Business Mathematics

Lecture 4

Matrix Algebra

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Introduction to Lecture 4

This lecture introduces you to matrix algebra and its applications to solving economics and business-related problems. We shall demonstrate operations in matrix algebra. However, basic matrix is a prerequisite to this lecture. These concepts are covered under basic mathematics. The examples in this lecture will however cover these basic operations. The lecture will form a foundation for lecture 5 on Input-output analysis.

Further Readings

These notes have been derived from diverse resources. These resources are recommended for further reading to gain more insights on the application of matrix algebra to business or commercial arithmetic, and other areas. The resources offer a detailed background introduction to matrix algebra that may not be covered in this lecture. These are (Jacques, 2006; Kahenya, 2017; Lay et al., 2016; Murray & Robert, 2009; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Carry out basic matrix operations.
(ii) Apply matrix algebra to solve business mathematics problems.

Introduction

Goods and even services can be displayed in a matrix format for faster and easier comprehension of their various attributes. Inventory is described as a matrix where the goods or services offered are stored. Goods have different attributes and configurations such as price, size, model, make, processes, finished, packaged, date of manufacturing, date of expiry among others. All these can be well represented in a matrix.

Therefore, the matrix structure is a management concept that can be used to enhance efficiency, collaboration, and decision-making not only in business setup but equally in other fields. In business, one can use matrix structure in;

Project management where employees are assigned to both functional sections and project teams simultaneously or basically assist in organizing and managing the personnel.

Resource allocation can be optimized and prioritized by applying matrix structures.

Equally complex problem solving can be facilitated by employing matrix structures. These structures can allow cross-functional problem-solving teams that can encourage diverse perspectives and fosters creativity in finding solutions.

Definition 1: Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order i.e. $m \times n$ then $A + B = C$ and $A - B = D$ for any i and j. Note that $A + B = B + A$ i.e. commutative holds for matrix addition. However, $A - B \neq B - A$.

Example 1: The sale of different type of soft drinks in a shop, in 3 days were recorded as below;

Soft drink	**Day 1**	**Day 2**	**Day 3**
Fanta	7	7	4
Coke	11	7	9
Sprite	15	6	3

This can be represented in a matrix form of order 3×3 where the rows are the type of soft drinks, and the columns are the days i.e.

$$Q = \begin{pmatrix} 7 & 7 & 4 \\ 11 & 7 & 9 \\ 15 & 6 & 3 \end{pmatrix}$$

Suppose each Fanta, coke, and sprite is sold at 20, 21 23 KES respectively, determine the amount that the shop received from these sales. Note that the price of the soft drinks can be represented as a matrix of order 1×3 i.e. $P = (20 \quad 21 \quad 23)$. Next we multiply P by Q to get

Total sales per day, $PQ = (20 \quad 21 \quad 23)\begin{pmatrix} 7 & 7 & 4 \\ 11 & 7 & 9 \\ 15 & 6 & 3 \end{pmatrix} = (716 \quad 425 \quad 338)$.

$$\text{Total Revenue, TR} = 716 + 425 + 338 = 1479$$

Example 2: Consider the sales of soft drinks at two different kiosks A and B, over a 3-day period.

	Day 1		**Day 2**		**Day 3**	
	Kiosk A	**Kiosk B**	**Kiosk A**	**Kiosk B**	**Kiosk A**	**Kiosk B**
Fanta	4	2	11	3	7	8
Coke	10	9	7	9	12	3
Sprite	19	13	6	11	13	9

i) Represent the sales in two different matrices.

ii) Determine the combined total sales for both kiosks A and B assuming that the prices of the soft drinks Fanta, coke, and sprite are 20 KES, 21 KES, and 23 KES respectively.

iii) Find the difference in sales of the two kiosks.

Solution:

(i) Kiosk $A = \begin{pmatrix} 4 & 11 & 7 \\ 10 & 7 & 12 \\ 19 & 6 & 13 \end{pmatrix}$ and Kiosk $B = \begin{pmatrix} 2 & 3 & 8 \\ 9 & 9 & 3 \\ 13 & 11 & 9 \end{pmatrix}$

(ii) The total number of drinks sold by both kiosks;

$$\text{Quantity}, Q = A + B = \begin{pmatrix} 4 & 11 & 7 \\ 10 & 7 & 12 \\ 19 & 6 & 13 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 8 \\ 9 & 9 & 3 \\ 13 & 11 & 9 \end{pmatrix} = \begin{pmatrix} 6 & 14 & 15 \\ 19 & 16 & 15 \\ 32 & 17 & 22 \end{pmatrix}$$

The total sales per drink will be

$$(20 \quad 21 \quad 23)\begin{pmatrix} 6 & 14 & 15 \\ 19 & 16 & 15 \\ 32 & 17 & 22 \end{pmatrix} = (1255 \quad 1007 \quad 1121)$$

Hence the total sales are $1255 + 1007 + 1121 = 3383$ KES

(iii) The difference in sales is

$$A - B = \begin{pmatrix} 4 & 11 & 7 \\ 10 & 7 & 12 \\ 19 & 6 & 13 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 8 \\ 9 & 9 & 3 \\ 13 & 11 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 8 & -1 \\ 1 & -2 & 9 \\ 6 & -5 & 4 \end{pmatrix}$$

Alternatively;

$$B - A = \begin{pmatrix} 2 & 3 & 8 \\ 9 & 9 & 3 \\ 13 & 11 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 11 & 7 \\ 10 & 7 & 12 \\ 19 & 6 & 13 \end{pmatrix} = \begin{pmatrix} -2 & -8 & -1 \\ -1 & 2 & -9 \\ -6 & 5 & -4 \end{pmatrix}$$

Notice the sign may not be significant in both cases. However, $A - B \neq B - A$. Note that matrix multiplication is not commutative i.e. $AB \neq BA$. Matrices must be compatible for matrix multiplication to be valid. That is given two matrices A and B of order $m \times n$ and $p \times q$, then AB is possible if and only if $n = q$. The resultant matrix will be of order $m \times q$.

Example 3: Given matrix $A = \begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 & -3 \\ 2 & 7 & 10 \end{pmatrix}$ then determine AB and BA.

Solution: Note that A is a 2×2 and B is a 2×3 hence AB is possible, but BA is invalid. The order of the resultant matrix of AB will be 2×3. Using Falk's scheme we have;

AB		5	1	−3
		2	7	10
2	7	24	51	64
4	−3	14	−17	−42

$$\begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 5 & 1 & -3 \\ 2 & 7 & 10 \end{pmatrix} = \begin{pmatrix} 24 & 51 & 64 \\ 14 & -17 & -42 \end{pmatrix}$$

Example 4: Two goods have their equilibrium prices as p_1 and p_2 respectively and satisfy the following system;

$$7p_1 + 3p_2 = 89$$
$$4p_1 + 9p_2 = 80$$

Determine the value of p_1 and p_2.

Solution: We can write the above system in matrix form to get; $\begin{pmatrix} 7 & 3 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 89 \\ 80 \end{pmatrix} \cdots$ (i)

We can use Cramer's rule to determine the values of p_1 and p_2 (see lecture 1) or we can use the inverse of the matrix method.

Let $A = \begin{pmatrix} 7 & 3 \\ 4 & 9 \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 89 \\ 80 \end{pmatrix}$ then we can write (i) as;

$$A\mathbf{p} = \mathbf{c}$$

$$\text{then } \mathbf{p} = A^{-1}\mathbf{c}$$

To get the inverse of matrix A we need first to find the determinant i.e.

$$\det A = \begin{vmatrix} 7 & 3 \\ 4 & 9 \end{vmatrix} = 63 - 12 = 51$$

Hence $A^{-1} = \frac{1}{51}\begin{pmatrix} 9 & -3 \\ -4 & 7 \end{pmatrix}$. Therefore, $\boldsymbol{p} = A^{-1}\mathbf{c} = \frac{1}{51}\begin{pmatrix} 9 & -3 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 89 \\ 80 \end{pmatrix} = \frac{1}{51}\begin{pmatrix} 561 \\ 204 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}$

$$\Rightarrow p_1 = 11, p_2 = 4$$

Definition 1: Transpose of a matrix

The transpose of a $m \times n$ matrix A is the $n \times m$ matrix denoted A^T whose columns are formed from the corresponding rows of matrix A e.g. given matrix $X = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 13 & 7 \end{pmatrix}$ then

$$X^T = \begin{pmatrix} 3 & 4 \\ 1 & 13 \\ 0 & 7 \end{pmatrix}$$

Example 5: The system below represents the prices of three interdependent goods.

$$\mathrm{p}_1 + 2\mathrm{p}_2 + \mathrm{p}_3 = 57$$
$$3\mathrm{p}_1 + \mathrm{p}_2 + 2\mathrm{p}_3 = 85$$
$$4\mathrm{p}_1 + 3\mathrm{p}_2 + 5\mathrm{p}_3 = 184$$

Determine the equilibrium prices p_1, p_2, p_3 using the inverse matrix method.

Solution: We need to write the system in matrix form i.e. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{pmatrix}\begin{pmatrix} \mathrm{p}_1 \\ \mathrm{p}_2 \\ \mathrm{p}_3 \end{pmatrix} = \begin{pmatrix} 57 \\ 85 \\ 184 \end{pmatrix}$

The system is of the form $\mathrm{A}\mathbf{p} = \mathbf{c}$ where $\mathrm{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} \mathrm{p}_1 \\ \mathrm{p}_2 \\ \mathrm{p}_3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 57 \\ 85 \\ 184 \end{pmatrix}$

Then $\mathbf{p} = \mathrm{A}^{-1}\mathbf{c} \Rightarrow \mathrm{A}^{-1} = \frac{1}{|\mathrm{A}|}\mathrm{Adj}\ (\mathrm{A})$ where $|\mathrm{A}| = \det \mathrm{A}, \mathrm{Adj}(\mathrm{A}) = \text{Adjoint A}$

The adjoint of a matrix A is the transpose of the matrix consisting of the cofactors of the entries of matrix A

From lecture 1, the $\det \mathrm{A} = 1\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = 1(-1) - 2(7) + 1(5) = -10$

Hence, $\mathrm{Adj}\ (\mathrm{A}) = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1 & -7 & 5 \\ -7 & 1 & 5 \\ 3 & 1 & -5 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1 & -7 & 3 \\ -7 & 1 & 1 \\ 5 & 5 & -5 \end{pmatrix}$

Then $\mathrm{A}^{-1} = \frac{1}{|\mathrm{A}|}\mathrm{Adj}\ (\mathrm{A}) = \frac{1}{-10}\begin{pmatrix} -1 & -7 & 3 \\ -7 & 1 & 1 \\ 5 & 5 & -5 \end{pmatrix}$

Therefore $\mathbf{p} = \mathrm{A}^{-1}\mathbf{c} = \frac{1}{-10}\begin{pmatrix} -1 & -7 & 3 \\ -7 & 1 & 1 \\ 5 & 5 & -5 \end{pmatrix}\begin{pmatrix} 57 \\ 85 \\ 184 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -100 \\ -130 \\ -210 \end{pmatrix} = \begin{pmatrix} 10 \\ 13 \\ 21 \end{pmatrix}$

Our equilibrium prices are $\{\mathrm{p}_1 \quad \mathrm{p}_2 \quad \mathrm{p}_3\} = \{10 \quad 13 \quad 21\}$

Definition 1: Echelon form

A rectangular matrix is said to be in echelon form if it has the following properties;

i) All non-zeros are above any rows of all zeros

ii) Each leading entry of a row is in a column to the right of the leading entry of the row above.

iii) All entries in a column below a leading entry are zeros

For example, matrix A $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ is in echelon form while matrix B $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{pmatrix}$ is not in echelon form.

To find the echelon form of a matrix, one needs to carry the elementary row operations.

Definition 2: Elementary Row Operations

There are 3 elementary row operations;

i) Interchanging two rows

ii) Multiplying a row by a non-zero scalar

iii) Adding a multiple of one row to another row.

Remark 1: We can write the echelon form of a matrix A and then find the determinant as product of the elements in the main diagonal.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{pmatrix} r_3 - 4r_1 \sim \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & -5 & 1 \end{pmatrix} r_2 - 3r_1 \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -5 & 1 \end{pmatrix} r_3 - r_2 \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 2 \end{pmatrix} = B$$

Matrix B is row equivalent to matrix A. Matrix B is an upper triangular matrix. The product of the elements in the main diagonal is -10 i.e. det A.

Remark 2: If matrices A and B are row equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets.

Example 1: Use Gaussian elimination method to solve the following system; $\begin{matrix} x + 2y - z = 2 \\ x - 3z = -8 \\ y - z = -2 \end{matrix}$

Solution: We can write the system in matrix form i.e. $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ -2 \end{pmatrix}$

The augmented matrix A of the system is $A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 0 & -3 & -8 \\ 0 & 1 & -1 & -2 \end{pmatrix}$ i.e. an augmented matrix consists of the coefficients matrix and the constant matrix.

Next row-reduce the matrix A to its row equivalent echelon form i.e.

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 0 & -3 & -8 \\ 0 & 1 & -1 & -2 \end{pmatrix} r_2 - r_1 \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -2 & -2 & -10 \\ 0 & 1 & -1 & -2 \end{pmatrix} 2r_3 + r_2 \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & -4 & -14 \end{pmatrix} \frac{1}{-2} r_2, \frac{1}{-4} r_3 \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{7}{2} \end{pmatrix} = B$$

Matrix B is row equivalent to matrix A and hence the solution set of matrix A is same as the solution set of matrix A

From Row 3: $z = \frac{7}{2}$

From Row 2: $y + z = 5 \Rightarrow y = 5 - z = 5 - \frac{7}{2} = 1.5$

From Row 1: $x + 2y - z = 2 \Rightarrow x = 2 - 2y + z = 2 - 3 + 3.5 = 2.5$

$\therefore \{x, y, z\} = \{2.5. 1.5, 3.5\}$

Definition 1: Consider the economies of two trading countries 1 and 2. Then the equation denoting the a trading model of each country, in the absence of government interference can be modeled by (Jacques, 2006, p. 498).

$$Y_i = C_i + I_i + X_i - M_i$$

Where Y_i- Investment of country i, C_i – consumption of country i, M_i- imports of country i, and X_i- exports of country i. Note that exports of one country must be equals to the imports of the other country i.e. $X_1 = M_2$ and $X_2 = M_1$.

It can also be assumed that imports are dependent of the national income, $M_i = m_i Y_i$ where the marginal propensity to import m_i, satisfies $0 < m_i < 1$.

Example 2: Consider the equations below that represents a model of two trading countries;

$$\begin{aligned} Y_1 &= C_1 + I_1 + X_1 - M_1 \\ C_1 &= 0.6Y_1 + 120 \\ M_1 &= 0.4Y_1 \end{aligned} \qquad \begin{aligned} Y_2 &= C_2 + I_2 + X_2 - M_2 \\ C_2 &= 0.7Y_2 + 90 \\ M_2 &= 0.3Y_2 \end{aligned}$$

Write Y_1 and Y_2 in terms of I.

Solution: $Y_1 = 0.6Y_1 + 120 + I_1 + X_1 - 0.4Y_1$ but $X_1 = M_2 = 0.3Y_2$

$\Rightarrow Y_1 = 0.6Y_1 + 120 + I_1 + 0.3Y_2 - 0.4Y_1$

Simplifying the above equation to get; $0.8Y_1 - 0.3Y_2 = 120 + I_1 \cdots$ (i)

Again, $Y_2 = 0.7Y_2 + 90 + I_2 + X_2 - 0.3Y_2$ but $X_2 = M_1 = 0.4Y_1$

$$\Rightarrow Y_2 = 0.7Y_2 + 90 + I_2 + 0.4Y_1 - 0.3Y_2$$

Simplifying the above equation to get; $0.6Y_2 - 0.4Y_1 = 90 + I_2 \cdots$ (ii)

We then write the systems (i) and (ii) in matrix form;

$$\begin{pmatrix} 0.8 & -0.3 \\ -0.4 & 0.6 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 120 + I_1 \\ 90 + I_2 \end{pmatrix}$$

We write the Augmented matrix A of the system and reduced it to echelon form

$$A = \begin{pmatrix} 0.8 & -0.3 & 120 + I_1 \\ -0.4 & 0.6 & 90 + I_2 \end{pmatrix} 2r_2 - r_2 \sim \begin{pmatrix} 0.8 & -0.3 & 120 + I_1 \\ 0 & 0.9 & 60 + 2I_2 - I_1 \end{pmatrix} = B$$

From Row 2: $0.9Y_2 = 60 + 2I_2 - I_1 \therefore Y_2 = \frac{60+2I_2-I_1}{0.9}$

From Row 1: $0.8Y_1 = 0.3Y_2 + 120 + I_1 = 0.3\left(\frac{60+2I_2-I_1}{0.9}\right) + 120 + I_1$

$$= 20 + \frac{2}{3}I_2 - \frac{1}{3}I_1 + 120 + I_1$$

$$= 140 + \frac{2}{3}I_2 + \frac{2}{3}I_1 = \frac{420 + 2I_2 + 2I_1}{3}$$

Therefore $\{Y_1, Y_2\} = \left\{\frac{420+2I_2+2I_1}{3}, \quad \frac{60+2I_2-I_1}{0.9}\right\}$ with I_i assumed to be a known value and is determined exogenously.

Exercise

1) Evaluate the following

a. $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 3 & 1 \\ 0 & 12 & 11 \end{pmatrix}$ b. $\begin{pmatrix} 2 & 3 & 19 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 11 & -2 \\ 5 & 9 \end{pmatrix}$ c. $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 11 & 34 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 & -8 \\ 3 & 18 & 3 \\ 1 & 4 & 1 \end{pmatrix}$

2) Identify two properties of a determinant and verify with an example

3) Compute the adjoint of the following matrices

a) $\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ -8 & 2 & 7 \end{pmatrix}$

4) Use Gaussian elimination method to solve the following systems

a. $\begin{aligned} P + 2Q &= 3 \\ 2P + 4Q &= 9 \end{aligned}$

b. $\begin{aligned} P_1 + 2P_2 + P_3 &= 5 \\ 2P_1 + 2P_2 + P_3 &= 6 \\ P_1 + 2P_2 + 3P_3 &= 9 \end{aligned}$

5) Use the formula $A^{-1} = \frac{1}{\det A} \text{Adj.} A$ to find the inverse of the following matrices (3, 5 mks)

a) $\begin{pmatrix} -3 & 5 \\ 7 & 11 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -3 & 1 \\ 5 & 1 & 2 \\ 1 & 2 & -5 \end{pmatrix}$

6) Solve the following using the inverse method $\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned}$

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

Kahenya, P. N. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). *Linear Algebra and its Application* (5th ed.). Pearson.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.

Werner, F., & Sotskov, Y. N. (2006). *Mathematics of Economics and Business*. Routledge: Taylor & Francis Group.