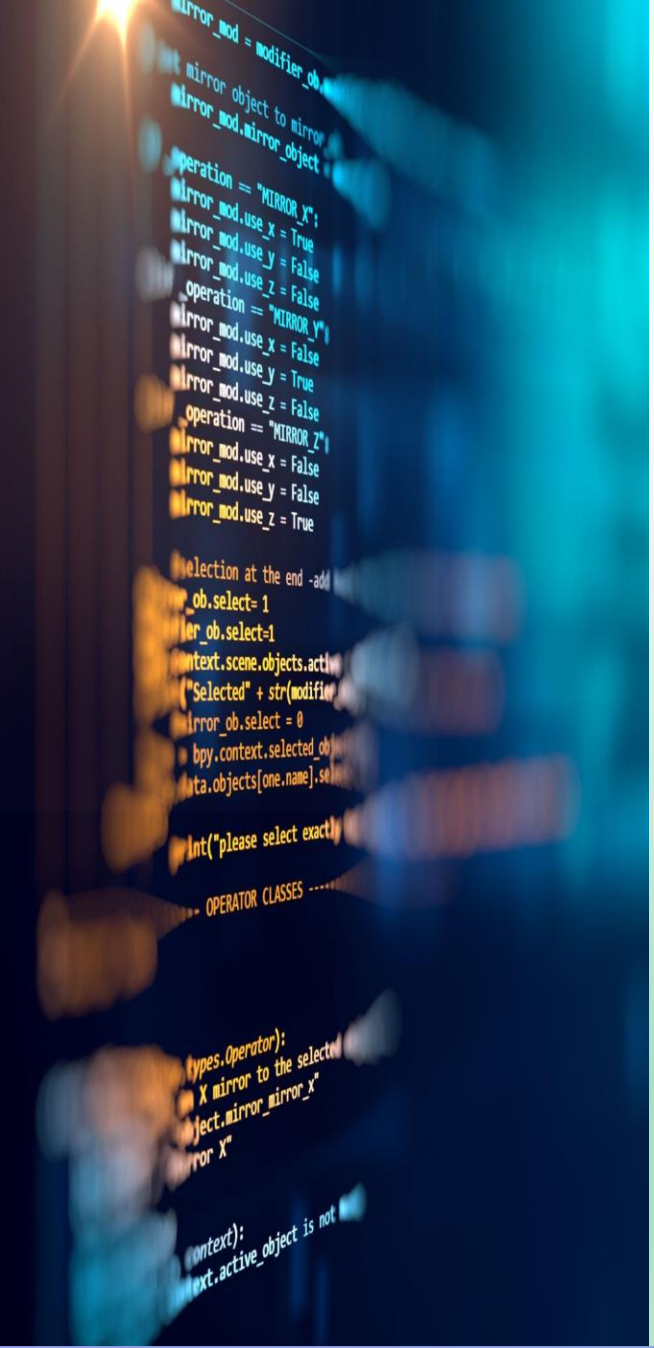




Parabola

Quadratic Functions Have The Following Characteristics

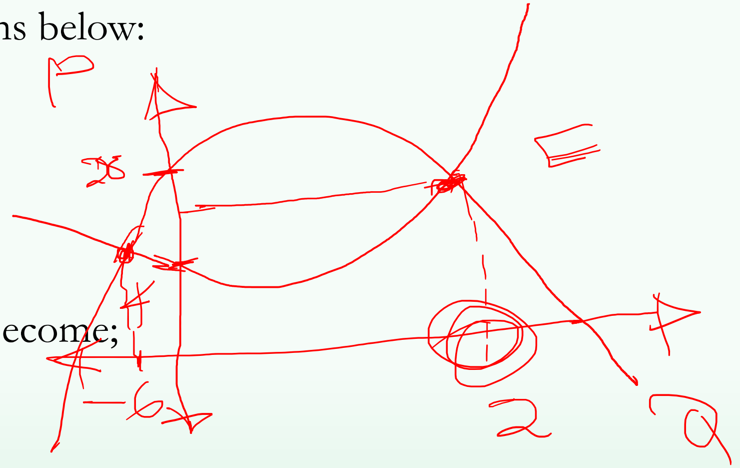
- Vertex $y = -3x^2 + 4x + 5$
- Axis of symmetry
- Intercepts
- Direction of opening $a>0$



Application of Quadratic Functions in Business Mathematics

- Quadratic functions are used to model and analyze real-world phenomena in various fields including algebra, calculus, economics, and computer science
- For the purposes of this lecture, we shall focus on their application in economics and business
- **Revenue and profit analysis:** The quadratic functions are used to model revenue and profit
- **Cost functions:** Quadratic functions can represent the relationship between quantity produced and total cost incurred in the production process
- **Optimization problems:** Quadratic functions are used in optimization problems. In business one is interested in maximizing profits, minimizing cost, or determining the maximum revenue or returns.
- **Break-even analysis:** another application of quadratic functions is in break-even analysis

Example 1: Determine the equilibrium price and quantity, given the supply and demand functions below:



$$P = Q_S^2 + 2Q_S + 4$$
$$P = -Q_D^2 - 6Q_D + 28$$

Where P, Q_S, Q_D are the price, quantity supplied, and quantity demanded respectively.

Solution: At equilibrium $Q_D = Q_S$ and hence we can let $Q_D = Q_S = Q$. Hence our equations become;

$$P = Q^2 + 2Q + 4$$
$$P = -Q^2 - 6Q + 28$$

$-28 + 6Q$ $\quad \Rightarrow Q^2 + 2Q + 4 = -Q^2 - 6Q + 28 \quad +6Q \; -28$

$$2Q^2 + 8Q - 24 = 0$$

We can use quadratic formula to find Q i.e. $\quad ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$Q = \frac{-8 \pm \sqrt{8^2 - 4(2)(-24)}}{2 \times 2} = \frac{-8 \pm 16}{4} = \frac{8}{4} \text{ or } -\frac{24}{4} = 2 \text{ or } -6$$

Hence the equilibrium Quantity $Q_2 = 2 \Rightarrow P = 2^2 + 2(2) + 4 = 12$

Definition: The profit function $\pi = TR - TC$ is a quadratic function, where TR is the Total Revenue and TC is the Total Cost.

- . Total revenue is the money received after selling a certain commodity while the Total cost is the amount of money used to produce this commodity. Hence the difference is the profit.
- Since the Total Revenue TR, is the amount of *money received after selling a number of commodity Q at a price P,* then TR is a product of P and Q i.e. $TR = PQ$

$TR = f(PQ)$

- Writing the profit function $\pi = TR - TC$ in terms of P and Q will result in a quadratic function.
- Understanding the total revenue function is essential for analyzing the behavior of firms and making decisions regarding pricing, production levels, and revenue maximization strategies.

Example : Suppose the demand function is given by $P = 100 - 4Q$, then express TR as a function of Q, determine Q when TR = 0, and determine the maximum value of TR

Solution:

From equation (i) above TR = PQ, hence we have

$$TR = (100 - 4Q)Q = 100Q - 4Q^2$$ QE

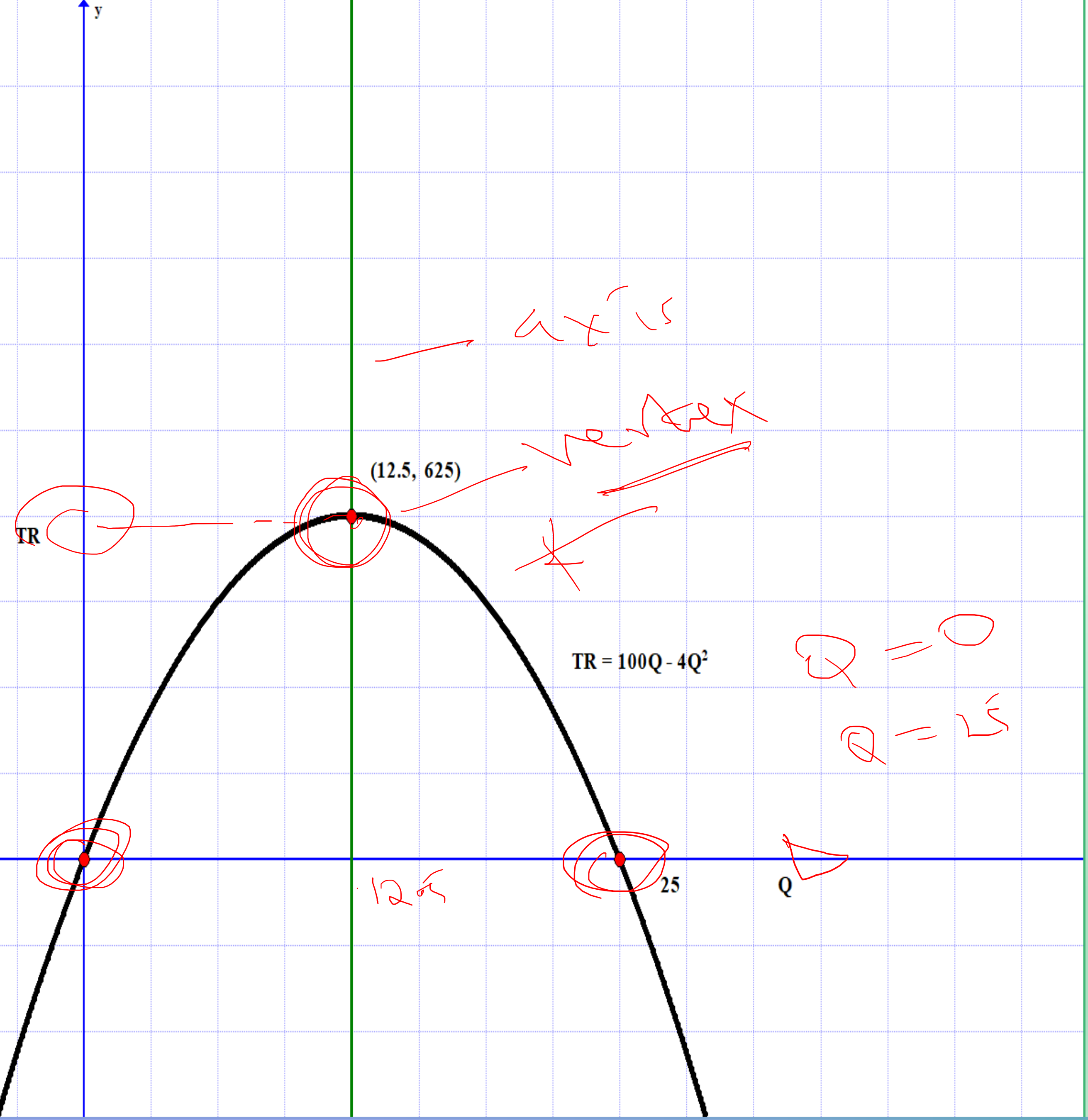
i.e. $TR = 100Q - 4Q^2 \cdots (*)$

$ax^2 + bx + c = 0$

From equation (*) the constant term is zero and hence when TR is zero Q is also zero. Hence we can solve the equation;

$$100Q - 4Q^2 = 0 \Rightarrow Q(100 - 4Q) = 0 \therefore Q = 25$$

$100 - 4Q = 0$
$+4Q \quad +4Q$
$\frac{100}{4} = \frac{4Q}{4}$
$25 \quad Q$



Example Contd:

The curve of $\mathrm{TR} = 100\mathrm{Q} - 4\mathrm{Q}^2$ is symmetrical at $\mathrm{Q} = 12.5$.

The maximum TR is attained at the point where the curve has the maximum value (local maximum/turning point) as shown in the graph below.

Hence when $\mathrm{Q} = 12.5$ we have;

$$\mathrm{TR} = 100\mathrm{Q} - 4\mathrm{Q}^2$$

$$= 100(12.5) - 4(12.5^2) = 625$$

The total cost of production depends on the amount of goods produced. Increasing the quantity Q of goods produced leads to an increase in the production cost. We have two types of cost:

- Fixed Cost FC: These are costs that take time to vary or remain constant over a considerable time such as the cost of land on which production firm is housed, equipment, skilled labor among similar costs
- Variable Cost VC: These are costs of production that vary with output such as the cost of the raw materials for producing the goods, energy, unskilled labor among other similar costs

It is clear then that the Total Variable Cost TVC is a function of Quantity Q of goods produced i.e.

$$TVC = (VC)Q$$

Hence the Total Cost is a linear combination of Fixed Cost FC and Total Variable Costs TVC i.e.

$TC = TVC + FC$

$$TC = (VC)Q + FC \text{ – Total Cost Function}$$

$$AC = \frac{TC}{Q} \text{ – Average Cost Function}$$

Example : A salesperson sells different types of chairs. He sells 4 types of chairs costing 12,250 KES, 14,050 KES, 17,100 KES, and 18400 KES. Determine the average cost of the phone sold.

Solution: Average cost of production = total cost of production over the number of items produced.

$$AC = \frac{12250 + 14050 + 17100 + 18400}{4}$$

$$= \frac{61800}{4}$$

$$= 15450 \text{ KES}$$

Example: Suppose that the fixed costs FC of producing Q items is 120, and that the variable costs VC is 7 per item. Express;

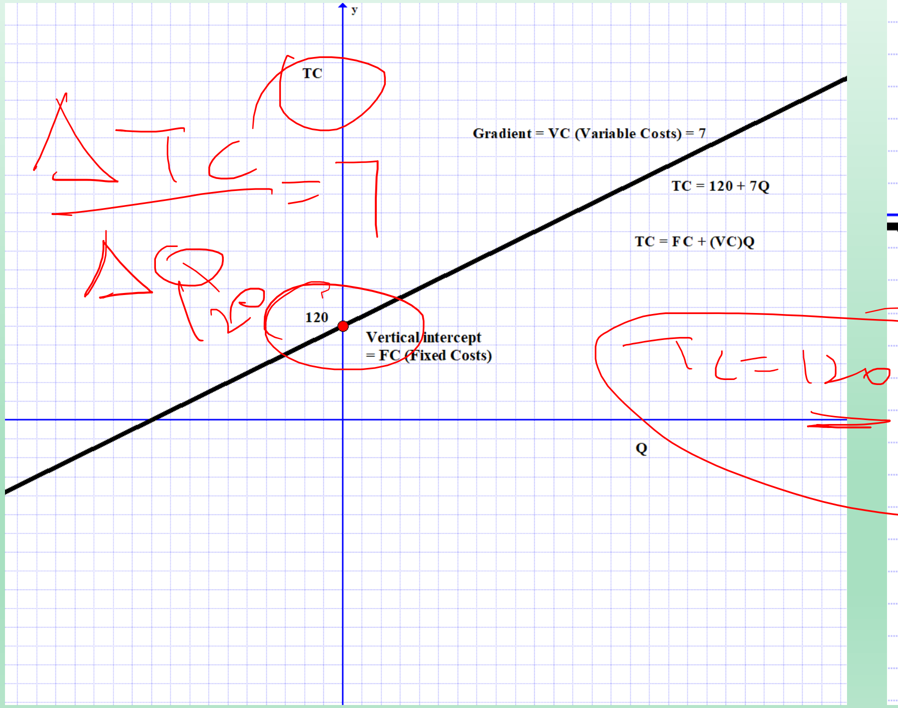
TC as a function of Q.

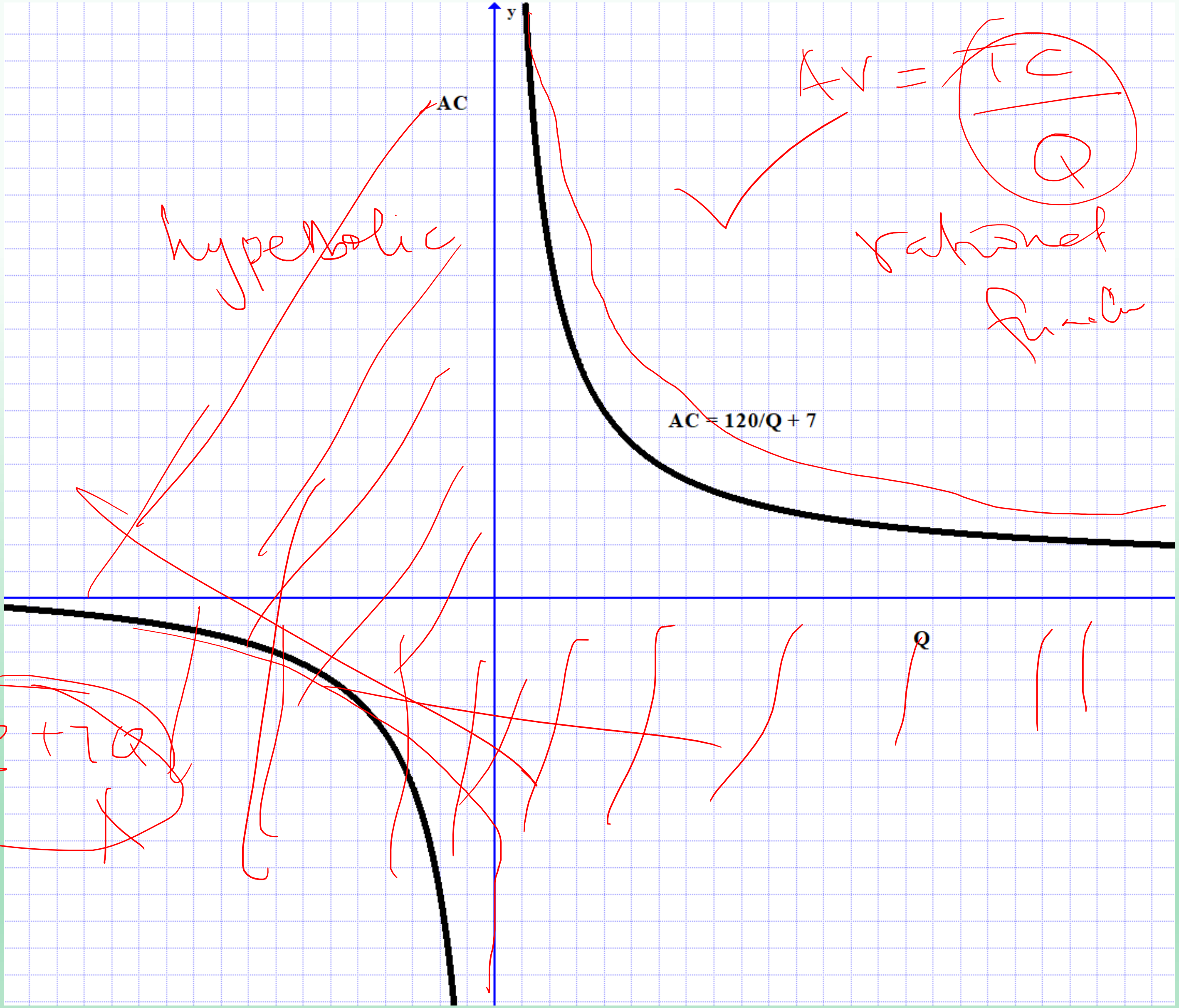
AC as a function of Q.

Solution:

By definition $\mathrm{TC} = \mathrm{FC} + (\mathrm{VC})\mathrm{Q}$. Hence we have; $\mathrm{TC} = 120 + 7\mathrm{Q}$

By definition $\mathrm{AC} = \frac{\mathrm{TC}}{\mathrm{Q}} \Rightarrow \mathrm{AC} = \frac{120+7\mathrm{Q}}{\mathrm{Q}} = \frac{120}{\mathrm{Q}} + 7$ i.e. $\mathrm{AC} = \frac{120}{\mathrm{Q}} + 7$


y
TC
Gradient = VC (Variable Costs) = 7
TC = 120 + 7Q
TC = FC + (VC)Q
120
Vertical intercept
= FC (Fixed Costs)
Q
ΔTC / ΔQ = 7
TC = 120 + 7Q


y
AC
AC = 120/Q + 7
Q
hyperbolic
Av = TC / Q
rational function

Example : Supposed that the fixed costs FC are 8, the variable costs are 2 per commodity, and the demand function is $P = 12 - 3Q$. Determine;

i) Profit function π in terms of Q.

ii) The amount of Q at the break-even point.

iii) Maximum profit.

$TC = FC + VCQ$

Solution:

i) By definition $\pi = TR - TC \cdots (*)$

Again, by definition we know that Total Cost is given as; $TC = FC + (VC)Q$ but FC = 8 and VC = 2

$$\Rightarrow TC = 8 + 2Q$$

$\pi = TR - TC$

Again $TR = PQ$

$$\Rightarrow TR = (12 - 3Q)Q = 12Q - 3Q^2$$

Hence our profit equation (*) becomes;

$$\pi = TR - TC$$

$$\pi = (12Q - 3Q^2) - (8 + 2Q) = 12Q - 3Q^2 - 8 - 2Q$$

$$\therefore \pi = 10Q - 3Q^2 - 8$$

$ax^2 + bx + c = 0$

i) The break-even happens when the profit function $\pi = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow 10Q - 3Q^2 - 8 = 0$$

$$Q = \frac{-10 \pm \sqrt{100 - 4 \times 3 \times 8}}{-2 \times 3} = \frac{-10 \pm 2}{-6} = -\frac{12}{-6} = 2 \text{ or } -\frac{8}{-6} = \frac{4}{3}$$

The break-even quantities are 2 and $\frac{4}{3}$

$a = -3$

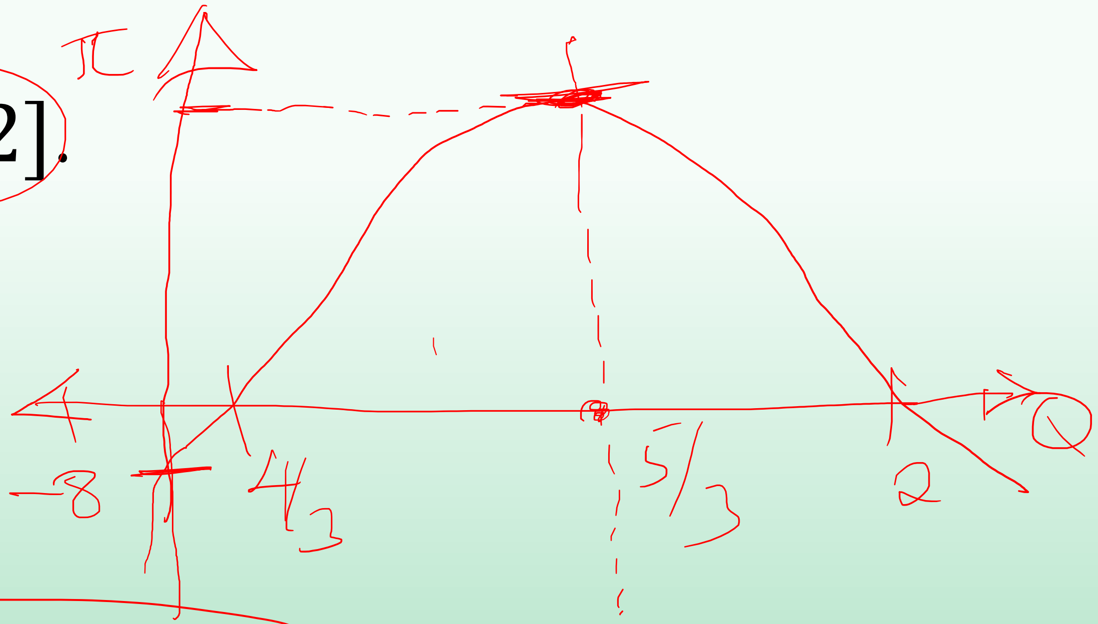
$b = 10$

$c = -8$

$Q = 2, 4/3$

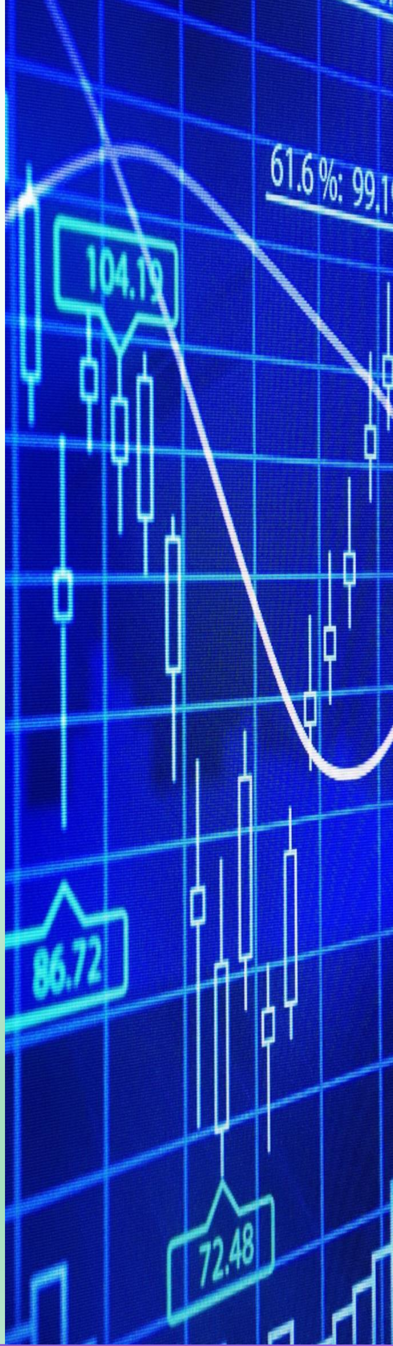
The maximum profit corresponds with the quantity Q at the midpoint of the interval $[\frac{4}{3}, 2]$.

$\pi = 10Q - 3Q^2 - 8$



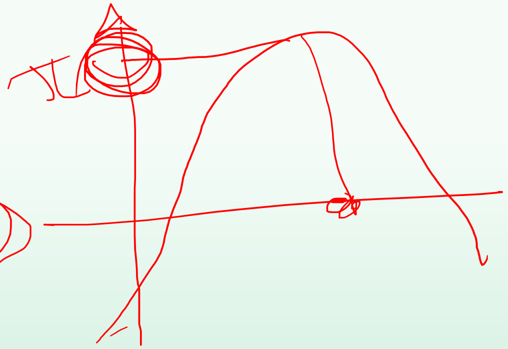
That is, $Q = \dfrac{2+\frac{4}{3}}{2} = \dfrac{5}{3}$

Hence, profit $\pi = 10\left(\frac{5}{3}\right) - 3\left(\frac{5}{3}\right)^2 - 8 = \frac{50}{3} - \frac{25}{3} - 8 = \frac{1}{3}$



Alternatively, we can find the vertex of the parabola $f(x) = ax^2 + bx + c$ using the formula;

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

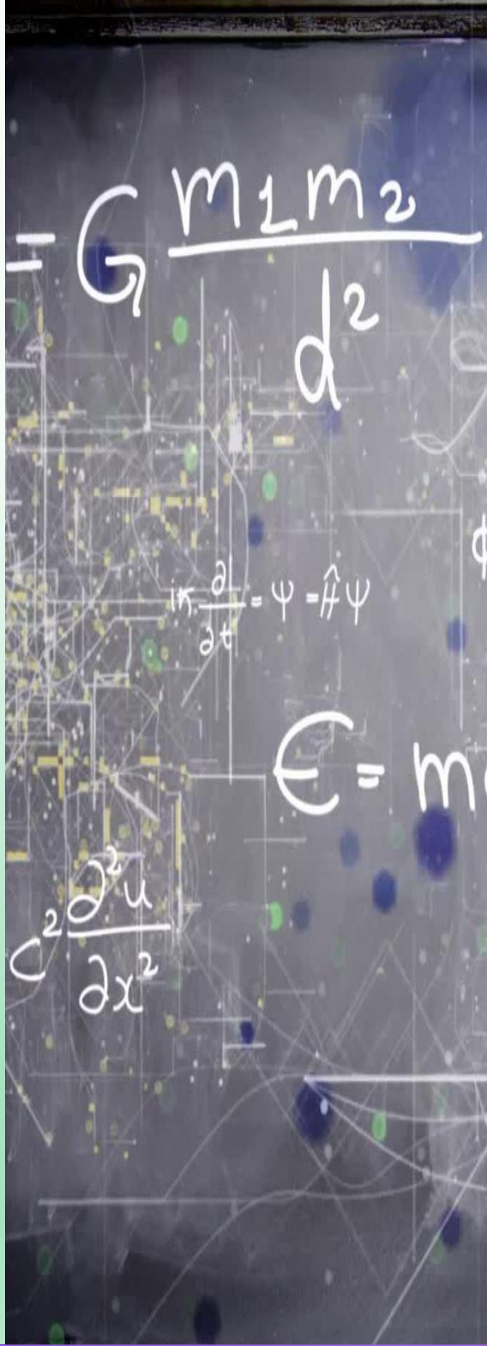


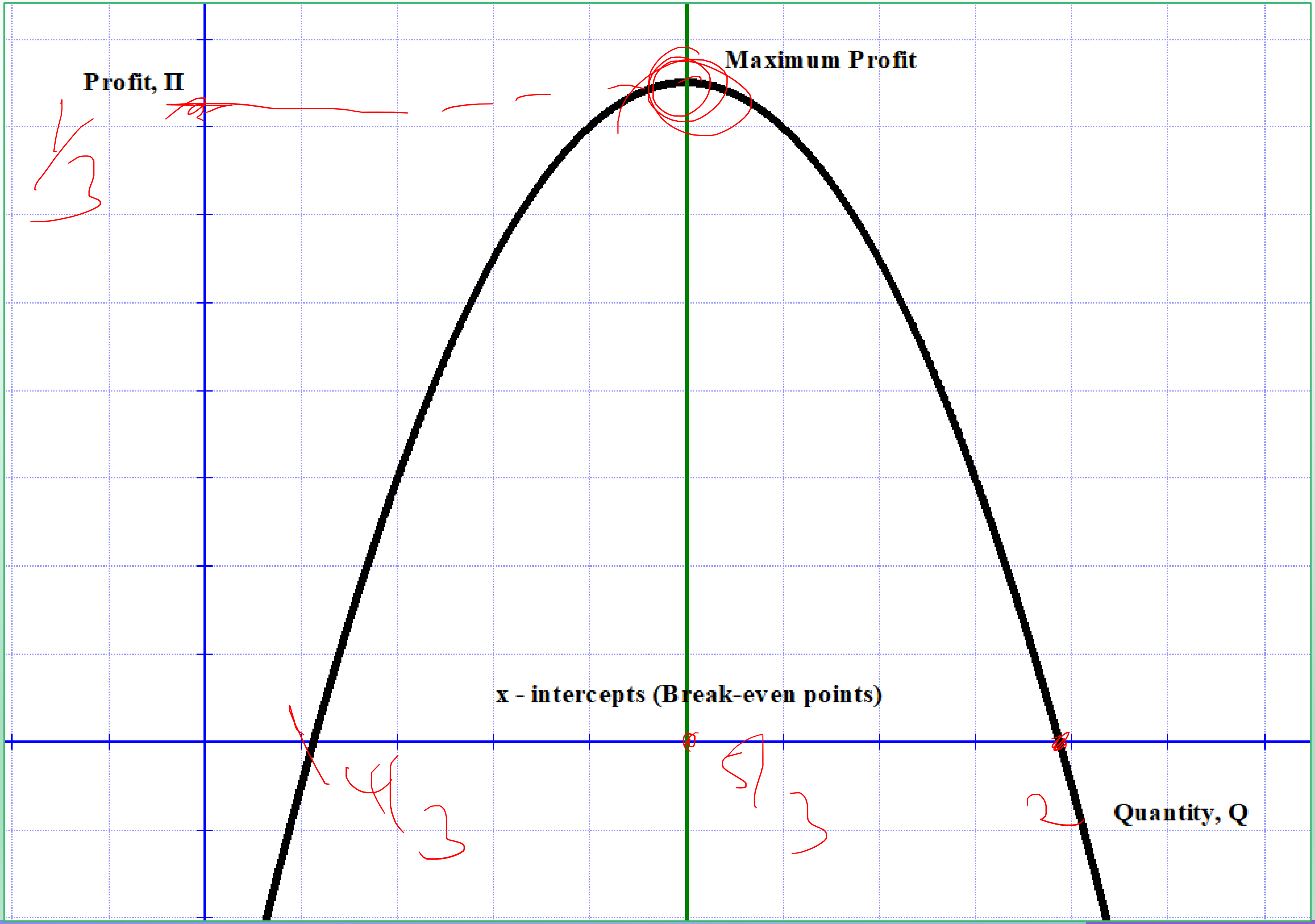
In our case $f(x)$ is the profit function $p(Q) = \pi = 10Q - 3Q^2 - 8$

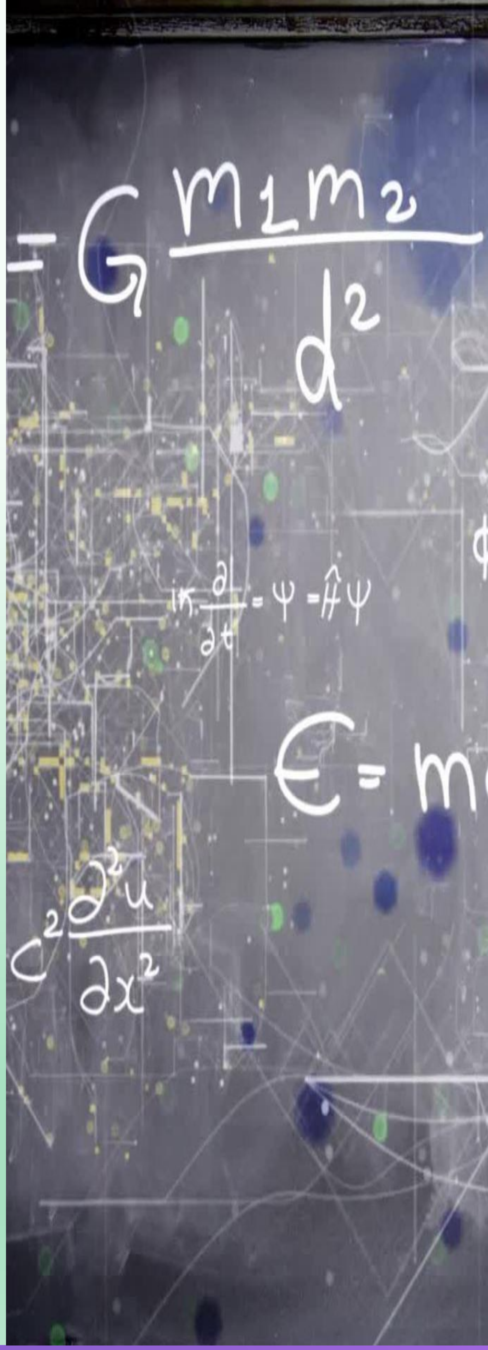
In particular the vertical coordinate $\frac{4ac-b^2}{4a}$ is of great interest since it represents the maximum profit.

Note in standard polynomial form; $a = -3, b = 10, c = -8$, hence

$$\frac{4ac - b^2}{4a} = \frac{4(-3)(-8) - 10^2}{4 \times -3} = \frac{-4}{-12} = \frac{1}{3}$$




Profit, Π
Maximum Profit
x - intercepts (Break-even points)
Quantity, Q



Example : A Kiosk has a fixed cost of 1500 KES and variable cost per item is 1750 KES. The kiosk sells its commodity at 5000 KES per item. Determine;

i) Total Cost function TC

ii) Total Revenue function TR

iii) Profit function π

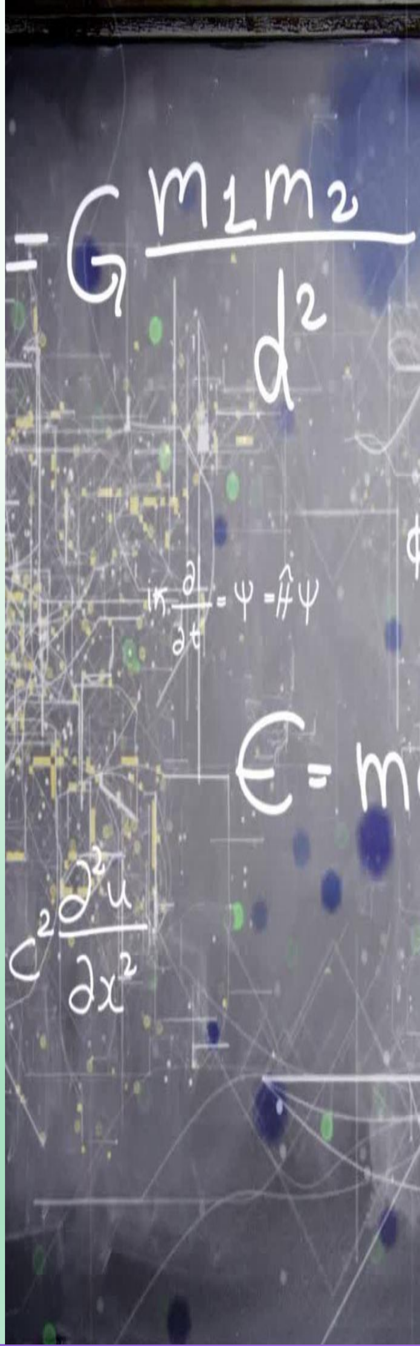
$TC = FC + TVC$

$TC = 1500 + 1750Q$

$TVC = (VC)Q$

Solution:

i) $TC = FC + TVC = 1500 + 1750Q$



Example : A Kiosk has a fixed cost of 1500 KES and variable cost per item is 1750 KES. The kiosk sells its commodity at 5000 KES per item. Determine;

i) Total Revenue function TR = PQ

ii) Profit function π

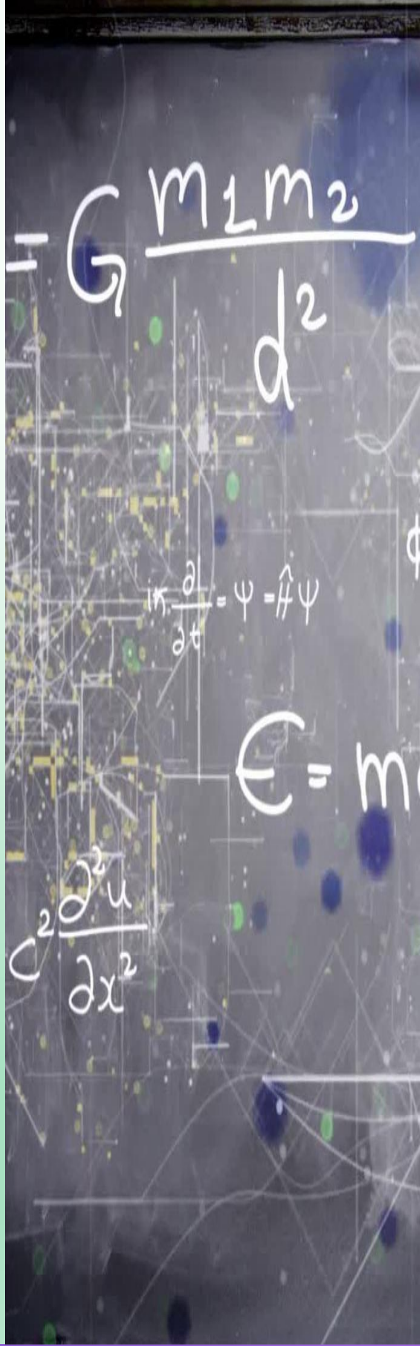
Solution:

i) Total Revenue function $TR = PQ = 5000Q$

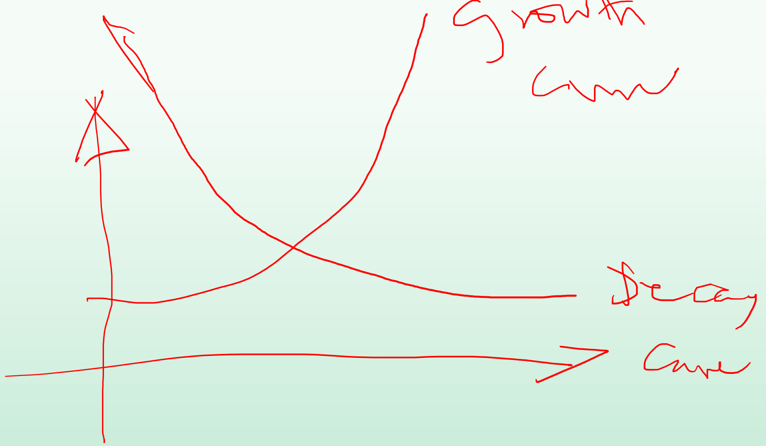
ii) Profit function $\pi = TR - TC = 5000Q - (1500 + 1750Q)$

$$= 5000Q - 1500 - 1750Q$$

$$\pi \quad = 3250Q - 1500$$



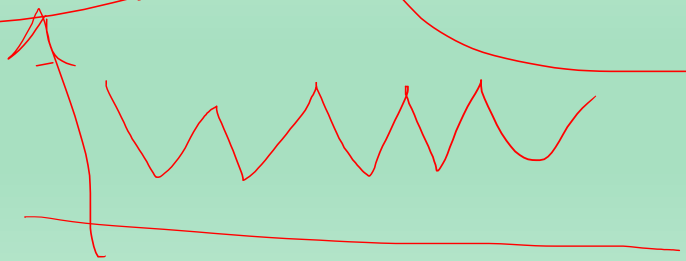
APPLICATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS IN BUSINESS MATHEMATICS



- Compound interest formula

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

- Supply and Demand Analysis: The demand curve of some products may follow an exponential decay curve while supply curve may follow an exponential growth function.
- . Logarithmic functions are used to determine elasticity of demand that helps understand how the change in prices affects demand.



APPLICATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS IN BUSINESS MATHEMATICS

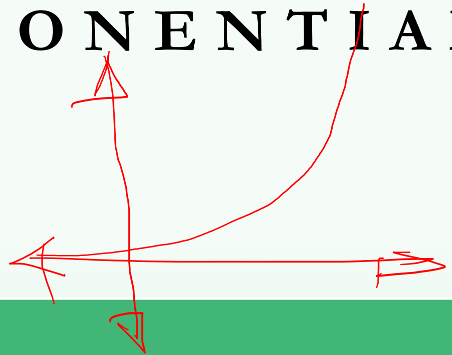
a) *Time value of money*

- Exponential and logarithmic functions are fundamental in understanding the time value of money.
- These functions are valuable when calculating present and future values of commodity or service, annuities, and amortization.

a) *Continuous growth or decay*

- Exponentials functions model situations of continuous growth and decay e.g. population growth, sales growth, or depreciation of assets.

APPLICATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS IN BUSINESS MATHEMATICS



Example 1: The economy of a country is predicted to grow continuously such the gross national product in trillion of KES, after years is modeled by $G_{NP} = 95e^{0.01t}$

Determine when it will be 102 trillion shilling

RECALL:

$$\text{If } a = x^y \Rightarrow \log_x a = y$$

as for $9 = 3^2$ $\quad \log_3 9 = 2$

here

Index Notation

log Notation

$$\log_e x = \ln x \text{ (Natural logarithm)}$$

$\log_e x = \ln x$ $\quad \log_e 3 = \ln 3$

$\frac{102}{95} = \frac{95 e^{0.01t}}{95}$ ← Index form

$1.073684 = e^{0.01t}$ | $\log_e 1.073684 = 0.01t$

Hence equation (*) above in logarithmic form is;

$$\log_e 1.073684 = 0.01t \Rightarrow \ln 1.073684 = 0.01t$$

$$\therefore t = \frac{\ln 1.073684}{0.01} \approx 7.11 \text{ years}$$

A small retail firm that started with a yearly revenue of 12000 KES has noted that its sales have been growing exponentially at a rate of 2 % per annum for the last 3 years. This can be modeled by'

$$R(t) = R_0(1+r)^t$$

- QF
- Hyperbolic Ae
- Exponential function

For t = 3 we have; $R(3) = 12000(1 + 0.02^3) \approx 12{,}734.496$

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Kahenya, P. N. (2021). *Basic Mathematics*. HUFOCW. https://www.hufocw.org/Course/854

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.