Business Mathematics

Lecture 5

Input-Output Analysis

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Introduction to Lecture 5

This lecture introduces you to input-output analysis in economics and business-related problems. It is a continuation of lecture 4 on matrix algebra on application of matrices.

Further Readings

These notes have been derived from diverse resources. These resources are recommended for further reading to gain more insights on the application of matrix algebra and input-output analysis to economics, and other areas. The resources offer a detailed background to not only matrix algebra but also an introduction to input-output analysis that may not be covered in this lecture. These are (Jacques, 2006; Lay et al., 2016; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Define terms used in input-output analysis.
(ii) Apply the Leontief input-output model.

Introduction

Input-output analysis is an economic model that is used to study interdependencies between different sectors of an economy. This model was developed by Wassily Leontief in the 1950s. It is a popular model in economics, planning, and policy analysis. The model recognizes that the economy is represented as a system of interconnected sectors. Each sector produces/supplies goods and services i.e. their outputs, using input from other sectors. Hence the model's interest is to understand how changes in one sector influences other sectors and by extension the overall economy of a country.

Terms Used in Input-Output Analysis

Let us assume that a country's economy is divided into n sectors that produces goods or services (outputs). Suppose **x** is the production vector in $\mathbb{R}^n$ that represents the output of each sector of a given period.

Again, suppose that there exists another sector of the economy that only consumes goods and services i.e. the open sector. Let vector **d** be a ***final demand*** vector that list the value of goods and services demanded by the open sector such as the consumer demand, government consumption, surplus production, export etc.

Note that as the 'various sectors produce goods *and services* to meet consumer demand, the producers create additional ***intermediate demand*** for goods *and services* they need as inputs for their own production'(Lay, 2003).

THE LEONTIEF MODEL (Lay, 2003)				
x		$C\mathbf{x}$		**d**
Amount Produced	=	Intermediate demand	+	Final demand

The *intermediate demand* plus *final demand* consists of the *total output*.

Total value of input is the amount of goods and services used in the production of other goods and services.

The input of an industry that originates from non-producing sectors such as labor, imports etc is referred to as the *primary input*.

In a *closed input-output model*, the entire production is consumed only by the productive industries in the economy. This model favors the intermediate demand only. While in an *open input-output model*, both the intermediate and final demand are satisfied. Both the producer and non-producer benefit.

Relevance of the model

As already pointed the model helps in

- To determine the output levels of each sector to meet both the intermediate demand and final demand.
- Understanding the interdependence of sectors in an economy. Understanding sectoral linkages is valuable in understanding supply chains, identifying bottlenecks, and assessing vulnerabilities to external shocks.
- Policy evaluation to evaluate the effectiveness of different policy interventions and investment strategies. This helps plan for economic development of a regional or country economy. Estimating the economic ripple effects of policy changes, policy makers, planners etc. it can help make informed decisions and prioritize investments that yield the highest economic returns.
- Assessing the impact of various shocks or policy changes in the economy e.g. the effects of an increase in government spending, a change in trade policy, introduction of new technologies etc.
- Understanding the effects of decision-making involving key sectors.
- Computing the gross national product GNP.

Basic Assumption of the model

For each sector there is a unit consumption vector in $\mathbb{R}^n$ *that lists the inputs needed per unit of output of the sector* (Lay, 2003).

The Leontief Input-output Model

Consider the Leontief model

$$\boxed{\underset{\text{Amount Produced}}{\mathbf{x}} = \underset{\text{Intermediate demand}}{\mathrm{C}\mathbf{x}} + \underset{\text{Final demand}}{\mathbf{d}}}$$

Where C is the given coefficients matrix that contains all the technical coefficients.

Then we have; $\mathrm{I}\mathbf{x} - \mathrm{C}\mathbf{x} = \mathbf{d} \Rightarrow (\mathrm{I} - \mathrm{C})\mathbf{x} = \mathbf{d}$

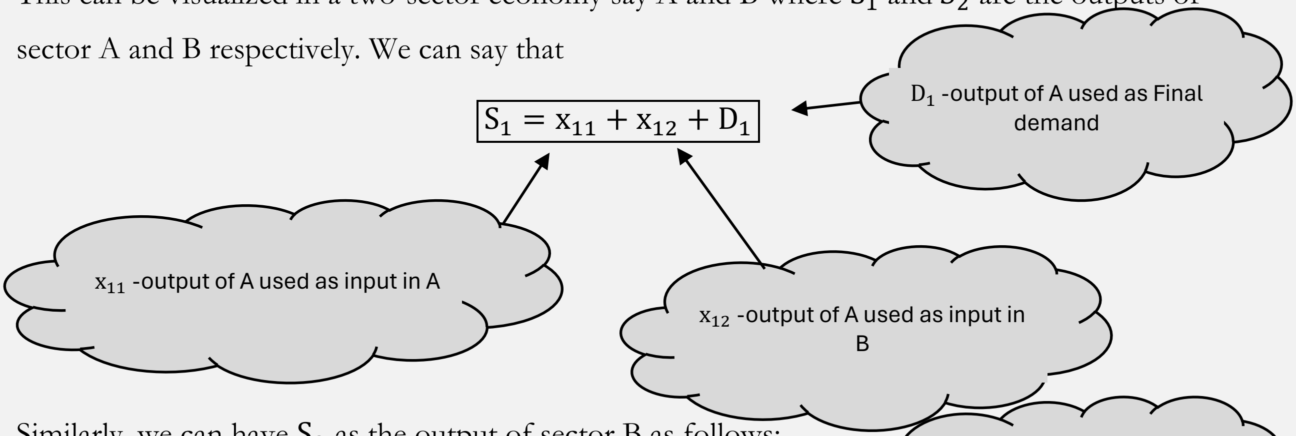
$$\therefore \boxed{\mathbf{x} = (\mathrm{I} - \mathrm{C})^{-1}\mathbf{d}}$$

where $(\mathrm{I} - \mathrm{C}) - \text{Leontief's matrix}$ with I as the identity matrix of the same order as matrix C.

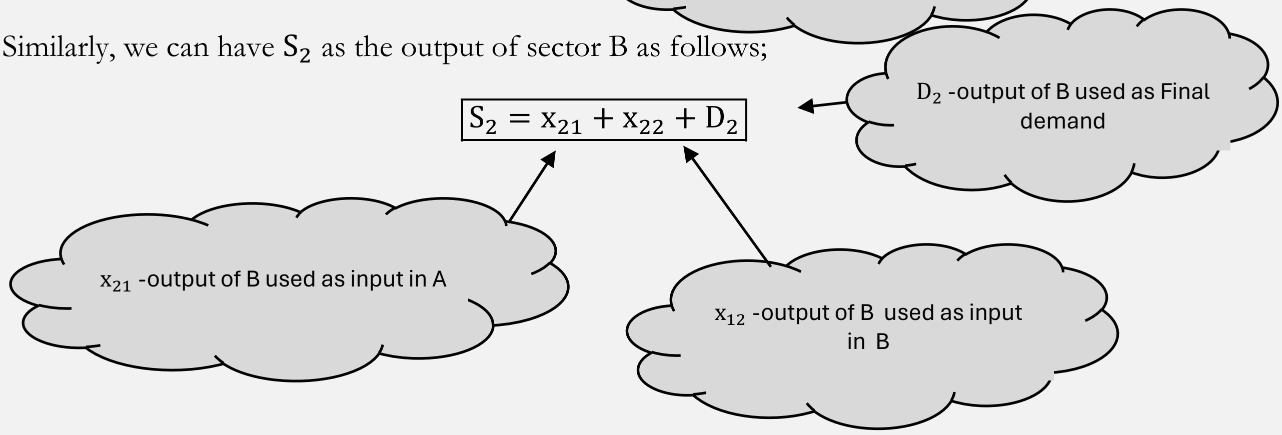
Example 1: We have seen that in the Leontief model the producing sectors in an economy are interdependent i.e. the output of a sector is used as input of other sectors, however a percentage of the output of each sector in the economy is also used for final consumption. That is output has two types of demand;

$$\boxed{\text{Output} = \text{Input demand} + \text{Final demand}}$$

This can be visualized in a two-sector economy say A and B where S_1 and S_2 are the outputs of sector A and B respectively. We can say that



Similarly, we can have S_2 as the output of sector B as follows;



Therefore, we have the outflow system of equation as

$$\begin{aligned} S_1 &= x_{11} + x_{12} + D_1 \\ S_2 &= x_{21} + x_{22} + D_2 \end{aligned} \cdots (*)$$

We can define;

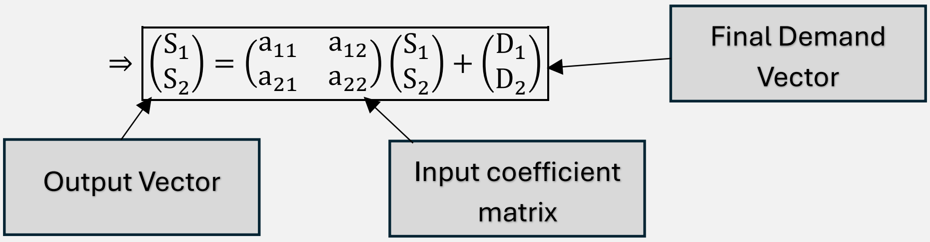
$$\frac{x_{ij}}{S_j} = a_{ij} \Rightarrow \boxed{x_{ij} = a_{ij}S_j}$$

With a_{ij} amount of output of i^{th} sector is used as input in the j^{th} sector to produce one unit of output of the j^{th} sector i.e. the *input coefficient.*

Therefore, we can rewrite the system (*) as follows;

$$\begin{aligned} S_1 &= a_{11}S_1 + a_{12}S_2 + D_1 \\ S_2 &= a_{21}S_1 + a_{22}S_2 + D_2 \end{aligned}$$

$$\Rightarrow \boxed{\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}}$$



$$\Rightarrow \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$$

$$\left(I - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \cdots (**)$$

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = C, \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \mathbf{x}, \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \mathbf{d}$ then (**) becomes $(I - C)\mathbf{x} = \mathbf{d}$

$$\Rightarrow \boxed{\mathbf{x} = (I - C)^{-1}\mathbf{d} \; - \text{Leontief input} - \text{output model}}$$

In general, for an economy with n producing interdependent sectors we have;

$$\begin{gathered} S_1 = a_{11}S_1 + a_{12}S_2 + \cdots \dots\dots\dots\dots\dots + D_1 \\ S_2 = a_{21}S_1 + a_{22}S_2 + \cdots \dots\dots\dots\dots\dots + D_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ S_n = a_{n1}S_n + a_{n2}S_n + \cdots + a_{nn}S_n + D_n \end{gathered}$$

This can be reduced to $\mathbf{x} = (I - C)^{-1}\mathbf{d}$ **–** *Static Leontief input-output model*. Since the components of final demand vector **d** are assumed to be autonomous.

Example 2: Consider a two-sector economy, agricultural A and manufacturing sector B.

Suppose the input-output coefficients and final demands for this economy are as follows;

X_{11}: 40% of A output is consumed by A

X_{12}: 20% of A output is consumed by B

X_{21}: 30% of B output is consumed by A

X_{22}: 50% of B output is consumed by B

	Agricultural A	*Manufacturing B*
Agricultural A	0.4	0.3
Manufacturing B	0.2	0.5

Table 1

The final demand for A is 200 units while the final demand for B is 300 units.

We can write the matrix of technical coefficients (technology matrix) C as; $C = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$

The final demand vector $\mathbf{d} = \begin{pmatrix} 200 \\ 300 \end{pmatrix}$

(i) Determine the total production from each sector that will satisfy both the intermediate and final demand.

Applying the Leontief model we have;

$$\therefore \mathbf{x} = (\mathrm{I} - \mathrm{C})^{-1}\mathbf{d}$$

$$\mathrm{I} - \mathrm{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.3 \\ -0.2 & 0.5 \end{pmatrix}$$

Next we find the determinant of $\mathrm{I} - \mathrm{C}$ i.e.

$$\Delta = (0.6 \times 0.5) - (0.2 \times 0.3) = 0.3 - 0.06 = 0.24$$

Therefore $(\mathrm{I} - \mathrm{C})^{-1} = \frac{1}{0.24}\begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.6 \end{pmatrix}$

Hence $\mathbf{x} = \frac{1}{0.24}\begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.6 \end{pmatrix}\begin{pmatrix} 200 \\ 300 \end{pmatrix} = \frac{1}{0.24}\begin{pmatrix} 190 \\ 220 \end{pmatrix} \approx \begin{pmatrix} 791.67 \\ 916.67 \end{pmatrix}$

$$\Rightarrow \mathrm{A} \approx 791.67 \text{ units}, \mathrm{B} \approx 916.67 \text{ units}$$

(ii) Suppose the primary input for sector A and B constitutes 20% and 30% respectively, determine the total primary input for the two sectors.

$$\mathrm{A} = 0.2 \times 791.67 \approx 158.33; \mathrm{B} = 0.3 \times 916.67 \approx 275 \Rightarrow \text{Total} \approx 433.33$$

(iii) Account for sector A output;

	Sector A
Consumed by itself	$0.4 \times 791.67 \approx$ 316.67
Sold to the other sector B	$0.3 \times 791.67 \approx 237.5$
Final demand	200
Total output	754.17

(iv) Account for sector B input

	Sector B
From sector A	$0.3 \times 916.67 \approx 275$
From itself	$0.5 \times 916.67 \approx 458.34$
Primary input	$0.3 \times 916.67 \approx 275$
Total input	1008.34

Example 3: Consider a 3-sector economy in a coastal county as shown below, determine the production level assuming the final demand is 100, 60, and 40 for manufacturing x_1, fishing x_2, and service x_3 sectors respectively.

	Input consumed per unit of output		
Purchased by	Manufacturing	Fishing	Service
Manufacturing	0.4	0.3	0.1
Fishing	0.3	0.4	0.2
Service	0.2	0.1	0.4

Table 2

Solution: by definition we have; $(I - C)\boldsymbol{x} = \boldsymbol{d} \cdots$ (i)

Hence $\mathrm{I} - \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.3 & -0.1 \\ -0.3 & 0.6 & -0.2 \\ -0.2 & -0.1 & 0.6 \end{pmatrix}$

Equation (i) becomes $\begin{pmatrix} 0.6 & -0.3 & -0.1 \\ -0.3 & 0.6 & -0.2 \\ -0.2 & -0.1 & 0.6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 100 \\ 60 \\ 40 \end{pmatrix} \cdots$ (ii)

We can solve for vector $\mathbf{x} = \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{pmatrix}$ using Gaussian elimination method. We reduced to echelon form the augmented matrix of (ii) i.e.

$$\mathrm{A} = \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ -0.3 & 0.6 & -0.2 & 60 \\ -0.2 & -0.1 & 0.6 & 40 \end{pmatrix}$$

$$\Rightarrow \mathrm{A} = \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ -0.3 & 0.6 & -0.2 & 60 \\ -0.2 & -0.1 & 0.6 & 40 \end{pmatrix} 3r_3 + r_1 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ -0.3 & 0.6 & -0.2 & 60 \\ 0 & -0.6 & 1.7 & 220 \end{pmatrix} 2r_2$$

$$+ r_1 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ 0 & 0.9 & -0.5 & 220 \\ 0 & -0.6 & 1.7 & 220 \end{pmatrix}$$

$$0.9r_3 + 0.6r_2 \sim \begin{pmatrix} 0.6 & -0.3 & -0.1 & 100 \\ 0 & 0.9 & -0.5 & 220 \\ 0 & 0 & 1.23 & 330 \end{pmatrix} = \mathrm{B}$$

Matrix B is in echelon form, and it is row equivalent to matrix A.

From row 3 of matrix B we have; $1.23\mathrm{x}_3 = 330 \therefore \mathrm{x}_3 = \frac{330}{1.23} \approx 268.3$

From row 2 we have; $0.9x_2 - 0.5x_3 = 220 \Rightarrow 0.9x_2 = 220 + 0.5\left(\frac{330}{1.23}\right) \approx 354.15 \therefore x_2 \approx 393.5$

From row 1 we have; $0.6x_1 - 0.3x_2 - 0.1x_3 = 100$

$$0.6x_1 = 100 + 0.3x_2 + 0.1x_3 \approx 100 + 0.3(393.5) - 0.1(268.3) = 191.22$$

$$x_1 \approx 318.7$$

Production levels for $\{x_1 \quad x_2 \quad x_3\} = \{318.7 \quad 393.5 \quad 268.3\}$

Example 4: (Closed input-output model) Consider as simple economy with 3-sectors Agriculture, Steel, and Petroleum, with their output as distributed among other sectors as shown below, where the entries in the column represent the fractional parts of a sector total output.

	Distribution of Output from:		
Purchased by	*Agriculture*	*Steel*	*Petroleum*
Agriculture	0.1	0.4	0.2
Steel	0.4	0.2	0.5
Petroleum	0.5	0.4	0.3

Table 3

Note that the columns of output add up to 1 since the entire output is consumed within these 3 sectors i.e. closed input-output model.

If we let Agriculture be x_1, steel x_2, and petroleum x_3 then from the 3 rows we have;

$$\begin{aligned} x_1 &= 0.1x_1 + 0.4x_2 + 0.2x_3 \\ x_2 &= 0.4x_1 + 0.2x_2 + 0.5x_3 \\ x_3 &= 0.5x_1 + 0.4x_2 + 0.3x_3 \end{aligned}$$

Amount produced ↑ (left side) Input Demand ↑ (right side)

We can rewrite the above to have a homogenous system;

$$\begin{aligned} -0.9x_1 + 0.4x_2 + 0.2x_3 &= 0 \\ 0.4x_1 - 0.8x_2 + 0.5x_3 &= 0 \\ 0.5x_1 + 0.4x_2 - 0.7x_3 &= 0 \end{aligned}$$

Next we write the augmented matrix of the system i.e.

$$\begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0.5 & 0.4 & -0.7 & 0 \end{pmatrix}$$

Then we row reduce the augmented matrix into its echelon form to get

$$\begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0.5 & 0.4 & -0.7 & 0 \end{pmatrix} 4r_3 - 5r_2 \sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.5 & 0 \\ 0 & 5.6 & -5.3 & 0 \end{pmatrix} 9r_2 + 4r_1 \sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0 & -5.6 & 5.3 & 0 \\ 0 & 5.6 & -5.3 & 0 \end{pmatrix}$$

$$r_3 + r_2 \sim \begin{pmatrix} -0.9 & 0.4 & 0.2 & 0 \\ 0 & -5.6 & 5.3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Note that from row 3 of matrix B, x_3 is a free variable.

From row 2; $-5.6x_2 = -5.3x_3 \Rightarrow x_2 \approx 0.95x_3$

From row 1; $-0.9x_1 = -0.4x_2 - 0.2x_3 = -0.4(0.95x_3) - 0.2x_3 = -0.58x_3 \therefore x_1 \approx 0.64x_3$

In conclusion, the equilibrium production levels is that agriculture must produce 64% of what petroleum produces, and steel must produce 95% of what petroleum produces.

Exercise

1) Theorem: Given a square matrix of order n and the identity matrix of the same order, if matrix A is row equivalent to I then $[A|I]$ is row equivalent to $[I|A^{-1}]$. Use this theorem to find A^{-1}

   a) $A = \begin{pmatrix} 5 & 3 \\ -2 & 7 \end{pmatrix}$

   b) $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -3 & 2 \\ 0 & 1 & 5 \end{pmatrix}$

2) Consider the technological coefficients matrix $A = \begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0.1 & 0.2 & 03 \\ 0.1 & 0.1 & 0.5 \end{pmatrix}$. Suppose that the total output is $x = \begin{pmatrix} 900 \\ 400 \\ 750 \end{pmatrix}$ determine the total internal demand. What is the total output required to mee the following external demand $d = \begin{pmatrix} 200 \\ 500 \\ 400 \end{pmatrix}$

3) The economy of a given country can be viewed from the goods sector and services sector. To produce 100 million KES worth of goods requires 20million KES worth of goods and 10 million KES worth of services. Again, to produce 100 million KES worth of services needs 70 million KES worth of goods and 15 million KES worth of KES. Generate an input-output matrix and hence calculate the value of goods and services needed in order to have an export surplus of 800 million KES worth of goods and 650 million KES worth of service.

4) A county X simple economy consists of three sectors; Agriculture A, Manufacturing B,, and Mining C. the input output matrix for this economy is given by

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0.1 & 0.4 & 0.3 \\ B & 0.2 & 0.1 & 0.4 \\ C & 0.3 & 0.5 & 0.1 \end{array}$$

   a. Find the number of product A consumed in the production of 120 million KES worth of goods B.
   b. Determine the total input required to produce 1 unit of product C
   c. Which sector consumes the greatest amount of mining products in the production of a unit of goods in the sector? Which the least consumer?
   d. What is the output from the economy required to meet an external demand of 250 million 450 million, and 372 million from sector A, B and C respectively.
   e. Calculate the total internal consumption in meeting the above demand?

5) It is given that Leontief's inverse matrix is:

$$(\mathrm{I} - \mathrm{A})^{-1} = \begin{bmatrix} 3.041.541.14 \\ 1.231.243.30 \\ 2.513.014.23 \end{bmatrix}$$

a. Determine the primary inputs of each industry.

b. Determine the technical coefficients matrix.

c. Find the matrix (I – A)

d. Find the total output and the primary input required by each industry if the final demand is 120, 230 and 240 for d_1, d_2, d_3 respectively

6) Attempt practice problems in (Jacques, 2006, p. 513).

References

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