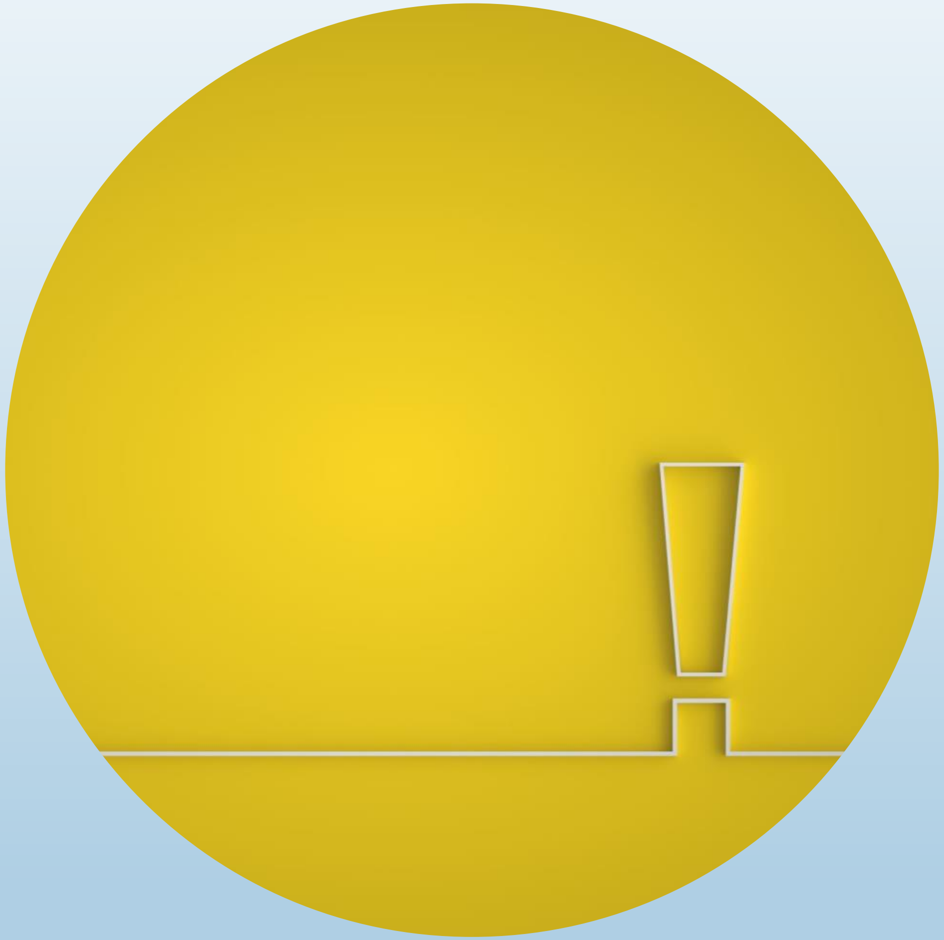
BUSINESS MATHEMATICS
Lecture 5
Input-output Analysis

Kahenya, N.P

Introduction to Lecture 5

- This lecture introduces you to input-output analysis in economics and business-related problems.
- It is a continuation of lecture 4 on matrix algebra on application of matrices.



Further Readings

- These notes have been derived from diverse resources.
- These resources are recommended for further reading to gain more insights on the application of matrix algebra and input-output analysis to business, and other areas.
- The resources offer a detailed background to matrix algebra and introduction to input-output analysis that may not be covered in this lecture.
- These are (Jacques, 2006; Kahenya, 2017; Lay et al., 2016; Murray & Robert, 2009; Werner & Sotskov, 2006).

Intended Learning Outcomes

- Define terms used in input-output analysis.
- Apply the Leontief input-output model.

Introduction

- Input-output analysis is an economic model that is used to study interdependencies between different sectors of an economy.
- This model was developed by Wassily Leontief in the 1950s.
- It is a popular model in economics, planning, and policy analysis.
- The model recognizes that the economy is represented as a system of interconnected sectors.
- Each sector produces/supplies goods and services i.e. their outputs, using input from other sectors.
- Hence the model's interest is to understand how changes in one sector influences other sectors and by extension the overall economy of a country.

Terms used in input-output analysis

- Let us assume that a country's economy is divided into n sectors that produces goods or services (outputs).
- Suppose $\mathbf{x}$ is the production vector in $\mathbb{R}^n$ that represents the output of each sector of a given period.
- Again, suppose that there exists another sector of the economy that only consumes goods and services i.e. the open sector.
- Let vector $\mathbf{d}$ be a ***final demand*** vector that list the value of goods and services demanded by the open sector such as the consumer demand, government consumption, surplus production, export etc.

Terms used in Input-output Analysis

- Note that as the 'various sectors produce goods *and services* to meet consumer demand, the producers create additional ***intermediate demand*** for goods *and services* they need as inputs for their own production'(Lay, 2003).

THE LEONTIEF MODEL (Lay, 2003)				
x		C**x**		**d**
Amount Produced	=	Intermediate demand	+	Final demand

- The intermediate demand plus final demand consists of the total output

Terms…contd.

- *Total value of input* is the amount of goods and services used in the production of other services.

- The input of an industry that originates from non-producing sectors such as labor, imports etc is referred to as the *primary input*.

- In a *closed input-output model*, the entire production is consumed only by the productive industries in the economy.

- This model favors the intermediate demand only.

- While in an *open input-output model*, both the intermediate and final demand are satisfied. Both the producer and non-producer benefit.

Relevance of the model

- To determine the output levels of each sector to meet both the intermediate demand and final demand.

- Understanding the interdependence of sectors in an economy. So as to understand supply chains, identifying bottlenecks, and assessing vulnerabilities to external shocks.

- Policy evaluation to evaluate the effectiveness of different policy interventions and investment strategies. This helps plan for economic development of a regional or country economy.

Relevance of the model..contd

- Estimating the economic ripple effects of policy changes, policy makers, planners etc. it can help make informed decisions and prioritize investments that yield the highest economic returns.
- Assessing the impact of various shocks or policy changes in the economy e.g. the effects of an increase in government spending, a change in trade policy, introduction of new technologies etc.
- Understanding the effects of decision-making involving key sectors.
- Computing the gross national product GNP.

Basic Assumption of the Model

For each sector there is a unit consumption vector in $\mathbb{R}^n$ that lists the inputs needed per unit of output of the sector (Lay, 2003).

The Leontief Input-Output model

Consider the Leontief model

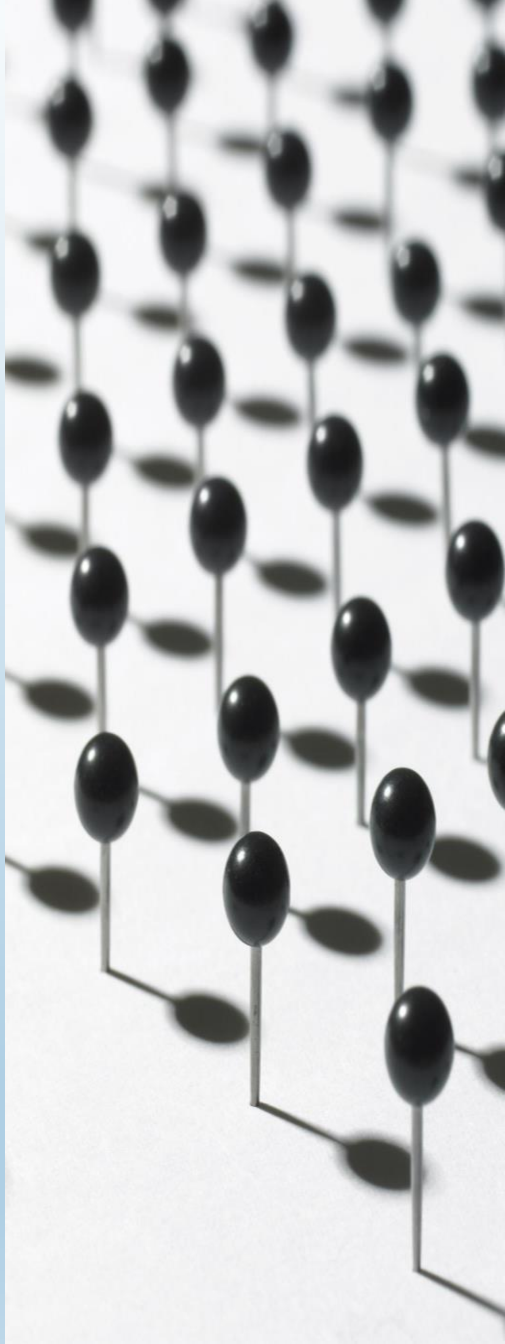
$$\underset{\text{Amount Produced}}{\mathbf{x}} = \underset{\text{Intermediate demand}}{\mathrm{C}\mathbf{x}} + \underset{\text{Final demand}}{\mathbf{d}}$$

Where C is the given coefficients matrix that contains all the technical coefficients.

Then we have; $\mathrm{I}\mathbf{x} - \mathrm{C}\mathbf{x} = \mathbf{d} \Rightarrow (\mathrm{I} - \mathrm{C})\mathbf{x} = \mathbf{d}$

$$\therefore \boxed{\mathbf{x} = (\mathrm{I} - \mathrm{C})^{-1}\mathbf{d}}$$

where $(\mathrm{I} - \mathrm{C}) - \text{Leontief's matrix}$ with I as the identity matrix of the same order as matrix C.



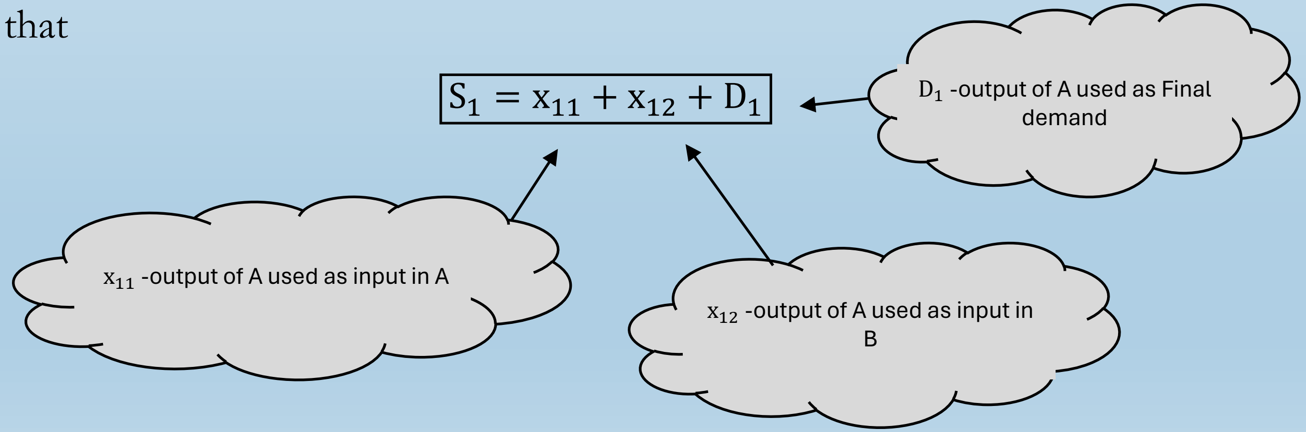
Example 1

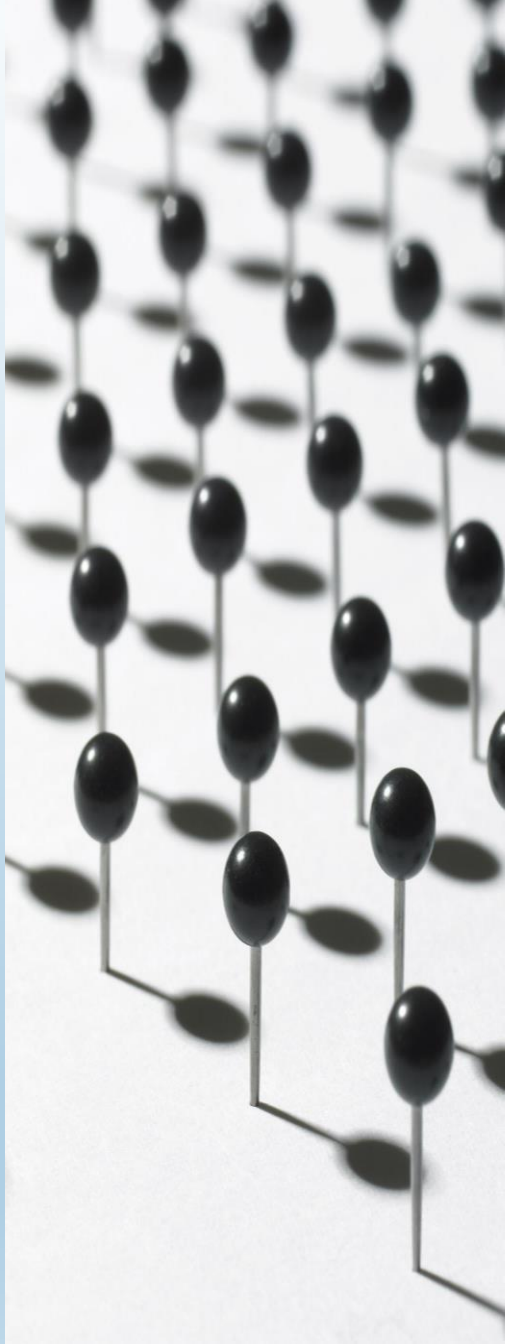
We have seen that in the Leontief model the producing sectors in an economy are interdependent i.e., the output of a sector is used as input of other sectors, however a percentage of the output of each sector in the economy is also used for final consumption.

That is output has two types of demand;

$$\boxed{\text{Output} = \text{Input demand} + \text{Final demand}}$$

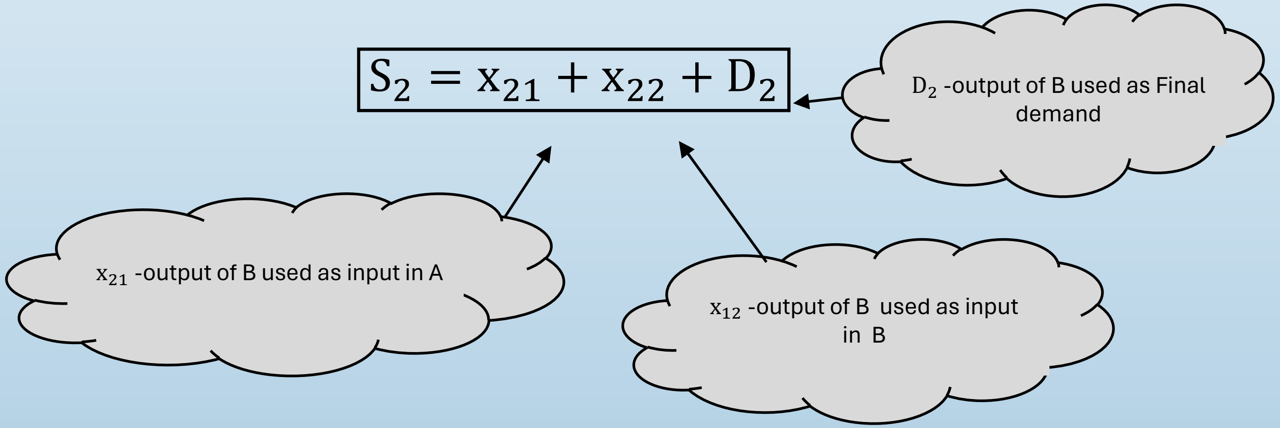
This can be visualized in a two-sector economy say A and B where S_1 and S_2 are the outputs of sector A and B respectively. We can say that





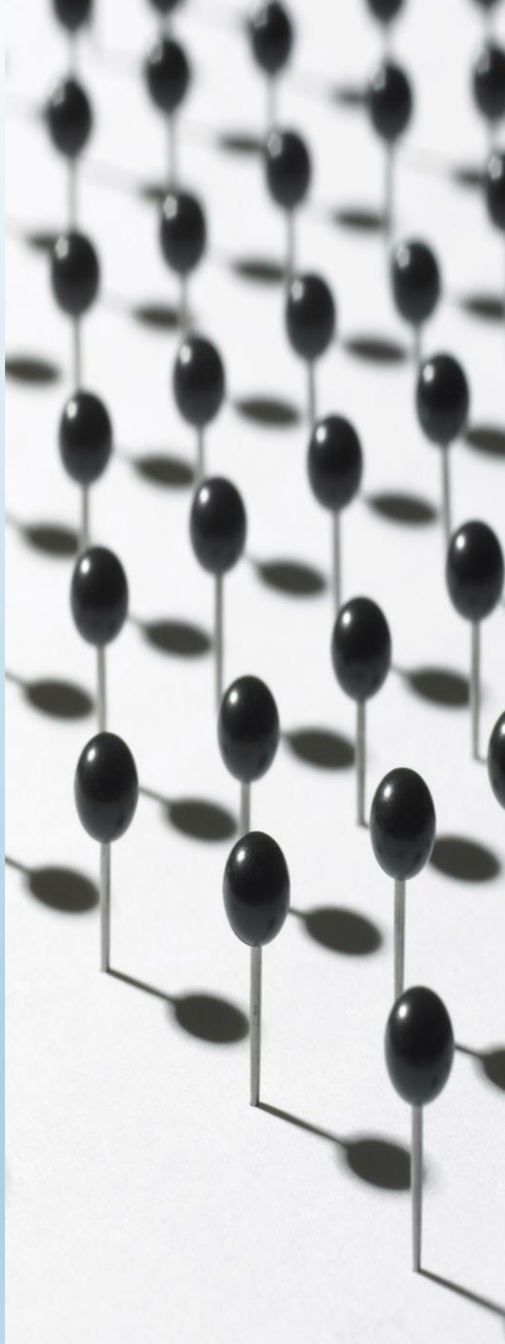
Example 1…contd.

Similarly, we can have X_2 as the output of sector B as follows;



Therefore, we have the outflow system of equation as

$$\begin{matrix} S_1 = x_{11} + x_{12} + D_1 \\ S_2 = x_{21} + x_{22} + D_2 \end{matrix} \dots (*)$$



Example 2…contd.

We can define;

$$\frac{x_{ij}}{S_j} = a_{ij} \Rightarrow \boxed{x_{ij} = a_{ij}S_j}$$

With a_{ij} amount of output of i^{th} sector is used as input in the j^{th} sector to produce one unit of output of the j^{th} sector i.e. the *input coefficient.*

Therefore, we can rewrite the system (*) as follows;

$$\begin{gathered} S_1 = a_{11}S_1 + a_{12}S_2 + D_1 \\ S_2 = a_{21}S_1 + a_{22}S_2 + D_2 \end{gathered}$$

Example 2…contd.

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$$

Output Vector

Input coefficient matrix

Final Demand Vector

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$$

$$\left(I - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \cdots (**)$$

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = C$, $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \mathbf{x}$, $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \mathbf{d}$ then (**) becomes $(I - C)\mathbf{x} = \mathbf{d}$

$$\Rightarrow \boxed{\mathbf{x} = (I - C)^{-1}\mathbf{d} \; - \text{Leontief input} - \text{output model}}$$

Example 2

Consider a two-sector economy, agricultural A and manufacturing sector B.

Suppose the input-output coefficients and final demands for this economy are as follows;

X_{11}: 40% of A output is consumed by A

X_{12}: 20% of A output is consumed by B

X_{21}: 30% of B output is consumed by A

X_{22}: 50% of B output is consumed by B

Example 2…contd.

The final demand for A_1 is 200 units while the final demand for B_1 is 300 units.

We can write the matrix of technical coefficients (technology matrix) C as; $C = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$

The final demand vector $\mathbf{d} = \begin{pmatrix} 200 \\ 300 \end{pmatrix}$

	Agricultural A	Manufacturing B
Agricultural A	**0.4**	**0.3**
Manufacturing B	**0.2**	**0.5**

Example2…contd.

Determine the total production from each sector that will satisfy both the intermediate and final demand.

Applying the Leontief model we have;

$$\therefore \mathbf{x} = (\mathrm{I} - \mathrm{C})^{-1}\mathbf{d}$$

$$\mathrm{I} - \mathrm{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.3 \\ -0.2 & 0.5 \end{pmatrix}$$

Next, we find the determinant of $\mathrm{I} - \mathrm{C}$ i.e.

$$\Delta = (0.6 \times 0.5) - (0.2 \times 0.3) = 0.3 - 0.06 = 0.24$$

Therefore $(\mathrm{I} - \mathrm{C})^{-1} = \frac{1}{0.24}\begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$

Hence $\mathbf{x} = \frac{1}{0.24}\begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.6 \end{pmatrix}\begin{pmatrix} 200 \\ 300 \end{pmatrix} = \frac{1}{0.24}\begin{pmatrix} 190 \\ 220 \end{pmatrix} \approx \begin{pmatrix} 791.67 \\ 916.67 \end{pmatrix}$

$$\Rightarrow \mathrm{A} \approx 791.67 \text{ units}, \mathrm{B} \approx 916.67 \text{ units}$$

Example 2…contd.

Suppose the primary input for sector A and B constitutes 20% and 30% respectively, determine the total primary input for the two sectors.

Solution:

We know that $A \approx 791.67$ units, $B \approx 916.67$ units

$$A = 0.2 \times 791.67 \approx 158.33; B = 0.3 \times 916.67 \approx 275$$

$$\Rightarrow \text{Total} \approx 433.33$$

Example 2...contd.

Account for A output

	Sector A
Consumed by itself	$0.4 \times 791.67 \approx 316.67$
Sold to the other sector B	$0.3 \times 791.67 \approx 237.5$
Final demand	200
Total output	754.17

Example 2…contd.

Account for sector B input

	Sector B
From Sector A	$0.3 \times 916.67 \approx 275$
From itself	$0.5 \times 916.67 \approx 458.34$
Primary input	$0.3 \times 916.67 \approx 275$
Total input	**1008.34**

Example 3

Consider a 3-sector economy in a coastal county as shown below.

Determine the production level assuming the final demand is 100, 60, and 40

for manufacturing x_1, fishing x_2, and service x_2 sectors respectively.

	Input consumed per unit of output		
Purchased by	Manufacturing	Fishing	Service
Manufacturing	0.4	0.3	0.1
Fishing	0.3	0.4	0.2
Service	0.2	0.1	0.4

Example 3…contd.

Solution: we have; $(\mathrm{I}-\mathrm{C})\mathbf{x}=\mathbf{d}\cdots(\mathrm{i})$

Hence $\mathrm{I}-\mathrm{C}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}0.4 & 0.3 & 0.1\\ 0.3 & 0.4 & 0.2\\ 0.2 & 0.1 & 0.4\end{pmatrix}=\begin{pmatrix}0.6 & -0.3 & -0.1\\ -0.3 & 0.6 & -0.2\\ -0.2 & -0.1 & 0.6\end{pmatrix}$

Equation (i) becomes $\begin{pmatrix}0.6 & -0.3 & -0.1\\ -0.3 & 0.6 & -0.2\\ -0.2 & -0.1 & 0.6\end{pmatrix}\mathbf{x}=\begin{pmatrix}100\\ 60\\ 40\end{pmatrix}\cdots(\mathrm{ii})$

We can solve for vector $\mathbf{x}=\begin{pmatrix}\mathrm{x}_1\\ \mathrm{x}_2\\ \mathrm{x}_3\end{pmatrix}$ using Gaussian elimination method.

We reduced to echelon form the augmented matrix of (ii) i.e.

Example 3…contd.

Solution: We reduced to echelon form the augmented matrix of (ii) i.e.

$$\Rightarrow \mathrm{A} = \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ -0.3 & 0.6 & -0.2 & 60 \\ -0.2 & -0.1 & 0.6 & 40 \end{pmatrix} 3r_2 + r_1 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ -0.3 & 0.6 & -0.2 & 60 \\ 0 & -0.6 & 1.7 & 220 \end{pmatrix} 2r_2 + r_1 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ 0 & 0.9 & -0.5 & 220 \\ 0 & -0.6 & 1.7 & 220 \end{pmatrix}$$

$$0.9r_3 + 0.6r_2 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ 0 & 0.9 & -0.5 & 220 \\ 0 & 0 & 1.23 & 330 \end{pmatrix} = \mathrm{B}$$

Matrix B is in echelon form, and it is row equivalent to matrix A.

Example 3…contd.

$$\begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ 0 & 0.9 & -0.5 & 220 \\ 0 & 0 & 1.23 & 330 \end{pmatrix} = \mathrm{B}$$

Matrix B is in echelon form, and it is row equivalent to matrix A.

From row 3 of matrix B we have; $1.23\mathrm{x}_3 = 330 \therefore \mathrm{x}_3 = \frac{330}{1.23} \approx 268.3$

From row 2 we have; $0.9\mathrm{x}_2 - 0.5\mathrm{x}_3 = 220 \Rightarrow 0.9\mathrm{x}_2 = 220 + 0.5\left(\frac{330}{1.23}\right) \approx 354.15 \therefore \mathrm{x}_2 \approx 393.5$

From row 1 we have; $0.6\mathrm{x}_1 - 0.3\mathrm{x}_2 - 0.1\mathrm{x}_3 = 100$

$$0.6\mathrm{x}_1 = 100 + 0.3\mathrm{x}_2 + 0.1\mathrm{x}_3 \approx 100 + 0.3(393.5) - 0.1(268.3) = 191.22$$

$$\mathrm{x}_1 \approx 318.7$$

Production levels for $\{\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_3\} = \{318.7 \quad 393.5 \quad 268.3\}$

Example 4- Closed Input-output Model

Consider as simple economy with 3-sectors Agriculture, Steel, and Petroleum, with their output as distributed among other sectors as shown below, where the entries in the column represent the fractional parts of a sector total output.

Purchased by	Distribution of Output from:		
	Agriculture	Steel	Petroleum
Agriculture	0.1	0.4	0.2
Steel	0.4	0.2	0.5
Petroleum	0.5	0.4	0.3

Note that the columns of output add up to 1 since the entire output is consumed within these 3 sectors i.e. closed input-output model.

Example 4- Closed Input-output Model

If we let Agriculture be x_1, steel x_2, and petroleum x_3, then from the 3 rows we have

$$\begin{aligned} x_1 &= 0.1x_1 + 0.4x_2 + 0.2x_3 \\ x_2 &= 0.4x_1 + 0.2x_2 + 0.5x_3 \\ x_3 &= 0.5x_1 + 0.4x_2 + 0.3x_3 \end{aligned}$$

Amount Produced

Input Demand

Example 4…contd.

We can rewrite the above to have a homogenous system;

$$-0.9x_1 + 0.4x_2 + 0.2x_3 = 0$$
$$0.4x_1 - 0.8x_2 + 0.5x_3 = 0$$
$$0.5x_1 + 0.4x_2 - 0.7x_3 = 0$$

Next, we write the augmented matrix of the system

$$\begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0.5 & 0.4 & -0.7 & 0 \end{pmatrix}$$

Then we row reduce the augmented matrix into its echelon form to get

Example 4...contd.

$$\begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0.5 & 0.4 & -0.7 & 0 \end{pmatrix} 4r_3 - 5r_2 \sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0 & 5.6 & -5.3 & 0 \end{pmatrix} 9r_2 + 4r_1$$

$$\sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0 & -5.6 & 5.3 & 0 \\ 0 & 5.6 & -5.3 & 0 \end{pmatrix} r_3 + r_2 \sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0 & -5.6 & 5.3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Example 4...contd.

$$\begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0 & -5.6 & 5.3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{B}$$

From row 2; $-5.6x_2 = -5.3x_3 \Rightarrow x_2 \approx 0.95x_3$

From row 1; $-0.9x_1 = -0.4x_2 - 0.2x_3 = -0.4(0.95x_3) - 0.2x_3 = -0.58x_3 \therefore x_1 \approx 0.64x_3$

In conclusion,

The equilibrium production levels is that agriculture must produce 64% of what petroleum products, and steel must produce 95% of what petroleum produces.

References

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