Business Mathematics

Lecture 3

Linear Programming

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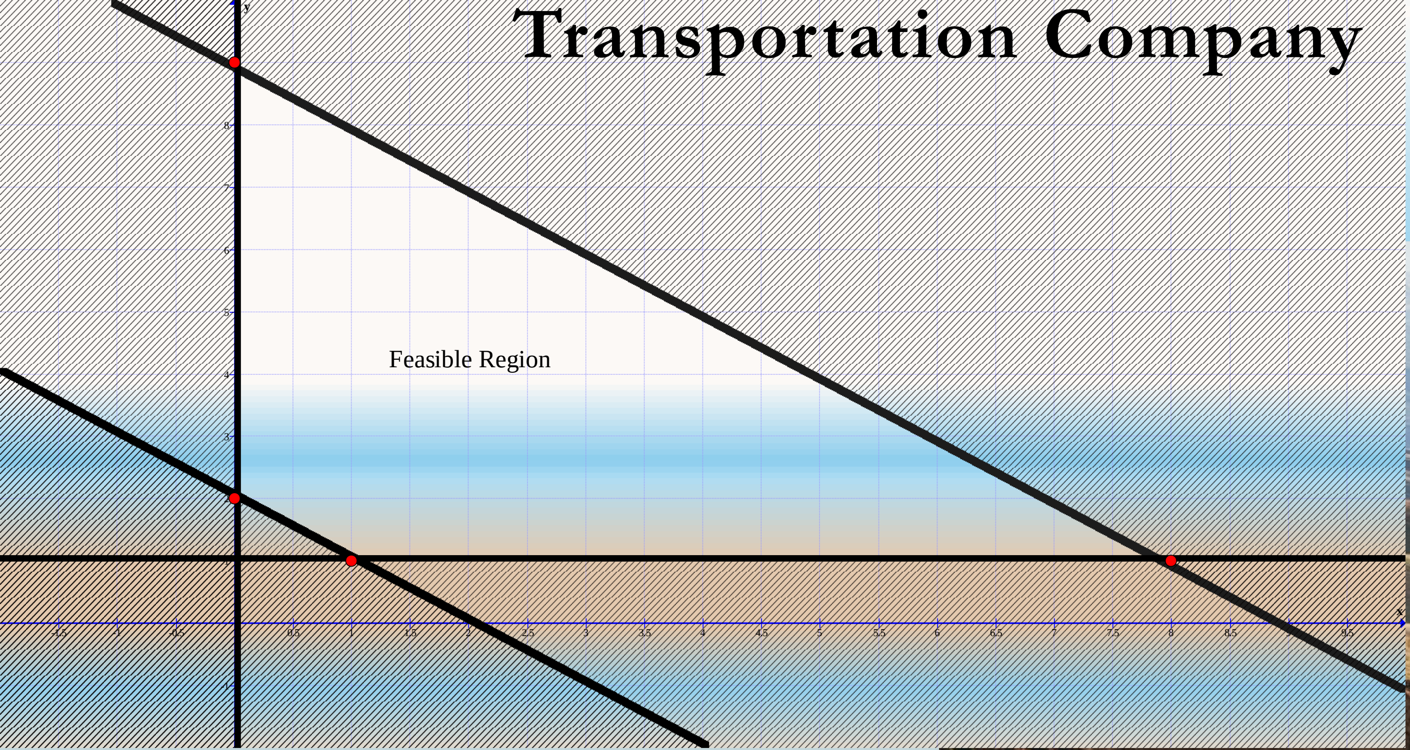
Introduction to Lecture 1

This lecture introduces you to linear programming and their applications to solving economics and business-related problems

We shall demonstrate how linear inequalities are used to model business problems

Functions of two variables can be optimized subject to some given constraints, and this can be demonstrated using linear inequalities

Transportation Company



Maximize profit $Z = 4x_1 + 3x_2 + 2x_3$

x_1 − efficiency of operations, x_2 − Demand & Pricing,
x_3 – Asset management

Subject to the constraints;

$$3x_1 + 2x_2 + 4x_3 \leq 30,$$
$$2x_1 + x_2 + x_3 \leq 10,$$
$$x_1 + 3x_2 + 2x_3 \leq 20,$$
$$x \geq 0, y \geq 0, z \geq 0$$

Courtesy: office 365 stock image

Intended Learning Outcomes

At the end of this lecture, you will be able to

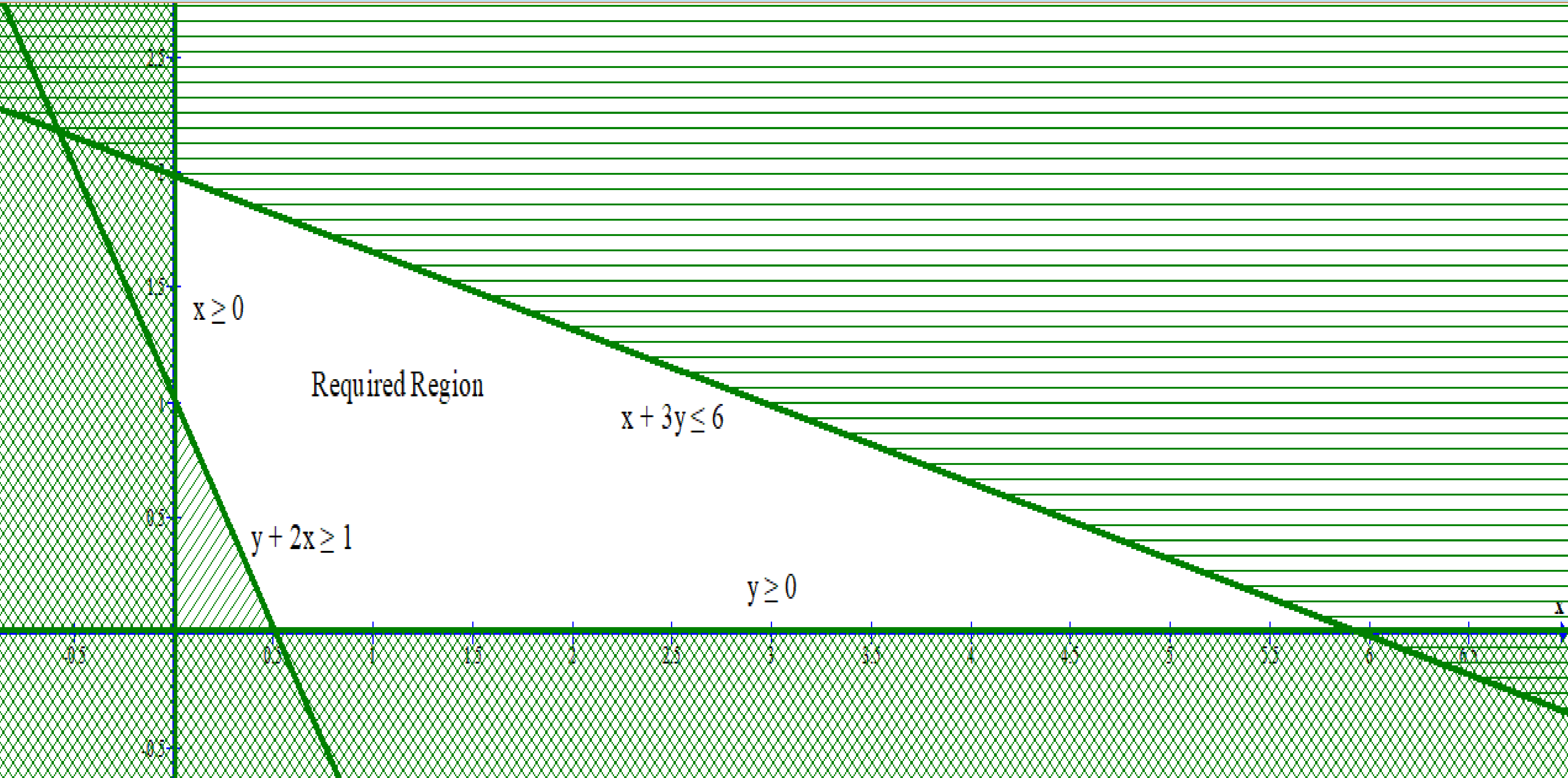
- ❑Define an objective function
- ❑Represent linear inequalities graphically
- ❑Solve business mathematics problems by applying linear programming

Example

Given the linear inequalities; $x+3y \leq 6$; $y+2x \geq 1$; $x \geq 0$; $y \geq 0$, represent the feasible region graphically

$x + 3y = 6$			
x	0	3	6
y	2	1	0
$y + 2x = 1$			
x	0	1	2
y	1	-1	-3

Solution



Definition

- In linear programming the main goal is to optimize the given problem
- Objective function is therefore an expression that defines the quantity to be maximized or minimized
- The purpose of the objective function is to determine the values for decision variables that optimize i.e. maximize or minimize, while satisfying all the given constraints
- It can be maximizing profit, minimizing cost, minimizing time among other variables

Example

- A tailor has 50 square metres of clothing materials in which he plans to make skirts and shorts, and maximize profit within some constraints
- What is the number of skirts and shorts that he has to make to maximize profit?

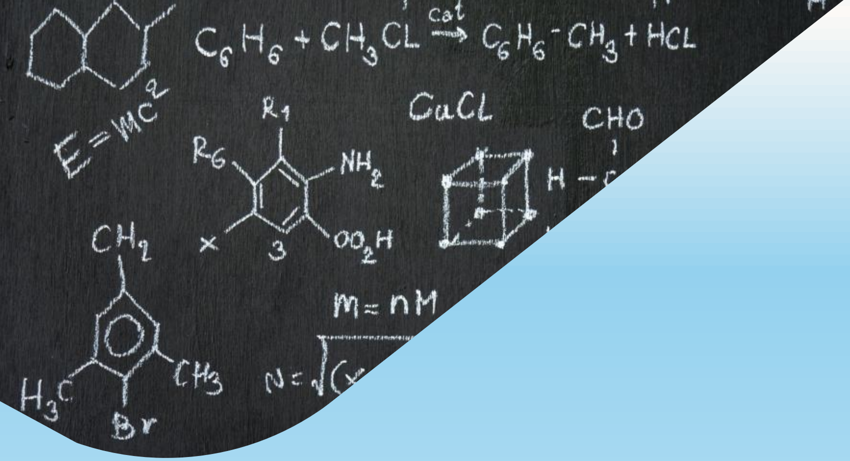
Objective: Maximize profit

Decision variables: The tailor makes x skirts and y shorts

Courtesy: office 365 stock image

Constraints

- A skirt requires 2 square metres of material while a short requires 0.5 square metres of materials i.e. $2x + 0.5y \leq 50$
- He must make at least 10 shorts, i.e. $y \geq 10$
- He has at least 30 people available to make the skirts and shorts. Each skirt requires 3 people while a short requires 1 person i.e. $3x + y \geq 30$
- Non-negativity constraint are $x \geq 0, y \geq 0$



Solution:

Objective function

- It is given that profit for selling one skirt is 200 KES and a short is 300 KES . Hence our Objective Function is

$$P = 200x + 300y$$

We need to maximize this function

- We can plot the inequalities on a plane as shown below
- We can use the vertices of the feasible region to determine the maximum profit using the profit function

$$P=200x+300y$$

- To maximize profit the tailor needs to make 100 shorts only so as to realize the maximum profit of 30,000 KES

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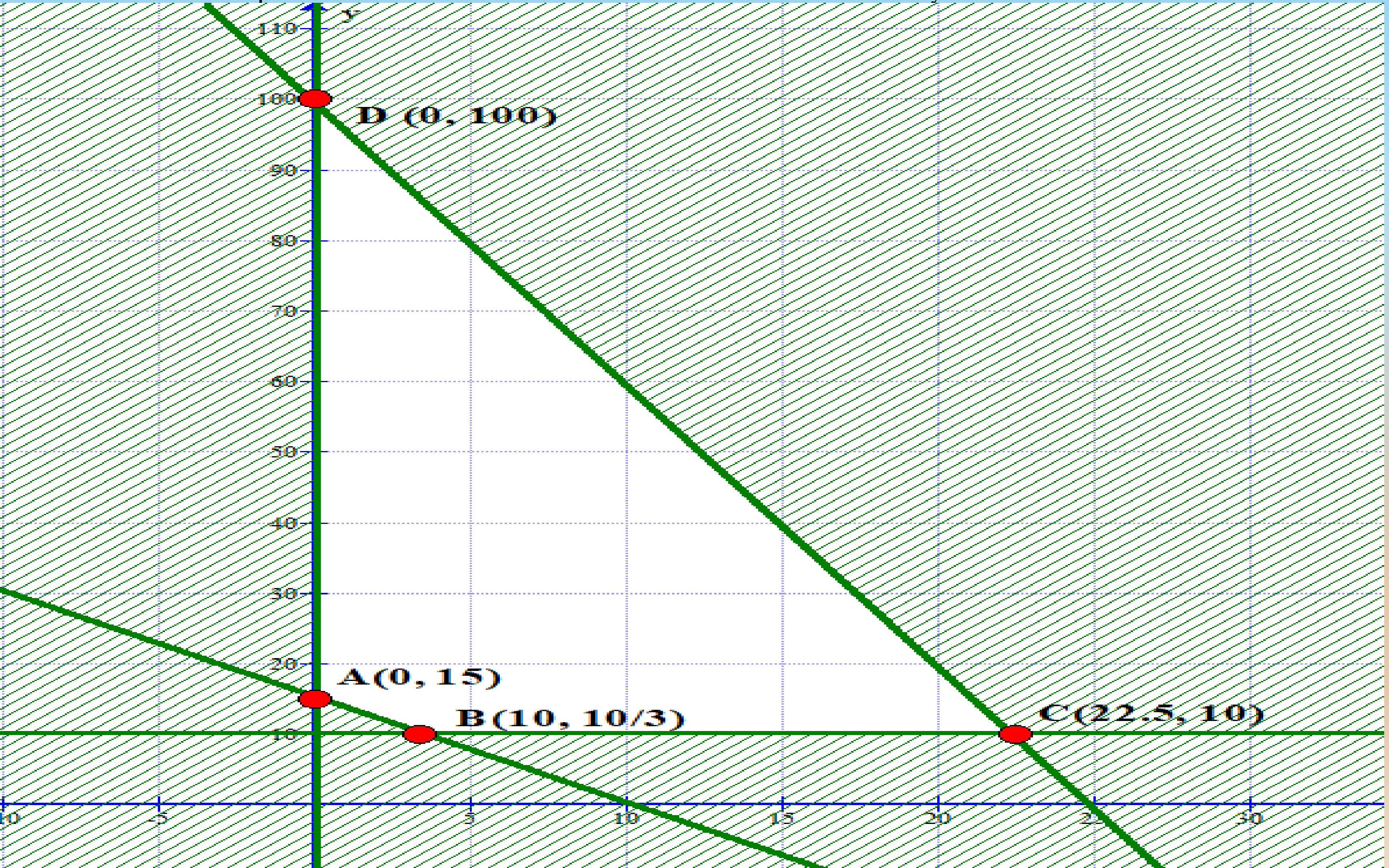

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Example

- A certain firm produces two types of iron-boxes x and y per day
- **Objective:** Maximize sales revenue
- **Decision variables:** The firm produces type x iron boxes and type y boxes per day

Constraints

- The firm can only ship out 300 iron-boxes i.e. $x+y \leq 300$
- The firm laborers can only offer at most 400 working hours
- Type x iron box requires 2 hours to make while type y requires 1 hour i.e. $2x + y \leq 400$

Non-negativity constraint

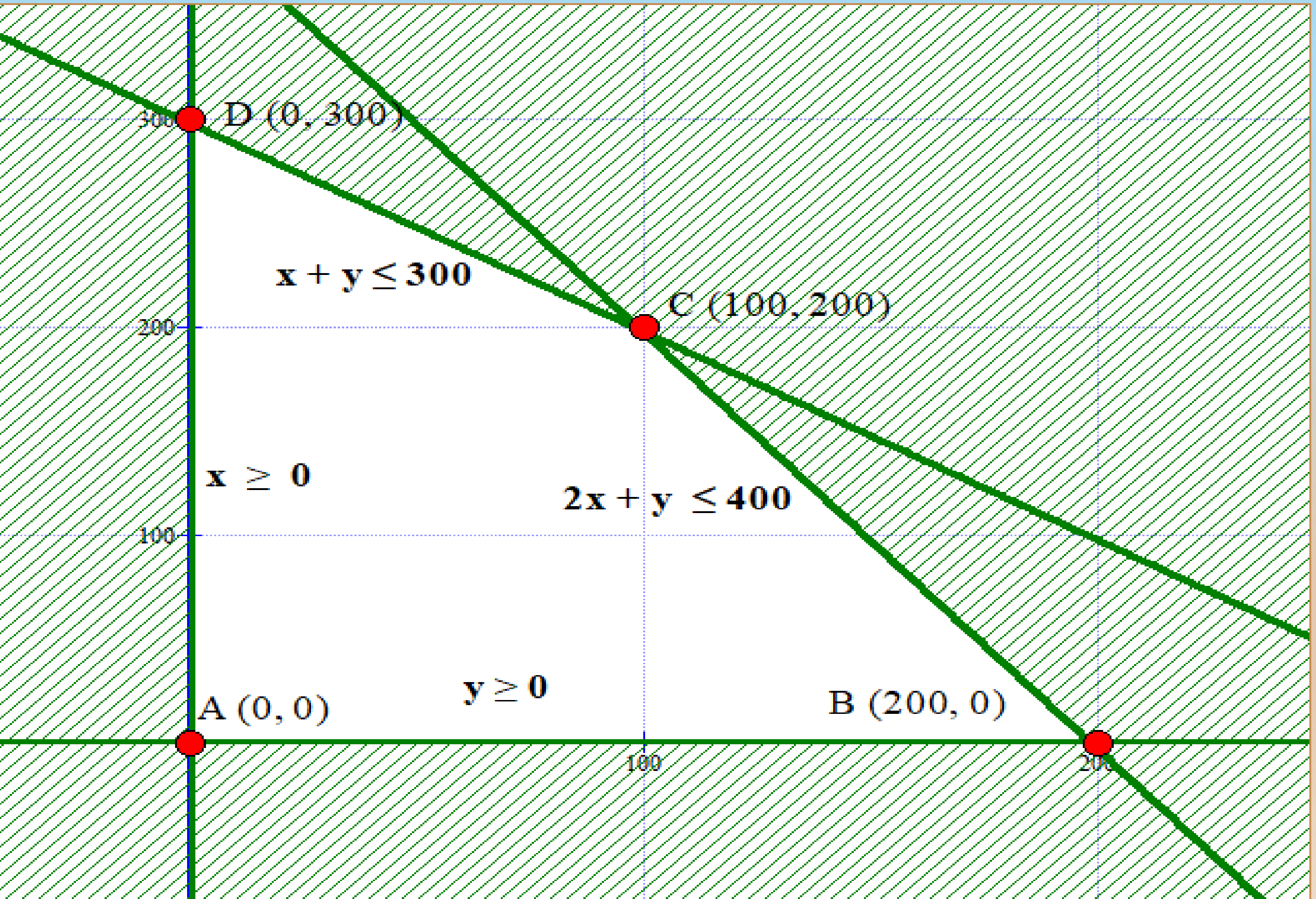
- These are $x \geq 0, y \geq 0$ since the firm cannot make negative number of iron boxes

Objective function

- It is given that type x iron-box sells at 4200 KES and type y sells at 3500 KES i.e. $P = 4200x + 3500y$

Solution:

We can plot the inequalities on the xy-plane as below We can use the sales revenue function $p = 4200x + 3500y$ to determine the maximum sales revenue.



Point	x	y	Sales Revenue
A(0, 0)	0	0	0
B(200,0)	840,000	0	840,000
C(100, 200)	420,000	700,000	1,120,000
D(0, 300)	0	1,050,000	1,050,000

This is done to improve on the objective function value until an optimal is reached

Step 1: Our objective function is $P=4200x+3500y$

Our constraints are: $2x+y \leq 400$ and $x+y \leq 300$

Next, we write the objective function as; $-4200x-3500y+P=0$

Next, we introduce two values S_1 and S_2 - slack values

We write the constraints as equalities $2x+y+S_1 = 400$ and $x+y+S_2 = 300$

Simplex Method

Initial tableau

	x	y	s_1	s_2	P	
s_1	2	1	1	0	0	400
s_2	1	1	0	1	0	300
	-4200	-3500	0	0	1	0

- The last row has negatives. Our aim is to have all positives or zero in this row to achieve our optimal.
- Next identify the smallest negative value on the last row i.e. -4200 to know the column that we need to pivot i.e. to make it 1 and the only non-zero in that column.

- Under the column x we have 2 and 1. We need to determine which of the two we shall pivot. To do this we need to determine which of this row has lowest ratio.
- For row 1 we take the ratio $\frac{400}{2} = 200$ and for row 2 we take the ratio $\frac{300}{1} = 300$.
- The smallest ratio is 200 for row 1. Hence, we need to pivot 2.

	x	y	s_1	s_2	P	
s_1	2	1	1	0	0	400
s_2	1	1	0	1	0	300
	-4200	-3500	0	0	1	0

	x	y	s_1	s_2	P	
x	**1**	0.5	0.5	0	0	200
s_2	1	1	0	1	0	300
	-4200	-3500	0	0	1	0

- Next, we have to make the rest of the values below it i.e. 1 and -4200 all zeros. This is achieved by following elementary row operations;
- To change Row 2, R_2; $(R_1 - R_2) \rightarrow R_2$
- To change Row 3, R_3; $(4200R_1 + R_3) \rightarrow R_3$. To get the tableau;

	x	y	s_1	s_2	P	
x	1	0.5	0.5	0	0	200
s_2	0	-0.5	0.5	-1	0	-100
	0	-1400	2100	0	1	840,000

- Note that when we get the pivot point corresponding to x, the row 1 is now

headed x instead of s_1

- Our third row is not yet optimal since we have a negative value – 1400.

- To determine which between 0.5 and – 0.5 will be our pivot point, we first check the ratio.

- For row 1 we have; $\frac{200}{0.5} = 400$ and for row 2 we have $\frac{-100}{-0.5} = 200$.

	x	y	s_1	s_2	P	
x	1	0.5	0.5	0	0	200
s_2	0	-0.5	0.5	-1	0	-100
	0	-1400	2100	0	1	840,000

The lowest ratio is 200 and hence we make -0.5 the pivot i.e. first, we multiply row 2 by – 2 to get;

	x	y	s_1	s_2	P	
x	1	0.5	0.5	0	0	200
y	0	1	-1	2	0	200
	0	-1400	2100	0	1	840,000

- Next, we have to make the rest of the values in column y to be zero i.e. 0.5 and -1400.
- This is achieved by following elementary row operations;
- To change Row R_3; $(1400R_2 + R_3) \rightarrow R_3$ To change Row 1, R_1; $(R_1 - \frac{1}{2} R_2) \rightarrow R_1$;

	x	y	s_1	s_2	P	
x	1	0.5	0.5	0	0	200
s_2	0	-0.5	0.5	-1	0	-100
	0	-1400	2100	0	1	840,000

to get the tableau below

	x	y	s_1	s_2	P	
x	1	0	0.25	-0.5	0	100
y	0	1	-1	2	0	200
	0	0	700	2800	1	1,120,000

Note that when we get the pivot point corresponding to y, the row 1 is now headed y instead of s_1

	x	y	s_1	s_2	P	
x	1	0	0.25	-0.5	0	100
y	0	1	-1	2	0	200
	0	0	700	2800	1	1,120,000

The last row, Row 3, is all positive, and hence this is the optimal.

Therefore, from Row 1 we can conclude that $x = 100$ and from Row 2 $y = 200$, and from Row 3 we get our maximum sales revenue as $1{,}120{,}000$.

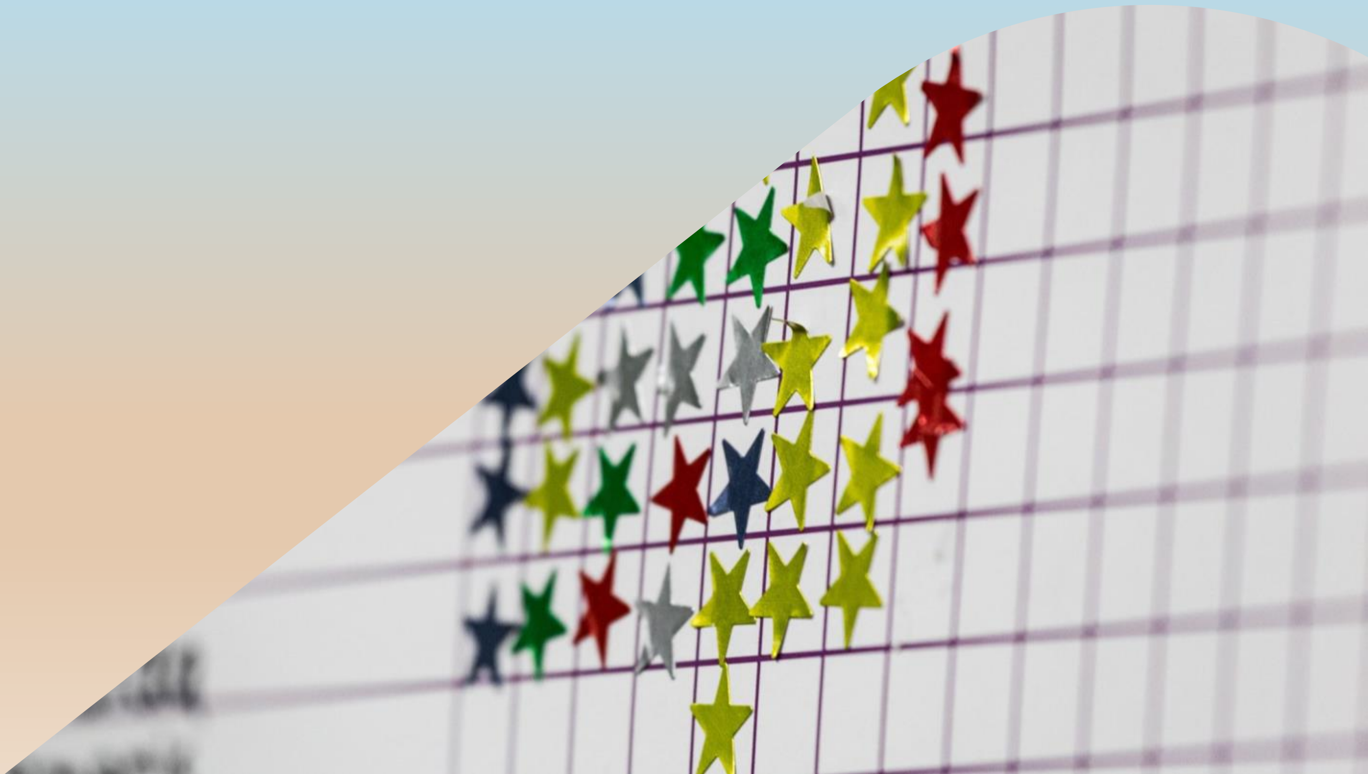
Example

Use the Simplex Method to work out the following:

Minimize $P = 3x + 2y$

Subject to the constraints:

$$2x + y \geq 10$$
$$x + 3y \geq 15$$
$$x \geq 0$$
$$y \geq 0$$



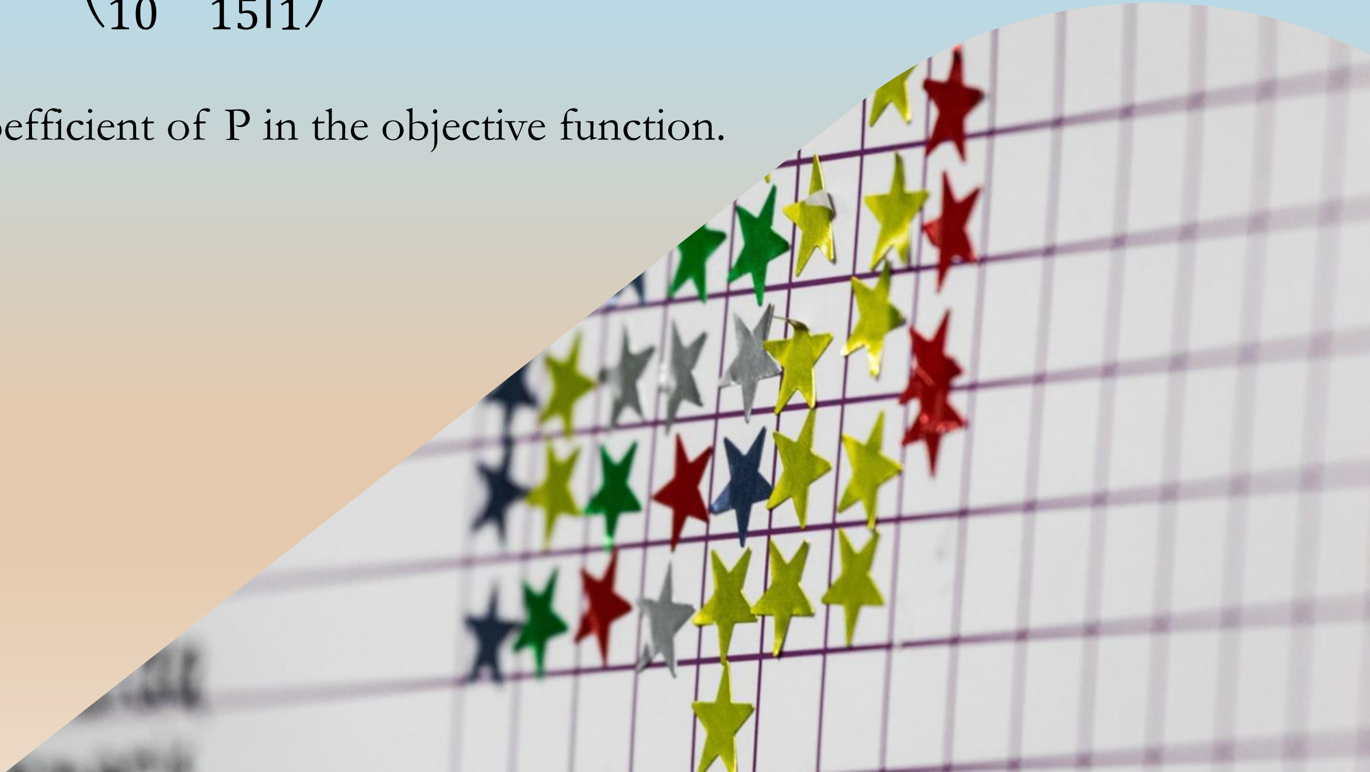
Example…contd…

We need to convert our case to a maximization case.

First, we write in matrix form and transpose the matrix i.e.

$$\left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 3 & 15 \\ 3 & 2 & 1 \end{array}\right)^{T} = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 10 & 15 & 1 \end{array}\right)$$

Note that the 1 in the last row, last column i.e. a_{33} is the coefficient of P in the objective function.



Example…contd…

$$\left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 3 & 15 \\ 3 & 2 & 1 \end{array}\right)^{T} = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 10 & 15 & 1 \end{array}\right)$$

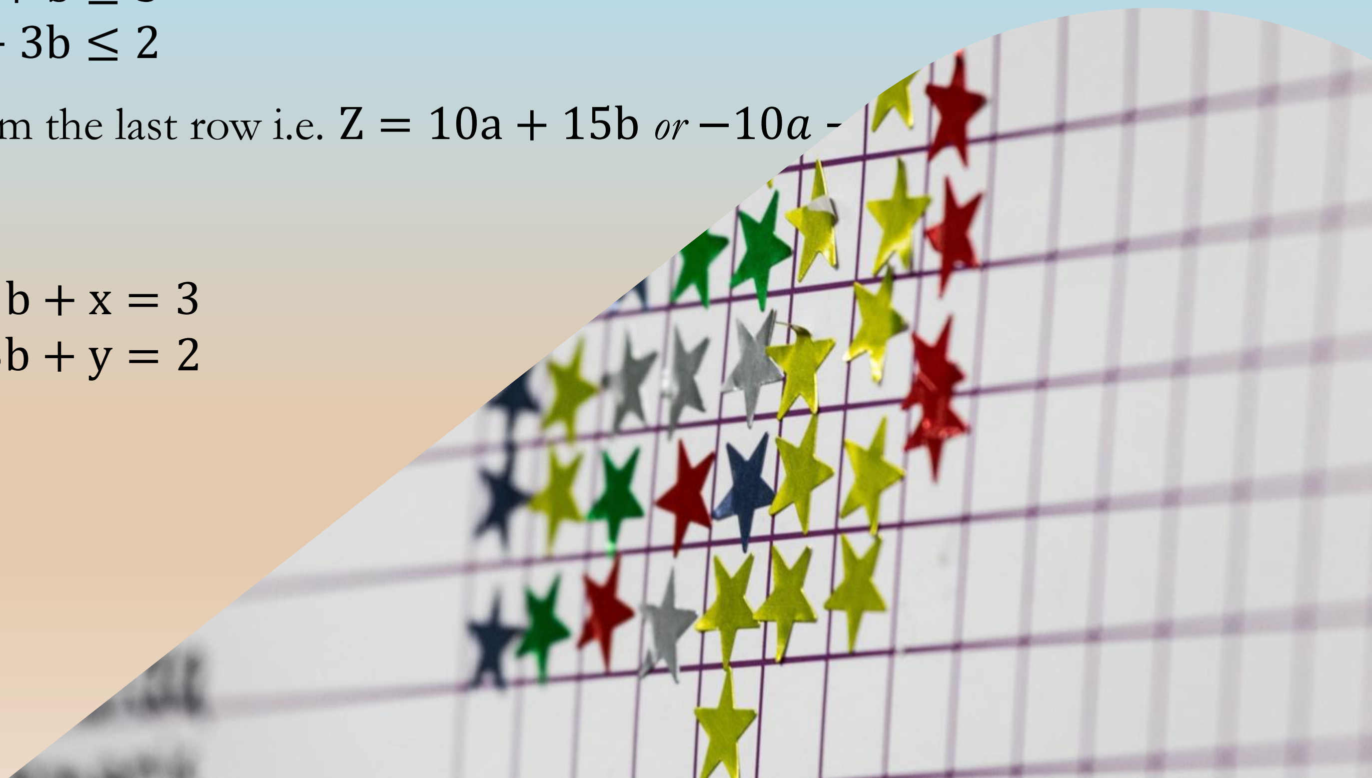
Thus, we have the constraints as;

$$2a + b \leq 3$$
$$a + 3b \leq 2$$

And the objective function that we need to maximize is from the last row i.e. $Z = 10a + 15b$ *or* $-10a -$

We introduce x and y as the slack variables to get;

$$2a + b + x = 3$$
$$a + 3b + y = 2$$



Initial tableau

	a	b	*x*	*y*	z	
	2	1	1	0	0	3
	1	3	0	1	0	2
	-10	-15	0	0	1	0

- Our aim is to ensure that the last row is all positive or zero.
- The smallest value is -15, hence we need to make either 1 or 3 in column b a pivot entry.

- We find the ratio for the two rows. For row 1, the ratio is $\frac{3}{1} = 3$ and for row 2, the ratio is $\frac{2}{3}$.

The smallest ratio is $\frac{2}{3}$ hence we work with row 2. Next multiply row 2 by $\frac{1}{3}$ to make 3 1 i.e. $\frac{1}{3}R_2 \rightarrow R_2$

	a	b	x	y	z	
	2	1	1	0	0	3
	1	3	0	1	0	2
	-10	-15	0	0	1	0

	a	b	x	y	z	
	2	1	1	0	0	3
	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$
	-10	-15	0	0	1	0

- For it to be a pivot entry, it must be the only none zero in the column.

Hence, we need to make 1 and -15 zero by applying the elementary operations;

$(15R_2 + R_3) \rightarrow R_3$ and $(R_1 - R_2) \rightarrow R_1$ to get;

	a	b	x	y	z	
	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{7}{3}$
	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$
	-5	0	0	5	1	10

Again, in Row 3 we have a negative 5. We need to make either $\frac{5}{3}$ or $\frac{1}{3}$ a pivot entry.

The ratio of row 1 is $\frac{7}{3} \div \frac{5}{3} = \frac{7}{3} \times \frac{3}{5} = \frac{7}{5}$ and the ratio of row 2 is $\frac{2}{3} \div \frac{1}{3} = \frac{2}{3} \times \frac{3}{1} = 2$.

This implies that we need to make $\frac{5}{3}$ the pivot entry. Hence, we carry out the following elementary operations; $\frac{3}{5} R_1 \rightarrow R_1$

	a	b	x	y	z	
	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{7}{5}$
	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$
	-5	0	0	5	1	10

Next, we ensure the entry is the only non-zero in the column by carrying out the following operations; $\left(\frac{1}{3}R_1 - R_2\right) \to R_2$ and $(5R_3 + R_3) \to R_3$ to get;

	a	b	x	y	z	
	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{7}{5}$
	0	-1	$\frac{1}{5}$	$-\frac{2}{5}$	0	$-\frac{1}{5}$
	0	0	3	4	1	17

The integers in our last row are all positive.

Therefore, $x = 3, y = 4$ and our minimum value is 17

References

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❖ The Simplex Method and the Dual : A Minimization Example ❖ (youtube.com)